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Convex integration and the L^p theory of elliptic equations

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Abstract. This paper deals with the L^p theory of linear elliptic partial differential equations with bounded measurable coefficients. We construct in two dimensions examples of weak and so-called very weak solutions, with critical integrability properties, both to isotropic equations and to equations in non-divergence form. These examples show that the general L^p theory, developed in [1, 24] and [2], cannot be extended under any restriction on the essential range of the coefficients. Our constructions are based on the method of convex integration, as used by S. Müller and V. Šverák in [30] for the construction of counterexamples to regularity in elliptic systems, combined with the staircase type laminates introduced in [15].

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1. Introduction

In the theory of the elliptic partial differential equations with bounded measurable coefficients the solutions are initially assumed to have square summable derivatives; for equations of non-divergence form the assumptions concern the second derivatives. As is well known (see [6, 7] and [26]), there is a range of exponents beyond $p = 2$ such that the derivatives are, in fact, L^p-integrable. Recent developments in the theory of planar quasiconformal mappings, in particular the area distortion theorem obtained by the first author in [1] and the invertibility of Beltrami operators proved in [4], have in two dimensions provided the precise range for these critical exponents, see [2, 24] and Theorems 1.1 and 1.5 below. For more information see also the monograph [3]. These ranges of exponents depend only on the ellipticity constants of the equation.

It is a natural question to ask if restricting the range of the coefficients could yield higher integrability for the gradients of the solutions. A basic result pointing in this direction is the work of L. C. Piccinini and S. Spagnolo [35]. There it is shown

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that if $\sigma(x) = \rho(x)I$, where ρ is a real valued function with $1/K \le \rho(x) \le K$, then u has a better Hölder regularity than in the case of a general σ.

In this article we present a general method for constructing examples which show that, in the contrary, for the L^p theory such improved regularity beyond the critical exponents is not possible.

Let us start by recalling the basic notations and the positive results.

Theorem 1.1. *Let $K \ge 1$, let $\Omega \subset \mathbb{R}^2$ be a bounded domain and let $\sigma(x) : \Omega \to \mathbb{R}^{2\times 2}_{\mathrm{sym}}$ be a measurable function such that, in the sense of quadratic forms,*

$$\frac{1}{K} I \le \sigma(x) \le KI \qquad a.e.\ \ x \in \Omega \tag{1.1}$$

Let $u \in W^{1,q}(\Omega)$, $q \ge \frac{2K}{K+1}$, be a weak solution of

$$\operatorname{div}(\sigma(x)\nabla u(x)) = 0 \quad in\ \Omega. \tag{1.2}$$

Then $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ for all $p < \frac{2K}{K-1}$.

Here $\mathbb{R}^{2\times 2}_{\mathrm{sym}}$ represents the space of 2×2 symmetric matrices with real entries.

Theorem 1.1 is a combination of results due to F. Leonetti and V. Nesi [24], the first author [1] and S. Petermichl and A. Volberg [34]. The proof is based on the fact [24] that locally any weak solution to (1.2) coincides with the real part of a K-quasiregular mapping, K being the ellipticity constant of (1.1). The results in [1] imply that any K-quasiregular mapping in $W^{1,q}$ belongs actually to the space $W^{1,p}$, whenever $\frac{2K}{K+1} < q < p < \frac{2K}{K-1}$. Finally, the end point case $q = \frac{2K}{K+1}$ was recently covered by S. Petermichl and A. Volberg (see [4, 13, 34]).

The classical examples built on the radial stretching[1] $u(x) = \Re(x|x|^{\frac{1}{K}-1})$ show that for general σ the range of exponents p, q cannot be improved without extra assumptions. On the other hand, the work of L. C. Piccinini and S. Spagnolo [35] suggests that for isotropic coefficients one might have better regularity.

Our first theorems show, however, that for Sobolev regularity one cannot improve either of the critical exponents $\frac{2K}{K+1}$, $\frac{2K}{K-1}$ even if the essential range of σ consists of only two isotropic matrices.

Theorem 1.2. *Let Ω be a bounded domain in $\mathbb{R}^2$ and let $K > 1$. There exists a measurable function $\rho_1 : \Omega \to \left\{\frac{1}{K}, K\right\}$ such that the solution $u_1 \in W^{1,2}(\Omega)$ to the equation*

$$\begin{cases} \operatorname{div}(\rho_1(x)\nabla u_1(x)) = 0 & in\ \Omega \\ u_1(x) = x_1 & on\ \partial\Omega \end{cases} \tag{1.3}$$

satisfies for every disk $B = B(x_0, r) \subset \Omega$ the condition

$$\int_B |\nabla u_1|^{\frac{2K}{K-1}} = \infty. \tag{1.4}$$

[1] Here the variable $x \in \mathbb{R}^2$ is identified with the complex number $x = x_1 + ix_2$.

Theorem 1.3. *For every $\alpha \in (0,1)$ there exists a measurable function $\rho_2 : \Omega \to \left\{\frac{1}{K}, K\right\}$ and a function $u_2 \in C^\alpha(\overline{\Omega})$ such that $u_2 \in W^{1,q}(\Omega)$ for all $q < \frac{2K}{K+1}$, $u_2(x) = x_1$ on $\partial\Omega$ with $x = (x_1, x_2)$ and*

$$\operatorname{div}(\rho_2(x)\nabla u_2(x)) = 0$$

in the sense of distributions, but for every disk $B = B(x_0, r) \subset \Omega$

$$\int_B |\nabla u_2|^{\frac{2K}{K+1}} = \infty. \tag{1.5}$$

As a particular consequence, Theorem 1.2 and Theorem 1.3 apply also to the quasiregular mappings, since u_1 and u_2 coincide with the real parts of (weakly) quasiregular mappings f_1 and f_2, respectively. Here both mappings satisfy

$$\partial_{\bar z} f = \pm k\, \overline{\partial_z f} \quad \text{a.e. in } \Omega, \tag{1.6}$$

where $k = \frac{K-1}{K+1}$. Indeed, in the proofs we construct f_1 and f_2 rather than u_1 and u_2. The same ideas yield extremal solutions also to the classical Beltrami equation

$$\partial_{\bar z} f = \pm k\, \partial_z f, \tag{1.7}$$

for details see Remark 3.21.

Remark 1.4. Our methods do not imply that ρ_1 and ρ_2 in the above theorems could be equal. Surprisingly, in the analogous problem for the Beltrami equation (1.7) one has a simple argument relating the corresponding results. Namely, let f be the very weak solution of (1.7) constructed in Theorem 3.18, with dilatation $\mu : \Omega \to \{\pm k\}$ and $u_2 = \Re f$ satisfying (1.5). If F is the classical homeomorphic solution to the same equation, with $F \in W^{1,2}_{\mathrm{loc}}$ and $\partial_{\bar z} F = \mu \partial_z F$, then $\int_B |\nabla F|^{\frac{2K}{K-1}+\varepsilon} = \infty$ for every $\varepsilon > 0$ and every ball $B \subset \Omega$. To see this, since $\nabla f \in L^q(B)$ for any $q < \frac{2K}{K+1}$, we would otherwise have $\nabla F \in L^{p_0}(B)$ and $\nabla f \in L^{q_0}(B)$ for some dual exponents p_0 and q_0. But then, for example by [23, Lemma 6.4], the composition $h = f \circ F^{-1}$ is in $W^{1,1}(FB)$ and h obeys the chain rule. Therefore $\partial_{\bar z} h = 0$ a.e. and by Weyl's lemma h is analytic. Then $f = h \circ F$ is also quasiregular, which contradicts the fact that $f \notin W^{1,2}(B)$.

Lastly we turn to the analogous results concerning linear elliptic equations in non-divergence form. The following theorem is due to K. Astala, T. Iwaniec and G. Martin in [2] where, answering a question of Pucci [36], quasiconformal techniques are applied to establish the precise L^p theory for planar equations in non-divergence form.

Theorem 1.5. *Let $K \geq 1$, let $\Omega \subset \mathbb{R}^2$ be a bounded domain and let $A(x) : \Omega \to \mathbb{R}^{2\times 2}_{\mathrm{sym}}$ be a measurable function such that for a.e. $x \in \Omega$*

$$\frac{1}{\sqrt{K}} I \leq A(x) \leq \sqrt{K} I, \quad \textit{with } \det A(x) \equiv 1. \tag{1.8}$$

Let $u \in W^{2,q}(\Omega)$, $q > \frac{2K}{K+1}$, be a solution of the equation

$$\operatorname{Tr}\Big(A(x)D^2u(x)\Big) = 0, \tag{1.9}$$

where $D^2u(x) = \big(\partial_{ij}u(x)\big)_{ij}$ is the Hessian matrix of u. Then $u \in W^{2,p}_{\mathrm{loc}}(\Omega)$ for all $p < \frac{2K}{K-1}$.

In (1.8) the second condition on $A(x)$ is a normalization which can always be made since A is bounded above and below and (1.9) is pointwise linear.

The key point in the proof is that under the assumptions of the theorem the complex gradient of the solution, $\partial_z u = (u_x, -u_y)$, is a K-quasiregular mapping. Therefore the ideas from [1,4,34] apply. In analogy with the result of Piccinini and Spagnolo it was recently shown by A. Baernstein II and L. V. Kovalev ([5]) that the complex gradients of solutions to equation (1.9) belong to a better Hölder space than general quasiregular mappings.

Concerning the sharpness of the theorem, an example due to C. Pucci built on an appropriate radial function shows that the L^p estimates fail at the lower critical exponent $q = \frac{2K}{K+1}$. Nevertheless, examples built on radial functions do not seem to work for the upper critical exponent. Here we not only provide required examples, but again show that the range of $A(x)$ is as simple as one can ask for.

Theorem 1.6. *Let $K \geq 1$ and let $\Omega \subset \mathbb{R}^2$ be a bounded domain.*

There exists a measurable

$$A_3 : \Omega \to \left\{ \begin{pmatrix} \frac{1}{\sqrt{K}} & 0 \\ 0 & \sqrt{K} \end{pmatrix}, \begin{pmatrix} \sqrt{K} & 0 \\ 0 & \frac{1}{\sqrt{K}} \end{pmatrix} \right\}$$

such that the solution $u_3 \in W^{2,2}(\Omega)$ to the equation

$$\begin{cases} \operatorname{Tr}(A_3(x)D^2u_3) = 0 & \textit{in } \Omega, \\ u_3(x) = x_1 & \textit{on } \partial\Omega \end{cases} \tag{1.10}$$

satisfies for every disk $B = B(x_0, r) \subset \Omega$ the condition

$$\int_B |D^2u_3|^{\frac{2K}{K-1}} = \infty. \tag{1.11}$$

Similarly, we have the counterpart of Theorem 1.3.

Theorem 1.7. *For every $\alpha \in (0, 1)$ there exists a measurable mapping*

$$A_4 : \Omega \to \left\{ \begin{pmatrix} \frac{1}{\sqrt{K}} & 0 \\ 0 & \sqrt{K} \end{pmatrix}, \begin{pmatrix} \sqrt{K} & 0 \\ 0 & \frac{1}{\sqrt{K}} \end{pmatrix} \right\}$$

and a function $u_4 \in C^{1,\alpha}(\overline{\Omega})$ *such that* $u_4 \in W^{2,q}(\Omega)$ *for all* $q < \frac{2K}{K+1}$,

$$\operatorname{Tr}(A_4(x)D^2u_4) = 0 \tag{1.12}$$

in the sense of distributions, but for every disk $B = B(x_0, r) \subset \Omega$

$$\int_B |D^2u_4|^{\frac{2K}{K+1}} = \infty. \tag{1.13}$$

Theorem 1.2 improves the results in [15]. These in turn have roots in [27], where G. Milton proposed that a suitable layered construction, using infinitely many scales, should yield extremal integrability properties. In [15] the second author interpreted Milton's idea from a different point of view, introduced the so-called staircase laminates and used Beltrami operators to complete the technical details left open by Milton. The method in [15] yields only a sequence of equations of the type (1.3), such that the corresponding solutions $\{u_j\}_{j=1}^{\infty}$ satisfy

$$\lim_{j\to\infty} \int_B |\nabla u_j|^{\frac{2K}{K-1}} = \infty.$$

Moreover, we do not know how to use that approach with Beltrami operators to show that the lower critical exponent $\frac{2K}{K+1}$ is sharp.

In this work we replace the use of Beltrami operators by *convex integration*. Generally, convex integration is a method for solving differential inclusions of the type

$$\nabla f(x) \in E, \quad \text{a.e. } x \tag{1.14}$$

where E is a given closed set of matrices. Roughly speaking the method consists of iteratively constructing layers within layers of oscillations, starting with an affine function whose gradient is in a suitably defined "hull" of E (see Section 2) and iteratively pushing the gradients towards E itself. The original method was introduced and developed by M. Gromov [18] as a very general and versatile method for solving partial differential inclusions related to underdetermined geometric problems. V. Šverák and S. Müller [30] adapted this method, combining with analysis of oscillations in the spirit of Tartar's compensated compactness [41], for constructing regularity counterexamples for elliptic systems (see also [40]). The survey [22] gives a very good overview of the techniques and results available in the Lipschitz setting.

The general existence theory for (1.14) has been developed also in [38, 39] as well as in [11, 12, 21, 43], where a different line of thought - following more closely the classical Baire category approach to solving ordinary differential inclusions - has been pursued. However, for us the techniques of Müller and Šverák are particularly useful since they not only give the existence of solutions to (1.14), but also provide us with solutions having "extremal" properties. Recently B. Kirchheim [21] has developed a powerful Baire category setting which combines both approaches in an elegant manner and also yields such extremal solutions.

Our plan of proof for Theorems 1.2, 1.3 and Theorems 1.6, 1.7 is as follows. We rewrite the equations as differential inclusions, in Lemmas 3.2 and 4.1, and then proceed with convex integration. The first step is to find a sequence of laminates (see Definition 2.2) with the required integrability properties. These will be called the staircase laminates, following [15, 16]. We remark that the construction of this type of laminates seems very flexible and adaptable to other situations. For example in [8] and [9] they have been used in connection with the problem of regularity of rank-one convex functions.

Once we find staircase laminates supported in the appropriate sets, we proceed in a different way for the lower and the upper critical exponents. For the upper critical exponents we are dealing with honest quasiregular mappings. This allows us to fit the set of solutions of (1.3) into a natural metric space setting. Then we can adapt the method of Kirchheim in [21] and use Baire category. This approach is based on the observation that points of continuity of the gradient are typically residual.

For the lower critical exponent we are not able to find a natural metric space setting. The reason is that the only norm which we are able to bound is the $W^{1,1}$ norm (see Remark 3.23). Therefore, we develop a version of the approach of Müller and Šverák [30], which works for unbounded sets and laminates. This approach is technically more involved than Baire category, but has the advantage that we get very precise information on the integrability of the gradient, namely that our solutions have gradient in the weak Lebesgue space $L^{\frac{2K}{K+1}}_{\text{weak}}$ (see Theorem 3.18). Finally, let us remark that the same proof works with minor modifications also for the upper critical exponents (see Remark 3.19), however we have preferred to use the Baire category method for its transparency and elegance.

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2. Preliminaries

We start by introducing the following notation. For matrices $A \in \mathbb{R}^{2\times 2}$ write $A = (a_+, a_-)$ where $a_+, a_- \in \mathbb{C}$ denote the conformal coordinates. That is, in the identification of vectors $v = (x, y) \in \mathbb{R}^2$ with the complex numbers $v = x + iy$, the coordinates are defined by the relation

$$Av = a_+ v + a_- \overline{v}$$

For future reference we record that multiplication of matrices in conformal coordinates reads as

$$AB = (a_+ b_+ + a_- \overline{b}_-,\ a_+ b_- + a_- \overline{b}_+), \tag{2.1}$$

and that $\mathrm{Tr}A = 2\Re a_+$, where $\Re z$ denotes the real part of the complex number z. Also

$$\begin{aligned} \det A &= |a_+|^2 - |a_-|^2, \\ |A|^2 &= 2|a_+|^2 + 2|a_-|^2, \\ \|A\| &= |a_+| + |a_-|, \end{aligned} \tag{2.2}$$

where $|A|$ and $\|A\|$ denote the Hilbert-Schmidt and the operator norm, respectively. In quasiconformal geometry it is important to measure how far a matrix is from being conformal. There are two classical notions quantifying this, the complex dilatation and the distortion $K(A)$. In our work it is most convenient to use the so called second complex dilatation, which is defined by

$$\mu_A = \frac{a_-}{\overline{a_+}} \tag{2.3}$$

Then distortion $K(A)$ of A is given by

$$K(A) = \frac{\|A\|^2}{|\det(A)|} = \left|\frac{1 + |\mu_A|}{1 - |\mu_A|}\right|. \tag{2.4}$$

These definitions extend naturally to mappings through the differential. If $f \in W^{1,1}(\Omega, \mathbb{R}^2)$, its second complex dilatation is

$$\mu_f(x) = \mu_{Df(x)} = \partial_{\overline{z}} f(x) \big/ \overline{\partial_z f(x)}$$

and its distortion function is $K_f(x) = K(Df(x))$, which are defined almost everywhere.

We use the notation $\mathcal{M}(\mathbb{R}^{m\times n})$ for the set of signed Radon measures on $\mathbb{R}^{m\times n}$ with finite mass. By the Riesz representation theorem, $\mathcal{M}(\mathbb{R}^{m\times n})$ can be identified with the dual of the space $C_0(\mathbb{R}^{m\times n})$ of continuous functions vanishing at infinity. Given $\nu \in M(\mathbb{R}^{m\times n})$ with finite first moment we use the notation

$$\overline{\nu} = \int_{\mathbb{R}^{m\times n}} A d\nu(A)$$

and call $\overline{\nu}$ the *barycenter* of ν.

Next, we turn to convex integration. The basic building block for us in solving partial differential inclusions is the following lemma. Here, and in the rest of the paper, we will say that a mapping $f : \Omega \to \mathbb{R}^2$, continuous up to the boundary, is *piecewise affine* if there exists a countable family of pairwise disjoint open subsets $\Omega_i \subset \Omega$ with $|\partial\Omega_i| = 0$ and

$$\Big|\Omega \setminus \bigcup_i \Omega_i\Big| = 0,$$

such that f is affine on each subset Ω_i. In a similar way we define *piecewise quadratic* functions $u : \Omega \to \mathbb{R}$, meaning that u is C^1 up to the boundary and coincides with a quadratic function on each Ω_i.

Lemma 2.1. *Let $\alpha \in (0, 1)$, $\varepsilon, \delta > 0$ and let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Let $A, B \in \mathbb{R}^{m\times n}$ with* $\operatorname{rank}(A - B) = 1$ *and suppose $C = \lambda A + (1 - \lambda)B$ for some* $\lambda \in (0, 1)$.

(i) *There exists a piecewise affine Lipschitz mapping $f : \Omega \to \mathbb{R}^m$ such that*

(a) $f(x) = Cx$ *if* $x \in \partial\Omega$,

(b) $[f - C]_{C^\alpha(\overline{\Omega})} < \varepsilon$,

(c) $|\{x \in \Omega : |\nabla f(x) - A| < \delta\}| = \lambda|\Omega|$,

(d) $|\{x \in \Omega : |\nabla f(x) - B| < \delta\}| = (1 - \lambda)|\Omega|$.

(ii) *If in addition $A, B \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$, then the map f in part* (i) *can be chosen so that $f = \nabla u$ for some piecewise quadratic $u \in W^{2,\infty}(\Omega)$.*

Proof. Part (i) of the lemma is standard in the literature (see [29, Lemma 3.1]), but usually with C^0 instead of C^α approximation in (b). For reader's convenience we recall the idea of this argument. We may assume $A - B = a \otimes e_n$ and define two auxiliary functions,

$$s(t) = \lambda(1 - \lambda) + t\big[(1 - \lambda)\chi_{(-\lambda,0)}(t) - \lambda\chi_{(0,1-\lambda)}(t)\big] \tag{2.5}$$

and, with $\delta' < \frac{1}{n|a|}\delta$,

$$w(x) = \delta'\left[s\left(\frac{x_n}{\delta'}\right) - \sum_{i=1}^{n-1}|x_i|\right].$$

Let now $f_0(x) = C(x) + w(x)a$. In the polytope $\Omega_0 = \{x \in \mathbb{R}^n : w(x) > 0\}$ the function f_0 satisfies (a) and (c),(d), while (b) is satisfied by choosing δ' sufficiently small.

For an arbitrary domain Ω, f is obtained by rescaling f_0. That is, fixing $r > 0$ we cover Ω by small copies of Ω_0 up to measure zero, so that

$$\Big|\Omega \setminus \bigcup_{i=1}^{\infty}(a_i + r_i\Omega_0)\Big| = 0 \tag{2.6}$$

with $r_i < r$, and then place in each copy the rescaled function

$$f_{r_i,a_i}(x) = r_i f_0(r_i^{-1}(x - a_i)) + C(a_i).$$

This implies (a), (c) and (d), but also (b) follows: As a Lipschitz function $f \in C^\alpha$ and its C^α norm decreases when scaling and letting $r \to 0$.

An alternative way to construct f would be by extending first s 1-periodically to $\mathbb{R}$ and then setting $f(x) = C(x) + \delta[s(x_n/\delta)]a$. Then $Df \in \{A, B\}$ a.e., but f does not have correct boundary values. These can be achieved by deforming f near $\partial\Omega$; see *e.g.* [21] which does this but obtains only $\operatorname{dist}(Df(x), [A, B]) < \delta$ in the exceptional set. However, arguing as in (ii) below one may further deform f to have $\operatorname{dist}(Df(x), \{A, B\}) < \delta$ almost everywhere.

Part (ii) is essentially done in [21], but we need to do some more work in order to obtain the *exact* volume fractions (c) and (d). Let $A, B, C \in \mathbb{R}^{2\times2}_{\text{sym}}$ be as in the lemma. Then, by [21, Proposition 3.4] for every $\epsilon, \delta > 0$ there exists a piecewise quadratic $u = u(\lambda, A, B) \in W^{2,\infty}(\Omega, \mathbb{R}^2)$ such that

(1.1) $u(x) = \frac{1}{2}\langle Cx, x\rangle$ if $x \in \partial\Omega$,

(1.2) $\left|\left\{x \in \Omega :\ D^2u(x) = A\right\}\right| > (1-\varepsilon)\lambda|\Omega|$,

(1.3) $\left|\left\{x \in \Omega :\ D^2u(x) = B\right\}\right| > (1-\varepsilon)(1-\lambda)|\Omega|$,

(1.4) $\operatorname{dist}\left(D^2u(x), [A, B]\right) < \delta$ a.e. in Ω.

However, we need a function u such that $D^2u(x)$ belongs to a δ neighborhood of $\{A, B\}$ instead of a neighborhood of the whole segment $[A, B]$. To achieve this, we iterate the construction of u to produce a sequence of piecewise quadratic functions $\{u_i\}_{i=1}^\infty$ with the following properties: writing

$$U_i = \left\{x \in \Omega :\ \operatorname{dist}\left(D^2u_i, \{A, B\}\right) < (1-2^{-i})\delta\right\}$$

we construct by induction the functions u_i so that

(2.1) $u_i(x) = \frac{1}{2}\langle Cx, x\rangle$ if $x \in \partial\Omega$,

(2.2) $\left|\left\{x \in \Omega : D^2u_i(x) = A\right\}\right| > (1-\varepsilon)\lambda|\Omega|$,

(2.3) $\left|\left\{x \in \Omega : D^2u_i(x) = B\right\}\right| > (1-\varepsilon)(1-\lambda)|\Omega|$,

(2.4) $u_j(x) = u_i(x)$ for $x \in U_i$ whenever $j \geq i$,

(2.5) $|\Omega \setminus U_{i+1}| \leq \frac{1}{4}|\Omega \setminus U_i|$ and $U_i \subset U_{i+1}$,

(2.6) $\operatorname{dist}\left(D^2u_i, [A, B]\right) < (1-2^{-i})\delta$ a.e. in Ω.

We can use (1.1)-(1.4) to define u_1, after replacing δ by $\delta/2$. Suppose then that u_i is given. Since u_i is piecewise quadratic, $\Omega \setminus U_i$ has a decomposition

$$\Omega \setminus U_i = \left(\bigcup_j \tilde{U}_j\right) \cup N$$

where $|N| = 0$ and $\tilde{U}_j$ are open sets on which $D^2 u_i = \tilde{C}$ for some constant matrix $\tilde{C} = \tilde{C}_j$ with $\operatorname{dist}(\tilde{C}, [A, B]) < (1 - 2^{-i})\delta$. Hence we can write

$$\tilde{C} = \tilde{\lambda} A + (1 - \tilde{\lambda}) B + \tilde{D},$$

where $0 < \tilde{\lambda} < 1$, and $|\tilde{D}| \le (1 - 2^{-i})\delta$. Therefore, if we set $\tilde{A} = A + \tilde{D}$ and $\tilde{B} = B + \tilde{D}$

$$\tilde{C} = \tilde{\lambda} \tilde{A} + (1 - \tilde{\lambda}) \tilde{B}, \tag{2.7}$$

with $|\tilde{A} - A|, |\tilde{B} - B| < (1 - 2^{-i})\delta$ and

$$\operatorname{rank}(\tilde{A} - \tilde{B}) = \operatorname{rank}(A - B) = 1.$$

Thus, we can use again [21, Proposition 3.4] in each $\tilde{U}_j$ to produce $\tilde{u_j}$ such that

(3.1) $\tilde{u_j}(x) = \frac{1}{2}\langle \tilde{C} x, x \rangle$ (modulo an affine function) if $x \in \partial \tilde{U}_j$

(3.2) $\left| \left\{ x \in \tilde{U}_j : D^2 \tilde{u_j} \in \left\{ \tilde{A}, \tilde{B} \right\} \right\} \right| > \frac{3}{4} |\tilde{U}_j|$.

(3.3) $\operatorname{dist}\left(D^2 u_i, [\tilde{A}, \tilde{B}] \right) < 2^{-i-1}\delta$ a.e. in $\tilde{U}_j$

Then the function u_{i+1} obtained from u_i by replacing u_i by $\tilde{u_j}$ on $\tilde{U}_j$ (modulo an affine function) fulfills the properties (1)-(6).

It follows from (2.1) and (2.6) that the sequence $\{u_i\}$ converges strongly in $W^{2,\infty}$ to a piecewise quadratic function u_λ with the following properties:

(4.1) $u_\lambda = \frac{1}{2}\langle C x, x \rangle$ on $\partial\Omega$

(4.2) $\left| \left\{ x \in \Omega : D^2 u_i(x) = A \right\} \right| > (1 - \varepsilon)\lambda |\Omega|$,

(4.3) $\left| \left\{ x \in \Omega : D^2 u_\lambda(x) = B \right\} \right| > (1 - \varepsilon)(1 - \lambda)|\Omega|$,

(4.4) $\operatorname{dist}\left(D^2 u_\lambda(x), \{A, B\} \right) < \delta$ a.e. in Ω.

The mapping u_λ fulfills all the desired properties, except possibly we do not obtain the exact volume proportions required in (c) and (d). If this is the case, we may assume without loss of generality that

$$\mu_{u_\lambda} = \frac{|\left\{ x \in \Omega : |D^2 u(x) - A| < \delta \right\}|}{|\Omega|}$$

satisfies $\lambda(1 - \epsilon) < \mu_{u_\lambda} < \lambda$. Fix $\epsilon = \frac{1}{2}$ so that

$$\frac{\lambda}{2} < \mu_{u_\lambda} < \lambda, \tag{2.8}$$

and choose $\hat{A} \in [A, B] \cap B(A, \delta)$ so that

$$C = \hat{\lambda} \hat{A} + (1 - \hat{\lambda}) B,$$

where $\hat{\lambda} = \lambda + \epsilon_1$ for some small $0 < \epsilon_1 < \delta/2$. For arbitrary $\epsilon_2 > 0$ we can repeat the above construction with $\hat{\lambda}$ in place of λ to obtain a function $u_{\hat{\lambda}}$ equal to $1/2\langle Cx, x\rangle$ on the boundary and such that $\operatorname{dist}(D^2 u_{\hat{\lambda}}(x), \{A, B\}) < \delta$ and

$$\mu_{u_{\hat{\lambda}}} > \hat{\lambda}(1 - \epsilon_2).$$

Choose $\epsilon_2 > 0$ so that

$$\mu_{u_{\hat{\lambda}}} > \lambda. \tag{2.9}$$

From (2.8) and (2.9) we deduce that

$$t \stackrel{\text{def}}{=} \frac{\mu_{u_{\hat{\lambda}}} - \lambda}{\mu_{u_{\hat{\lambda}}} - \mu_{u_\lambda}}$$

satisfies $t \in (0, 1)$, and $\lambda = t\mu_{u_{\hat{\lambda}}} + (1 - t)\mu_{u_\lambda}$. Finally we divide Ω in two regions $\Omega_\lambda, \Omega_{\hat{\lambda}}$ with $|\Omega_\lambda| = t|\Omega|$ and $|\Omega_{\hat{\lambda}}| = (1 - t)|\Omega|$. We put $\Omega_\lambda = \cup_i \Omega_{i,\lambda} \cup N$ and $\Omega_{\hat{\lambda}} = \cup_i \Omega_{i,\hat{\lambda}} \cup N$ where $\Omega_{i,\lambda}, \Omega_{i,\hat{\lambda}}$ are rescaled copies of Ω as in (2.6). In each $\Omega_{i,\lambda}$ we place a rescaled copies of u_λ and in each $\Omega_{i,\hat{\lambda}}$ a rescaled copy of $u_{\hat{\lambda}}$. This defines our final mapping u. It is easy to see that the volume fraction $\mu_u = t\mu_{\hat{\lambda}} + (1 - t)\mu_\lambda = \lambda$, in other words the function u satisfies the volume requirements exactly, and hence satisfies (a), (c) and (d). To obtain (b) we perform the same rescaling and covering as in part (i). □

The matrices A and B in Lemma 2.1 are said to be *rank-one connected* and the corresponding measure $\lambda\delta_A + (1 - \lambda)\delta_B \in \mathcal{M}(\mathbb{R}^{m\times n})$ is called a *laminate of first order*. Because the lemma provides piecewise affine maps, the construction can be iterated by modifying the map f in the regions where it is affine. For example, suppose we have two further rank-one connected matrices C_1, C_2 such that

$$B = \lambda' C_1 + (1 - \lambda')C_2. \tag{2.10}$$

Then in the open set $\{|\nabla f(x) - B| < \delta\}$, f can be replaced by a map (again given by the lemma) whose gradient oscillates on a much smaller scale between neighborhoods of C_1 and C_2.

Notice that, as in (2.7), (2.10) implies that if $|\nabla f(x) - B| < \delta$, then $\nabla f(x) = \lambda'\tilde{C_1} + (1-\lambda)\tilde{C_2}$ where $\tilde{C_1}, \tilde{C_2}$ are rank-one connected and they lie in corresponding neighborhoods of C_1 and C_2. Thus, on each region where f is affine, we can apply Lemma 2.1 to obtain the new mapping. On the level of the gradient distribution this amounts to replacing δ_B by $\lambda'\delta_{C_1} + (1-\lambda')\delta_{C_2}$. This type of iteration motivates the following definition ([10, 30, 32]).

Definition 2.2. The family of *laminates of finite order* $\mathcal{L}(\mathbb{R}^{m\times n})$ is the smallest family of probability measures in $\mathcal{M}(\mathbb{R}^{m\times n})$ with the properties

(i) $\mathcal{L}(\mathbb{R}^{m\times n})$ contains all Dirac masses.

(ii) Suppose $\sum_{i=1}^N \lambda_i\delta_{A_i} \in \mathcal{L}(\mathbb{R}^{m\times n})$ and $A_1 = \lambda B + (1-\lambda)C$ where $\lambda \in [0,1]$ and $\operatorname{rank}(B-C) = 1$. Then the probability measure

$$\sum_{i=2}^N \lambda_i\delta_{A_i} + \lambda_1(\lambda\delta_B + (1-\lambda)\delta_C)$$

is also contained in $\mathcal{L}(\mathbb{R}^{m\times n})$.

The process of obtaining new measures via (ii) is called *splitting*.

Proposition 2.3. *Let $\nu = \sum_{i=1}^N \alpha_i\delta_{A_i} \in \mathcal{L}(\mathbb{R}^{m\times n})$ be a laminate of finite order with barycenter $\overline{\nu} = A$. Then, for every $\alpha \in (0,1)$, $0 < \delta < \min|A_i - A_j|/2$ and every bounded open set $\Omega \subset \mathbb{R}^n$, there exists a piecewise affine Lipschitz mapping $f : \Omega \to \mathbb{R}^m$ such that*

(i) $f(x) = Ax$ *on* $\partial\Omega$,
(ii) $[f - A]_{C^\alpha(\overline{\Omega})} < \delta$,
(iii) $|\{x \in \Omega : |\nabla f(x) - A_i| < \delta\}| = \alpha_i|\Omega|$ *for each i, thus*
(iv) $\operatorname{dist}(\nabla f(x), \operatorname{spt}\nu) < \delta$ *a.e. in* Ω.

Moreover, if $A_i \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$, then the map f can be chosen so that $f = \nabla u$ for some $u \in W^{2,\infty}(\Omega)$.

Proof. The proof is by induction using Lemma 2.1, which proves the result for first order laminates. For higher order laminates the precise argument is given in [30, Lemma 3.2] with the C^0 norm instead of the C^α and matrices in $\mathbb{R}^{n\times n}$. The case of symmetric matrices is handled using part ii) of Lemma 2.1 instead of part i). ☐

Finally, we recall the definition of certain semiconvex envelopes of sets of $m \times n$ matrices (for more information see [22, 28, 32]).

Definition 2.4. Let $E \subset \mathbb{R}^{m\times n}$ be a closed set. The *polyconvex hull* of E is given by

$$E^{pc} = \left\{\overline{\nu} :\ \nu \in \mathcal{M}(\mathbb{R}^{m\times n}),\ \operatorname{spt}\nu \subset E \text{ and } \det(\overline{\nu}) = \int_{\mathbb{R}^{m\times n}} \det(A)\,d\nu(A)\right\}.$$

Similarly, the *lamination hull* of E is given by

$$E^{lc} = \{\overline{\nu} :\ \nu \text{ is a laminate of finite order with } \operatorname{spt}\nu \subset E\},$$

and the *first lamination hull* is

$$E^{lc,1} = \{\overline{\nu} :\ \nu \text{ is a laminate of first order with } \operatorname{spt}\nu \subset E\}.$$

In particular, since the determinant is an affine function on the rank-one lines of $\mathcal{M}(\mathbb{R}^{m\times n})$, we have $E^{lc,1} \subset E^{lc} \subset E^{pc}$.

3. Isotropic equations

For proving Theorems 1.2 and 1.3 we start, as in [30], by transforming the set of solutions to the isotropic PDE into solutions to a suitable differential inclusion.

Definition 3.1. For a set $\Delta \subset \mathbb{C} \cup \{\infty\}$, let

$$E_\Delta = \left\{ A \in \mathbb{R}^{2\times 2} : A = (a_+, a_-) \text{ with } a_- = \mu \overline{a}_+ \text{ for some } \mu \in \Delta \right\}, \tag{3.1}$$

i.e. E_Δ is the set of matrices with the second complex dilatation belonging to Δ. If $\mu = \infty$ the condition $a_- = \mu \overline{a_+}$ means that $a_+ = 0$. In particular, for $\Delta = \{0\}$, let E_0 denote the set of conformal matrices. Similarly, E_∞ is the set of anti-conformal matrices.

From (2.1) we see that E_Δ is invariant under precomposition by conformal mappings, *i.e.* that

$$E_\Delta = E_\Delta A \quad \text{for all } A \in E_0 \setminus \{0\}. \tag{3.2}$$

Lemma 3.2. *Let $K \geq 1$ with $k = \frac{K-1}{K+1}$ and let $1 \leq p \leq \infty$. Suppose $\Omega \subset \mathbb{R}^2$ is a bounded domain. Then $u \in W^{1,p}(\Omega, \mathbb{R})$ is a weak solution to*

$$\operatorname{div}(\rho(x)\nabla u(x)) = 0$$

for some coefficient function $\rho \in L^\infty(\Omega, \left\{K, \frac{1}{K}\right\})$ if and only if $u = f_1$ where $f = (f_1, f_2) : \Omega \to \mathbb{R}^2$ satisfies

$$\nabla f \in E_{\{k,-k\}}. \tag{3.3}$$

Proof. It is convenient to identify $\mathbb{R}^2$ with $\mathbb{C}$, so that $f_1 = \Re(f)$. Accordingly, let us write $f = u + iv$. Then

$$\begin{aligned} 2\overline{\partial_z f} &= \partial_x u + \partial_y v + i(\partial_y u - \partial_x v), \\ 2\partial_{\overline{z}} f &= \partial_x u - \partial_y v + i(\partial_y u + \partial_x v). \end{aligned}$$

Hence the condition $\nabla f \in E_{\{k,-k\}}$, or $\partial_{\overline{z}} f = \mu \overline{\partial_z f}$ with $\mu \in \{-k, k\}$, is equivalent to the system

$$\begin{aligned} (1-\mu)\partial_x u &= (1+\mu)\partial_y v, \\ (1-\mu)\partial_y u &= -(1+\mu)\partial_x v. \end{aligned}$$

In other words

$$\frac{1-\mu}{1+\mu}\nabla u = J\nabla v, \tag{3.4}$$

where $J = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. But in a simply connected 2-dimensional domain Ω an L^p vector-field is divergence-free if and only if it has the form $J\nabla v$ for some Sobolev function v. Hence (3.4) is equivalent to

$$\operatorname{div}\left(\frac{1-\mu}{1+\mu}\nabla u\right) = 0. \qquad \square$$

3.1. Upper critical exponent

Our proof of Theorem 1.2 makes strong use of the ideas developed by Kirchheim in [21, Chapter 3.3]. First, we define an appropriate closed and bounded subset $X \subset W^{1,2}$ containing certain weak solutions to (3.3). Since bounded subsets of $W^{1,2}$ are metrizable in the weak topology w, we deduce that (X, w) is a metric space. In particular, we are then in a setting where one can apply Baire category methods. Indeed we shall show, similarly as in [21], that functions in X are points of continuity of the map $\nabla : (X, w) \to L^2$ only if they satisfy the inclusion (3.3). From this we deduce, using the fact that ∇ is a Baire-1 mapping, that the solutions to (3.3) form in X a *residual* set, *i.e.* its complement is of first Baire category. Finally we show that the set of functions in X for which (1.4) holds in any fixed ball $B(x, r)$ is also residual, and for this we use the staircase laminate construction introduced in [15]. Combining these facts gives Theorem 1.2.

We will define the set X starting from the differential inclusion (3.3). We then need X to be weakly closed, and if we would analyze this property more systematically we would arrive at the concept of G-closure due to Spagnolo. Indeed it is not difficult to see that the set Δ_k introduced below is nothing but the interior of the G-closure of $\{-k, k\}$, with respect to equations of the type (1.6); in fact similar computations as those in Lemma 3.5 can be found in [14] and [33].

Definition 3.3. For each $0 < k < 1$ let

$$\Delta_k = \left\{ re^{i\phi} \in \mathbb{C} : r < k,\ r^2 \cos^2 \phi > \frac{(1-r^2)(r^2-k^4)}{(1-k^2)^2} \right\},$$

cf. Figure 3.1. Also, using the notation (3.1), define $\mathcal{U} = E_{\Delta_k} \subset \mathbb{R}^{2\times 2}$.

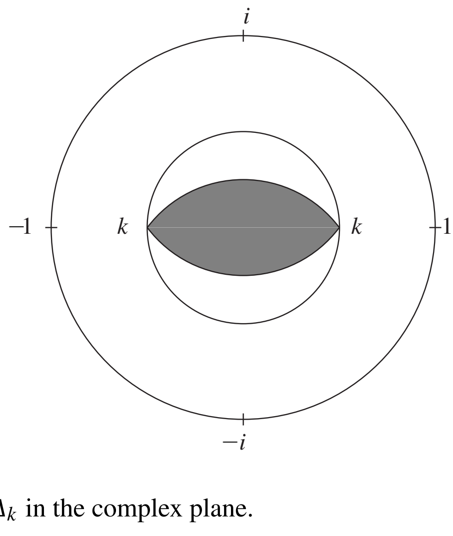

Figure 3.1. The set Δ_k in the complex plane.

Note that Δ_k, and hence $\mathcal{U}$, is open and that $E_{\{k,-k\}} \subset \overline{\mathcal{U}}$.

Definition 3.4. Let X be the closure in the weak topology of $W^{1,2}$ of the set

$$X_0 = \left\{ f \in W^{1,\infty}(\overline{\Omega}, \mathbb{R}^2) : \begin{array}{l} \bullet\ f \text{ piecewise affine} \\ \bullet\ \nabla f(x) \in \mathcal{U} \text{ a.e.} \\ \bullet\ f(x) = x \text{ on } \partial\Omega \end{array} \right\}. \tag{3.5}$$

Certainly, $f = Id \in X_0$ so the set X is not empty. *In Lemma* 3.7 *we show that X is a bounded subset of $W^{1,2}$ and hence it is metrizable in the weak topology.*

Lemma 3.5. *Let $E_{\{k,-k\}}$ be as in Lemma* 3.2 *and write $E^{lc,1}_{\{k,-k\}} = (E_{\{k,-k\}})^{lc,1}$ for its first lamination hull. Then*

$$E^{lc,1}_{\{k,-k\}} = E^{pc}_{\{k,-k\}} = E_{\overline{\Delta_k}},$$

with Δ_k given in Definition 3.3.

Proof. Since automatically $E^{lc,1}_{\{k,-k\}} \subset E^{pc}_{\{k,-k\}}$ we only need to prove the reverse inclusion. Hence let $W \in E^{pc}_{\{k,-k\}}$. Then by definition $W = \overline{\nu}$ for some probability measure ν supported on $E_{\{k,-k\}}$ with ν satisfying

$$\det(\overline{\nu}) = \int \det(A)\, \mathrm{d}\nu(A). \tag{3.6}$$

The crucial information to use here is that $\det(A) = (1-k^2)|a_+|^2$ for every $A \in E_{\{k,-k\}}$. Thus $F \mapsto \det F$ is a convex function when restricted to $E_{\{k,-k\}}$.

Let us now write

$$\nu = \lambda \nu_k + (1-\lambda)\nu_{-k}$$

where $\nu_{\pm k}$ are probability measures with $\operatorname{spt} \nu_k \subset E_k$ and $\operatorname{spt} \nu_{-k} \subset E_{-k}$ and barycenters

$$Y = \int A\, \mathrm{d}\nu_k(A) \in E_k \ \text{ and } \ Z = \int A\, \mathrm{d}\nu_{-k}(A) \in E_{-k},$$

respectively. Then (3.6) reads as

$$\det(W) = \lambda \int_{E_k} \det(A) d\nu_k(A) + (1-\lambda) \int_{E_{-k}} \det(A) d\nu_{-k}(A).$$

By Jensen's inequality and the convexity of $\det_{|E_{\pm k}}$ it follows that

$$\det W \geq \lambda \det Y + (1-\lambda) \det Z. \tag{3.7}$$

On the other hand in two dimensions the determinant is a quadratic form, and by direct calculation we obtain

$$\det(\lambda Y + (1-\lambda)Z) = \lambda \det(Y) + (1-\lambda)\det(Z) - \lambda(1-\lambda)\det(Z - Y).$$

Recalling that

$$W = \bar{\nu} = \lambda\bar{\nu}_k + (1-\lambda)\bar{\nu}_{-k} = \lambda Y + (1-\lambda)Z,$$

we deduce $\det(Y - Z) \leq 0$.

Now consider pairs tY and sZ with $t, s > 0$ such that for some $\lambda' \in (0, 1)$ we have $\lambda'(tY) + (1-\lambda')(sZ) = W$. Linear independence gives

$$\lambda' = \frac{\lambda}{t} \text{ and } s = \frac{1-\lambda}{1-\lambda'},$$

so that such pairs are parametrized by $t \in (\lambda, \infty)$ with $s = s(t) \to \infty$ as $t \to \lambda$. Let $d(t) = \det(tY - s(t)Z)$. By the calculation above $d(1) \leq 0$ and since $\det Y, \det Z > 0$ we have $d(t) \to +\infty$ as $t \to \lambda$. Thus there exists $t_0 \in (\lambda, 1]$ such that $d(t_0) = 0$. But then $t_0 Y \in E_k$ and $s(t_0)Z \in E_{-k}$ are rank-one connected, and so $W \in E^{lc,1}_{\{k,-k\}}$.

Next we obtain an explicit description of the lamination hull. A matrix $W = (w_+, w_-)$ lies in the first lamination hull of $E_{\{k,-k\}}$ if and only if there exists $\lambda \in [0, 1]$, and matrices $Y = (y_+, y_-) \in E_k$ and $Z = (z_+, z_-) \in E_{-k}$ such that

$$W = \lambda Y + (1-\lambda)Z \text{ and } |y_+ - z_+| = |y_- - z_-|$$

Substituting $y_- = k\overline{y}_+$, $z_- = -k\overline{z}_+$, $w_- = \mu\overline{w}_+$ and writing $t = (1-2\lambda)$ a calculation gives that $W \in E^{lc,1}_{\{k,-k\}}$ if and only if

$$|\mu + kt| = k|k + t\mu|$$

for some $t \in [-1, 1]$.

Let $p(t) = |\mu + kt|^2 - k^2|k + t\mu|^2$. Then $p(t)$ is a quadratic polynomial in t, with leading term $k^2(1 - |\mu|^2)t^2$, and moreover $X \in E^{lc,1}$ if and only if $p(t)$ has a root in the interval $[-1, 1]$. Notice that

$$p(1) = (1-k^2)|\mu + k|^2 \geq 0, \quad p(-1) = (1-k^2)|\mu - k|^2 \geq 0.$$

Therefore if p is concave it has no roots in $[-1, 1]$. So we may assume $|\mu| < 1$ and then if p has a root in $[-1, 1]$, the minimum of p also lies in $[-1, 1]$. The discriminant of p is:

$$D = 4k^2 \left\{ (1-k^2)^2(\Re\mu)^2 - (1-|\mu|^2)(|\mu|^2 - k^4) \right\}$$

and the minimum is at

$$t_0 = -\frac{(1-k^2)\Re\mu}{k(1-|\mu|^2)}.$$

Suppose $D \geq 0$. Then if $|\mu| > k$,

$$t_0^2 = \frac{(1-k^2)^2(\Re\mu)^2}{k^2(1-|\mu|^2)^2} \geq \frac{|\mu|^2 - k^4}{k^2(1-|\mu|^2)} > 1$$

whereas if $D \geq 0$ and $|\mu| \leq k$, then

$$|t_0| = \frac{(1-k^2)|\Re\mu|}{k(1-|\mu|^2)} \leq \frac{(1-k^2)}{(1-|\mu|^2)} \leq 1.$$

We have shown that p has a root in $[-1, 1]$ if and only if $D \geq 0$ and $|\mu| \leq k$. In turn, these inequalities are a precise description of $\overline{\Delta_k}$. $\square$

Remark 3.6. Lemma 3.5 implies that for each $A \in \overline{\mathcal{U}}$ there are rank-one connected $B, C \in E_{\{k,-k\}}$ such that $A \in [B, C] \subset \overline{\mathcal{U}}$. Later we will need the further information that in case A lies in the interior, *i.e.* $A \in \mathcal{U}$, then (except for the endpoints) the whole interval $(B, C) \subset \mathcal{U}$.

Namely, consider the half-interval $(B, A]$, and for $t \in (0, 1)$ let

$$W = tA + (1-t)B.$$

We claim that $W \in \mathcal{U}$. A simple continuity argument shows that for any $\epsilon > 0$ there exists $\delta > 0$ so that for any $\tilde{W}$ with $|W - \tilde{W}| < \delta$ there exists $\tilde{B} \in E_k$ with $|B - \tilde{B}| < \epsilon$ and such that $\det(\tilde{B} - \tilde{W}) = 0$. Let

$$\tilde{A} = \frac{1}{t}(\tilde{W} - (1-t)\tilde{B}),$$

so that $\tilde{W} = t\tilde{A} + (1-t)\tilde{B}$ and

$$|A - \tilde{A}| \leq \frac{1}{t}(|W - \tilde{W}| + \epsilon).$$

This last inequality implies that for sufficiently small $\epsilon > 0$ and for $\tilde{W}$ sufficiently close to W we have that $\tilde{A} \in \mathcal{U}$. But as $\det(\tilde{A} - \tilde{B}) = 0$ and $\overline{\mathcal{U}}$ is lamination convex, we deduce that $\tilde{W} \in \overline{\mathcal{U}}$ for all $\tilde{W}$ sufficiently close to W. This implies that $W \in \mathcal{U}$.

3.1.1. *Points of continuity of* ∇

Next we collect the functional analytic facts needed in the sequel.

Lemma 3.7. *The space (X, w) is compact and metrizable, and for any $f \in X$ we have $\nabla f(x) \in \overline{\mathcal{U}}$ a.e. in Ω. In particular f is K-quasiconformal with $f = Id$ on $\partial\Omega$. The metric d on X is equivalent to the metrics induced by the L^∞ and L^2 norms on X. Furthermore, the set of continuity points of the map $\nabla : (X, w) \to L^2(\Omega, \mathbb{R}^{2\times 2})$ is residual in (X, w).*

Proof. The key fact behind our argument is that we are dealing with elliptic equations, in particular with quasiregular mappings. Let X_0 be as in Definition 3.4. To prove that its weak closure (X, w) is metrizable we need to show that X_0 is bounded in $W^{1,2}$.

There is no loss of generality in assuming that $\Omega \subset B(0, \frac{1}{2})$. Therefore for $f \in X_0$ the Lipschitz mapping $\tilde{f} = f\chi_\Omega + x\chi_{B(0,1)\setminus\Omega}$ is a well defined K-quasiregular mapping (actually K-quasiconformal) by the definition of $\mathcal{U}$. Thus, $\Re\tilde{f} = u$ is a weak solution to

$$\operatorname{div}(\sigma(x)\nabla u(x)) = 0 \qquad \text{in } B(0, 1),$$

where σ is a measurable matrix function satisfying (1.1). This is seen similarly as in the proof of Lemma 3.2; for details see *e.g.* [2]. Testing the equation with $v(x) = u(x) - x_1$ yields the estimate

$$\int_{B(0,1)} |\nabla u|^2 \leq C(K),$$

and using the same argument for $\Im(\tilde{f})$ we obtain that

$$\int_{B(0,1)} |\nabla \tilde{f}|^2 \leq C(K). \tag{3.8}$$

Finally, the Sobolev embedding theorem (or alternatively, the maximum principle) yields the required bound,

$$\|f\|_{W^{1,2}(\Omega)} \leq C(K) \tag{3.9}$$

for f in X_0. By the weak lower semicontinuity of the $W^{1,2}$ norm we have the same bound for f in X.

The estimates show that in the weak topology induced by $W^{1,2}$, the space (X, w) is metrizable with metric $d = d_X$. By the compactness of the Sobolev embedding, (X, w) embeds continuously in $L^2(\Omega)$. Since (X, w) itself is compact, the embedding is an homeomorphism between $(X, \|\,\|_{L^2})$ and (X, w).

Next, X_0 is a bounded family of K-quasiconformal mappings, and as such sets are normal, also all $f \in X$ are K-quasiconformal. We may hence use properties of these maps. For instance, the weak convergence in $W^{1,2}$ and uniform convergence are equivalent notions for bounded families of K-quasiconformal maps, hence (X, w) is homeomorphic to $(X, \|\,\|_{L^\infty})$. As a second example, by the higher integrability of quasiregular mappings [1] we have that for $f \in X$

$$\int |\nabla f|^p \leq C(K, p) \tag{3.10}$$

for every $2 < p < \frac{2K}{K-1}$.

To prove that $\nabla f(x) \in \overline{\mathcal{U}}$ a.e. for any $f \in X$ we will use the weak continuity of $F \mapsto \det F$ in $W^{1,p}$ for $p > 2$ [10].

Given $f \in X$ we have a sequence $\{f_n\} \subset X_0$ with $f_n \rightharpoonup f$ in $W^{1,2}(\Omega)$. Then also $f_n \rightharpoonup f$ in $W^{1,p}(\Omega)$ for some $p > 2$ by (3.10). Thus, $\det(\nabla f_n) \rightharpoonup \det(\nabla f)$ in $L^{\frac{p}{2}}$. Using Mazur's Lemma there exists $0 \leq \lambda_n^j \leq 1$ with $\sum_j \lambda_n^j = 1$ such that as $n \to \infty$,

$$\sum_j \lambda_n^j (\nabla f_j(x), \det(\nabla f_j(x))) \to (\nabla f(x), \det(\nabla f(x)) \text{ a.e.}$$

For almost every x the convex combinations define a sequence of probability measures

$$\nu_x^n = \sum_j \lambda_n^j \delta_{\nabla f_j(x)} \in \mathcal{M}(\mathbb{R}^{2\times 2})$$

converging to probability measures ν_x supported in $E_{\{k,-k\}}$ such that

$$\nabla f(x) = \int_{E_{\{k,-k\}}} A d_{\nu_x}(A), \text{ and } \det(\nabla f(x)) = \int_{E_{\{k,-k\}}} \det(A) d_{\nu_x}(A).$$

According to the definition in Section 2 this means that for almost every x, $\nabla f(x) \in \mathcal{U}^{pc}$ for any $f \in X$, and $\mathcal{U}^{pc} = \overline{\mathcal{U}}$ by Lemma 3.5. The measure ν_x is the *gradient Young measure* generated by the sequence $\{f_n\}$ and in fact the above reasoning is a typical application of Young measure theory, see for example [28].

We now turn to the category concepts. For a reference see [21] or [31]. We define $(\Delta_h)_{ij}(x) = \frac{f^i(x+he_j)-f^i(x)}{h}$ to be matrix of differential quotients of f (recall that we have extended functions in X by the identity outside Ω so that $\Delta_h f(x)$ is defined a.e. in Ω). For all Sobolev functions $f \in W^{1,2}$ we have that

$$\lim_{h\to\infty} \|\nabla f - \Delta_h(f)\|_{L^2(\Omega,\mathbb{R}^{2\times 2})} = 0.$$

On the other hand, each Δ_h is a continuous operator from $L^2(\Omega,\mathbb{R}^2) \to L^2(\Omega,\mathbb{R}^{2\times 2})$. Since (X, w) is homeomorphic to $(X, \|\|_{L^2})$ it follows that Δ_h is continuous as a map from $(X, w) \to L^2(\Omega, \mathbb{R}^{2\times 2})$. Therefore $\nabla : (X, w) \to L^2(\Omega, \mathbb{R}^{2\times 2})$ is a pointwise limit of continuous mappings. This is the definition of a Baire-1 mapping, also called a *function of first Baire class*. It is part of Baire's theorem that the points of continuity of such functions form a residual set in (X, w) (See [21, page 53], [31, Theorem 7.3].) $\square$

Lemma 3.8. *The set of points of continuity in (X, d) of ∇ satisfy $\nabla f(x) \in E_{\{k,-k\}}$ almost everywhere.*

The proof is exactly as in [21, Proposition 3.17]. Here we reproduce the argument for the reader's convenience. The main point is that if f is piecewise affine with an affine piece $A \in \overline{\mathcal{U}} \setminus E_{\{k,-k\}}$, then (since $\overline{\mathcal{U}} = E^{lc,1}_{\{k,-k\}}$) there exists a rank-one segment through A in $\overline{\mathcal{U}}$ of length proportional to the dist$(A, E_{\{k,-k\}})$. This permits us, with the help of Lemma 2.1, to produce a perturbation of f showing that it cannot be a point of continuity.

Proof. Suppose for a contradiction that the set $\left\{x \in \Omega : \nabla f(x) \in \overline{\mathcal{U}} \setminus E_{\{k,-k\}}\right\}$ has positive measure, where $f \in X$ is a point of continuity of ∇. Then by Lusin's theorem there exists a compact $\Omega_0 \subset \Omega$ (with $|\Omega_0| = m > 0$) such that ∇f is continuous on Ω_0 and $\nabla f(\Omega_0) \cap E_{\{k,-k\}} = \emptyset$. Since $\nabla f(\Omega_0)$ is compact,

$$\varepsilon = \frac{1}{2}\text{dist}\,(\nabla f(\Omega_0), E_{\{k,-k\}}) > 0. \tag{3.11}$$

Let

$$\mathcal{V} = N_\varepsilon\,(\nabla f(\Omega_0)) \cap \overline{\mathcal{U}}$$

be the $\varepsilon-$neighborhood of $\nabla f(\Omega_0)$ in $\overline{\mathcal{U}}$. Since $\overline{\mathcal{U}} = E^{lc,1}_{\{k,-k\}}$, to any $A \in \mathcal{V}$ there exists a rank-one segment connecting E_k to E_{-k} and containing A. In fact, as dist$\,(A, E_{\{k,-k\}}) \geq \varepsilon$ by (3.11), there exists a rank-one matrix $C_A \in \mathbb{R}^{2\times 2}$ with $|C_A| \geq \varepsilon$ such that

$$[A - C_A, A + C_A] \subset \overline{\mathcal{U}}. \tag{3.12}$$

Moreover, according to Remark 3.6 we have

$$A \in \mathcal{U} \Rightarrow [A - C_A, A + C_A] \subset \mathcal{U}. \tag{3.13}$$

Lastly, since f is a point of continuity of $\nabla : (X, w) \to L^2(\Omega, \mathbb{R}^{2\times 2})$, there exists $\delta > 0$ such that

$$\|\nabla g - \nabla f\|_{L^2} < \frac{1}{8}\varepsilon\sqrt{m} \text{ whenever } d_X(g, f) < \delta \text{ and } g \in X. \tag{3.14}$$

We now show that (3.12) - (3.14) give the desired contradiction. For this, take a sequence of piecewise affine functions $X_0 \ni f_n \to f$ in (X, d). Since f is a point of continuity of ∇, we have $\nabla f_n \to \nabla f$ in measure, *i.e.* $|\{x : |\nabla f_n(x) - \nabla f(x)| > \varepsilon\}| \leq \varepsilon^{-2}\|\nabla f_n - \nabla f\|^2_{L^2} \to 0$ as $d(f_n, f) \to 0$. Therefore there exists n and $\Omega_1 \subset \Omega_0$ with $|\Omega_1| > \frac{m}{2}$ so that

$$d(f_n, f) < \frac{\delta}{2},\ \|\nabla f_n - \nabla f\|_{L^2} < \frac{1}{8}\varepsilon\sqrt{m} \text{ and } \nabla f_n(\Omega_1) \subset \mathcal{V} \cap \mathcal{U}.$$

Furthermore, as Ω is covered up to measure zero by open sets on which f_n is affine, there exists finite number of disjoint open sets $G_i \subset \Omega$ such that $f_n(x) = A_i x + a_i$ for $x \in G_i$,

$$[A_i - C_i, A_i + C_i] \subset \mathcal{U}$$

for some rank-one matrix C_i with $|C_i| > \varepsilon$ and $|\bigcup G_i| > \frac{m}{2}$.

Next note that by Lemma 3.7 there is no loss of generality in replacing d by the sup-norm $\|\,\|_{L^\infty}$ on X. For each i, Lemma 2.1 (with $A = C_i$, $B = -C_i$) supplies a piecewise affine function $\phi_i \in W^{1,\infty}_0(G_i, \mathbb{R}^m)$ such that $\|\phi_i\|_\infty < \frac{\delta}{2}$ and

$$\text{dist }(\nabla\phi_i(x), \{\pm C_i\}) < \min\left\{\frac{\varepsilon}{2}, \text{dist }(\{A_i \pm C_i\}, \partial\mathcal{U})\right\} \tag{3.15}$$

Let $g = f_n + \sum \chi_{G_i}\phi_i$. Then by (3.15), $\nabla g(x) \in \mathcal{U}$ *i.e.* $g \in X_0$, and

$$d(g, f) \leq d(g, f_n) + d(f_n, f) < \frac{\delta}{2} + \frac{\delta}{2} = \delta.$$

On the other hand, using Cauchy-Schwarz and (3.15)

$$\int_{\bigcup G_i} |\nabla g - \nabla f_n|^2 \geq \frac{1}{|\bigcup G_i|}\Big(\sum_i \int_{G_i} |\nabla\phi_i|\Big)^2 > \frac{\varepsilon^2}{4}\Big|\bigcup G_i\Big| > \frac{\varepsilon^2 m}{8}.$$

Hence

$$\|\nabla g - \nabla f\|_{L^2} \geq \|\nabla g - \nabla f_n\|_{L^2} - \|\nabla f_n - \nabla f\|_{L^2} > \frac{1}{8}\varepsilon\sqrt{m},$$

which is in contradiction with (3.14). □

Corollary 3.9. *The set of mappings f in X such that $\nabla f(x) \in E_{\{k,-k\}}$ is residual.*

3.1.2. *Staircase laminates and integrability*

We will complete the argument for Theorem 1.2 by producing a sequence of laminates supported in $\mathcal{U}$ such that their $\frac{2K}{K-1}$-moment diverges. We use the staircase construction introduced in [15].

Proposition 3.10. *Every $A \in \mathcal{U}$ is the center of mass of a sequence of laminates of finite order $\nu_n \in \mathcal{L}$ supported in $\mathcal{U}$ such that*

$$\lim_{n\to\infty} \int_{\mathbb{R}^{2\times 2}} |\lambda|^{\frac{2K}{K-1}} d\nu_n = \infty \tag{3.16}$$

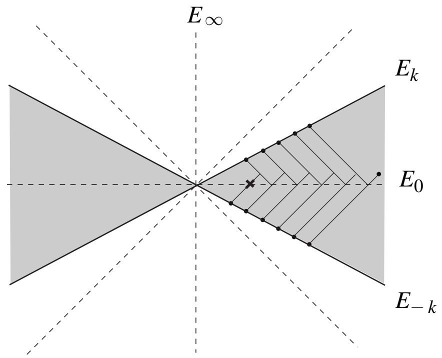

Figure 3.2. The strong staircase: the black dots are the support of the measure ν_n and the cross is it's barycenter $\overline{\nu}_n$. The shaded area is the set of diagonal matrices in $\mathcal{U}$.

Proof. For the case where $A = I$, the identity matrix, the result was shown by the second author in [15]. The idea was to construct so called staircase laminates, laminate measures ν_n satisfying (3.16) and supported in $E_{\{k,-k\}} \cap \mathcal{D}$, where $\mathcal{D}$ denotes diagonal matrices (see Figure 3.2).

For later purposes we sketch the construction of these measures. Set in conformal coordinates

$$C_k = \frac{1}{1+k}(1, k), \quad C_{-k} = \frac{1}{1+k}(1, -k), \quad P_n = \left(n + \frac{1}{2}, \frac{1}{2}\right),$$

and let

$$\lambda_1^n = \frac{1+k}{2kn+1+k}, \quad \lambda_2^n = \frac{(1+k)}{2k(n+1)}. \tag{3.17}$$

Since in conformal coordinates $I = (1, 0)$, we have

$$\begin{aligned} nI &= \lambda_1^n nC_{-k} + (1 - \lambda_1^n)P_n, \\ P_n &= \lambda_2^n(n+1)C_k + (1 - \lambda_2^n)(n+1)I, \end{aligned}$$

and moreover that the pairs nC_{-k}, P_n and $(n+1)C_k$, $(n+1)I$, respectively, are rank-one connected. Thus,

$$\nu_{nI} = \left(\lambda_1^n \delta_{nC_{-k}} + (1 - \lambda_1^n)\left(\lambda_2^n \delta_{(n+1)C_k} + (1 - \lambda_2^n)\delta_{(n+1)I}\right)\right) \tag{3.18}$$

is a laminate.

Now we set $\nu_1 = \nu_{1I}$. It contains a term $(1 - \lambda_1^1)(1 - \lambda_2^1)\delta_{2I}$. The laminate ν_2 is obtained by replacing δ_{2I} by ν_{2I}. In general, ν_n contains a term $\Pi_{i=1}^n(1 - \lambda_1^i)(1 - \lambda_2^i)\delta_{(n+1)I}$ and ν_{n+1} is obtained from it by replacing δ_{nI} by ν_{nI}. By taking logarithms, it follows from (3.17) that

$$\Pi_{i=1}^n(1 - \lambda_1^i)(1 - \lambda_2^i) \approx n^{\frac{-2K}{K-1}}.$$

Then (3.16) follows.

The measures ν_n are supported in $E_{\{k,-k\}} \subset \partial\mathcal{U}$. We shift them to obtain measures supported in the interior of $\mathcal{U}$. Declare, $\tilde{\nu_n}(\cdot) = \nu_n(\cdot + (1, 0))$. As spt $\tilde{\nu}_n \subset \mathcal{U} \cap \mathcal{D}$ we have the claim for $A = (2, 0)$. Next, for matrices $Q \in E_0$ the claim follows from the conformal invariance of $\mathcal{U}$, replacing the laminates $\tilde{\nu} = \sum \lambda_j \delta_{A_j}$ by $Q_\# \tilde{\nu} = \sum \lambda_j \delta_{A_j Q}$.

Finally, for $A \in \mathcal{U} \backslash E_0$ we use the fact that $\mathcal{U}$ is a lamination convex, *cf.* Lemma 3.5. We claim that for every $A \in \mathcal{U}$ there is a rank-one segment $[P, Q]$, contained in $\mathcal{U}$ and with one end point $Q \in E_0$, such that $A = \lambda P + (1 - \lambda)Q$ and $\lambda \in [0, 1)$. Indeed, writing $A = (a_+, a_-)$ in conformal coordinates, let $Q = (a_+ - a_-, 0) \in E_0$. Clearly rank$(A - Q) \leq 1$, since $A - Q = (a_-, a_-)$. Because $\mathcal{U}$ is lamination convex and contains E_0, it also contains the whole segment $[A, Q]$. Furthermore, as $\mathcal{U}$ is open, A is interior point for an extended segment $[P, Q] \in \mathcal{U}$. The required laminates can then be defined as

$$\tilde{\nu}_n = \lambda \delta_P + (1 - \lambda) Q_{\#} \nu_n,$$

where ν_n are the laminates with barycenter I constructed above. $\square$

Proposition 3.11. *For every ball $B = B(x_0, r_0) \subset \Omega$ the set*

$$X_{B,M} = \left\{ f \in X : \int_B |\nabla f|^{\frac{2K}{K-1}} \leq M \right\}$$

is closed and has no interior in X.

Proof. By lower-semicontinuity of the Dirichlet norms $X_{B,M}$ is closed in the weak topology of $W^{1,2}(\Omega)$.

Suppose for a contradiction that $X_{B,M}$ has nonempty interior for some B and M. Then there exists $f \in X_0 \cap X_{B,M}$ and $\varepsilon > 0$ such that

$$\int_B |\nabla g|^{\frac{2K}{K-1}} \leq M$$

whenever $g \in X$ with $d(g, f) < \varepsilon$.

Take any subdomain $\Omega_0 \subset B$ where f is affine, say $f(x) = Ax$ with $A \in \mathcal{U}$. By Proposition 3.10 there exists a laminate $\nu \in \mathcal{L}$ with barycenter $\overline{\nu} = A$ such that $\int_{\mathbb{R}^{2\times 2}} |\lambda|^{\frac{2K}{K-1}} d\nu > 2M$. Let ϕ_j be the corresponding mapping obtained by applying Proposition 2.3 to ν, Ω_0 and $\epsilon = \delta = \frac{1}{j}$ with j large enough. By property (iii) of the proposition it holds that

$$\int_{\Omega_0} |\nabla \phi_j|^{\frac{2K}{K-1}} \geq \int_{\mathbb{R}^{2\times 2}} |\lambda|^{\frac{2K}{K-1}} d\nu - \left(\frac{1}{j}\right)^{\frac{2K}{K-1}} \geq M.$$

Combining this together with property (iv) we deduce that $f_j = f + \chi_{\Omega_0}(\phi_j - A) \in X \setminus X_{B,M}$. However by property (ii) $\| f_j - f \|_\infty \leq \frac{1}{j}$ and hence $\lim_{j\to\infty} d(f_j, f) = 0$. This is a contradiction. $\square$

Corollary 3.12. *The set of points in X such that $\int_B |\nabla f|^{\frac{2K}{K-1}} < \infty$ for some ball $B = B(x_0, r) \subset \Omega$ is of first category in (X, d).*

Proof. This follows since

$$\left\{ f \in X : \int_B |\nabla f|^{\frac{2K}{K-1}} < \infty \text{ for some } B = B(x_0, r) \right\} = \bigcup_{M=1}^{\infty} \bigcup_{i=1}^{\infty} X_{B_i,M}$$

where B_i is an enumeration of balls in Ω with rational centers and radii. Therefore since each $X_{B_i,M}$ is of first category, the (countable) union is also of first category. $\square$

Combining Corollaries 3.9 and 3.12 yields the following result, proving Theorem 1.2.

Theorem 3.13. *Let $K > 1$ and $k = \frac{K-1}{K+1}$. For any bounded open set $\Omega \subset \mathbb{R}^2$ there exists a mapping $f \in W^{1,2}(\Omega; \mathbb{R}^2)$ with the following properties:*

(i) *$f(x) = x$ on $\partial\Omega$,*
(ii) *$\nabla f(x) \in E_{\{k,-k\}}$ a.e. in Ω,*
(iii) *for any ball $B \subset \Omega$ we have $\int_B |\nabla f(x)|^{\frac{2K}{K-1}} dx = \infty$.*

3.2. Lower critical exponents

In the following $J = (0, 1)$ in conformal coordinates, *i.e.*

$$J \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ -x_2 \end{pmatrix},$$

is the complex conjugation. This subsection consists essentially of two parts. First we deal with the geometry of $E_{\{k,-k\}}$, in particular we show that any matrix A lies on a rank-one connection between $E_{\{k,-k\}}$ and E_∞ whose length is proportional to $|A|$. Then we define the staircase laminates in Lemma 3.16. However, in contrast with the case of the upper exponent, we have no *apriori* bounds on the gradient, and so it becomes crucial to know precisely where the gradients of the approximating sequence lie.

In the second part we proceed with convex integration. The setting is quite general once the specific geometric properties have been established. In Proposition 3.17 we show the existence of piecewise affine maps f with the desired integrability property given by (3.33), which solve the inclusion up to a small L^∞ error. Moreover, the size of the error can be made to depend on $|\nabla f|$. Then in Theorem 3.18 we show that the L^∞ error can be successively removed.

The general scheme of passing from approximate solutions to exact solutions is to define an approximating sequence of piecewise affine maps whose gradients lie in smaller and smaller neighborhoods of the set $E_{\{k,-k\}}$. Following Gromov's original terminology such neighboring sets are called *in-approximations*. In our case the in-approximations are the sets

$$\mathcal{O}_n = \left\{ A \in \mathbb{R}^{2\times 2} : \operatorname{dist}(A, E_{\{k,-k\}}) < 2^{-n} \min(1, |A|^{-k}) \right\},$$

so that the approximating sequence satisfies $\nabla f_n \in \mathcal{O}_n$. Because f_n is piecewise affine, we can modify it locally (on affine pieces) to obtain f_{n+1}. The main estimate is in (3.53). It guarantees that the limit mapping lies in the weak Lebesgue space $L^{q_K}_w$, where $q_K = \frac{2K}{K+1}$ is the lower critical exponent.

Having made the size of the error depend on the size of the gradient gives us a very general method to solve differential inclusions in the L^p setting, applicable also in higher dimensions, see [17]. For the specific case, when the critical exponent satisfies $q_K < 2$, some simplifications could have been made. However we have preferred to present here the general method.

We begin with two simple lemmas regarding the geometry of rank-one lines in $\mathbb{R}^{2\times 2}$:

Lemma 3.14. *Let* $A, B \in \mathbb{R}^{2\times 2}$ *with* $\det B \neq 0$ *such that* $\det(A - B) = 0$. *Then*

$$|B| \leq \sqrt{2}K(B)|A|,$$

where $K(B)$ *is the distortion of* B *introduced in (2.4). In particular, if* $A \in \mathbb{R}^{2\times 2}$ *and* $Q \in E_k$ *such that* $\det(A - Q) = 0$, *we have*

$$\operatorname{dist}(A, E_k) \leq |A - Q| \leq (1 + \sqrt{2}K)\operatorname{dist}(A, E_k).$$

Proof. We work in conformal coordinates, writing

$$A = (a_+, a_-), \quad B = (b_+, b_-).$$

First we assume that $\det B > 0$. From (2.2) we see that in this case there exists $\mu \in \mathbb{C}$ with $|\mu| < 1$ so that

$$b_- = \mu b_+.$$

From (2.4) we obtain $|\mu| = \frac{K(B)-1}{K(B)+1}$.

Since $\det(A - B) = 0$, one has

$$|b_+ - a_+| = |\mu b_+ - a_-|\,.$$

Consequently,

$$|b_+| - |a_+| \leq |\mu|\,|b_+| + |a_-|, \quad \textit{i.e.} \quad |b_+| \leq \frac{|a_+| + |a_-|}{1 - |\mu|}.$$

Hence, using (2.2)

$$\begin{aligned} |B|^2 = 2(1 + |\mu|^2)|b_+|^2 &\leq 2\frac{1 + |\mu|^2}{(1 - |\mu|)^2}\,(|a_+| + |a_-|)^2 \leq 2\frac{(1 + |\mu|)^2}{(1 - |\mu|)^2}|A|^2 \\ &= 2K(B)^2|A|^2. \end{aligned}$$

In case $\det B < 0$ we use matrices AJ and BJ in the above computation, and arrive at the same conclusion.

Lastly, let $A \in \mathbb{R}^{2\times 2}$ and $Q \in E_k$, and let $Q_0 \in E_k$ such that $\operatorname{dist}(A, E_k) = |A - Q_0|$. Then, applying the first part of the lemma with $A - Q_0$ and $Q - Q_0$ we obtain that

$$|Q - Q_0| \leq \sqrt{2}K|A - Q_0|.$$

Hence

$$\begin{aligned} |A - Q| &\leq |A - Q_0| + |Q - Q_0| \leq (1 + \sqrt{2}K)|A - Q_0| \\ &= (1 + \sqrt{2}K)\operatorname{dist}(A, E_k). \end{aligned}$$

□

Lemma 3.15. *Let $K > 1$ and $k = \frac{K-1}{K+1}$. Then every $A \in \mathbb{R}^{2\times 2} \setminus \{0\}$ lies on a rank-one segment connecting E_∞ and E_k. To be precise, there exist matrices $P \in E_\infty \setminus \{0\}$ and $Q \in E_k \setminus \{0\}$ with $\det(P-Q) = 0$ such that $A \in [P, Q]$. Moreover,*

$$\frac{1}{c_K}|A| \leq |P-Q|, |P|, |Q| \leq c_K|A|, \tag{3.19}$$

where $c_K > 1$ depends only on K. The same holds if we replace E_k with E_{-k}.

Proof. It suffices to prove the lemma for E_k. Suppose in conformal coordinates $A = (a, b)$. Then for every $t \in \mathbb{R} \setminus \{0\}$,

$$A = (a, k\overline{a}) + \frac{1}{t}(0, tb - tk\overline{a}) = P + \frac{1}{t}Q_t$$

where $P \in E_k$ and $Q_t \in E_\infty$. Here P, Q_t are rank-one connected if and only if $|a| = |k\overline{a} + t(k\overline{a} - b)|$. By elementary geometry, there is precisely one positive $t = t_0 > 0$ for which this happens.

Note that $s := 1 + 1/t > 1$. Hence we may write

$$A = \frac{1}{s}(sP) + \frac{1}{ts}(sQ_t),$$

where clearly $sP \in E_k$, $sQ_t \in E_\infty$ are rank-one connected; also $s^{-1} + (ts)^{-1} = 1$.

Concerning the estimates in (3.19), first of all observe that

$$\operatorname{dist}(A, E_\infty) + \operatorname{dist}(A, E_k) \leq |A-P| + |A-Q| = |P-Q|.$$

Since E_∞ and E_k are linearly independent, we obtain,

$$\frac{1}{c_K}|A| \leq |P-Q|.$$

Moreover, from Lemma 3.14 we have

$$|P| \leq c_K|A|, \quad |Q| \leq c_K|A|, \quad |Q| \leq c_K|P|, \text{ and } |P| \leq c_K|Q|,$$

hence from the triangle inequality

$$|P-Q| \leq |P| + |Q| \leq (1 + c_K)\min(|P|, |Q|),$$

from which the remaining inequalities in (3.19) follow. □

With the next lemma, we construct one step in our staircase laminate. The steps of the staircase are the sets

$$\mathcal{S}_n = nJSO(2) = \left\{(0, ne^{i\theta}) : \theta \in [0, 2\pi]\right\}. \tag{3.20}$$

The measures are more complicated than in the case of upper exponent (3.18) since we want the center of mass to be any matrix in a neighborhood of $\mathcal{S}_n$.

Lemma 3.16. *Let* $0 < r < \frac{1}{2}$*. Then there exists a constant* c_K *such that for each* $n \in \mathbb{N}$ *and* $A \in \mathbb{R}^{2\times 2}$ *with*

$$\operatorname{dist}(A, \mathcal{S}_n) < r \tag{3.21}$$

there exists a laminate ν_A *of third order with the following properties:*

- $\overline{\nu}_A = A$,
- $\operatorname{spt}\nu_A \subset E_{\{k,-k\}} \cup \mathcal{S}_{n+1}$,
- $\operatorname{spt}\nu_A \subset \left\{\xi \in \mathbb{R}^{2\times 2} : c_K^{-1} n < |\xi| < c_K n\right\}$.

Moreover, it holds that

$$\left(1 - c_K \frac{r}{n}\right)\beta_n \le \nu_A(\mathcal{S}_{n+1}) \le \left(1 + c_K \frac{r}{n}\right)\beta_{n+2}, \tag{3.22}$$

where

$$\beta_n = 1 - \frac{1+k}{n}. \tag{3.23}$$

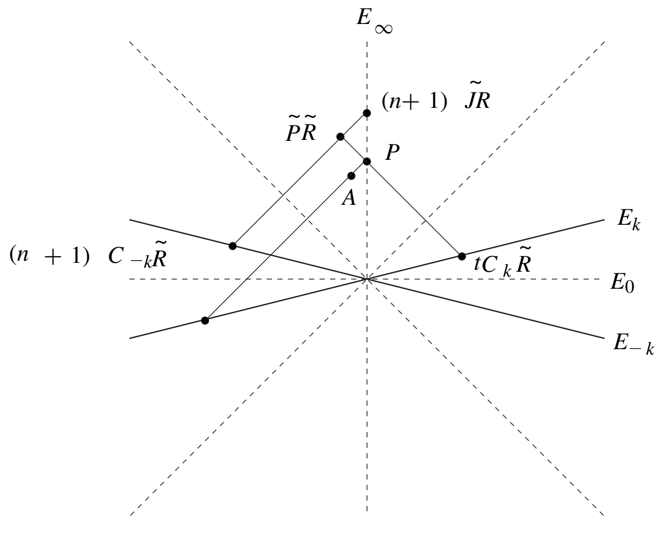

Figure 3.3. One weak step towards infinity.

Proof. Using Lemma 3.15 there exists $P \in E_\infty$ and $Q \in E_k$ with $\operatorname{rank}(P-Q) = 1$ such that $A = \lambda_1 P + (1-\lambda_1)Q$ for some $\lambda_1 \in [0,1]$ and so that (3.19) holds. By assumption (3.21) there exists a matrix $R \in SO(2)$ such that $|A - nJR| < r$. Lemma 3.14, applied to $A - nJR$ and $P - nJR$, together with the fact that $P - nJR \in E_\infty$, yields

$$|P - nJR| < \sqrt{2}r. \tag{3.24}$$

Hence $|P - A| < 3r$, and furthermore $|P - Q| > \frac{n}{c_K}$ by (3.19). But then

$$\lambda_1 = \frac{|A - Q|}{|P - Q|} \geq 1 - \frac{|P - A|}{|P - Q|} \geq 1 - c_K \frac{r}{n}. \tag{3.25}$$

Now JP is conformal, so that $JP = t\tilde{R}$ for some $t > 0$ and $\tilde{R} \in SO(2)$. Thus $P = tJ\tilde{R}$, and so (3.24) gives

$$n|JR| - \sqrt{2}r < t|J\tilde{R}| < n|JR| + \sqrt{2}r.$$

Therefore, (since $|JR| = |J\tilde{R}| = \sqrt{2}$)

$$|t - n| < r. \tag{3.26}$$

Define the matrices

$$C_k = \frac{1}{1+k}(1, k), \quad C_{-k} = \frac{1}{1+k}(-1, k) \tag{3.27}$$

in conformal coordinates. Note that $C_{\pm k} \in E_{\pm k}$ and $\det(J - C_{\pm k}) = 0$. Let

$$\tilde{P} = \left(-\frac{1 - (t - n)}{2}, n + \frac{1 + (t - n)}{2}\right).$$

By direct calculation

$$\begin{aligned} tJ &= \lambda_2 t C_k + (1 - \lambda_2)\tilde{P}, \\ \tilde{P} &= \lambda_3 (n + 1) C_{-k} + (1 - \lambda_3)(n + 1) J, \end{aligned}$$

where

$$\lambda_2 = \frac{1 + k - (t - n)(1 + k)}{2n + 1 + k + (t - n)(1 - k)}, \tag{3.28}$$

$$\lambda_3 = \frac{(1 - t + n)(1 + k)}{2(n + 1)}. \tag{3.29}$$

Moreover $\det(tC_k - \tilde{P}) = 0$. Therefore according to Definition 2.2

$$\begin{aligned} \nu_A = &\lambda_1 \Big(\lambda_2 \delta_{tC_k\tilde{R}} + (1 - \lambda_2)\left(\lambda_3 \delta_{(n+1)C_{-k}\tilde{R}} + (1 - \lambda_3)\delta_{(n+1)J\tilde{R}}\right)\Big) \\ &+ (1 - \lambda_1)\delta_Q \end{aligned}$$

is a laminate with barycenter A and

$$\nu_A\left(\left\{(n + 1)J\tilde{R}\right\}\right) = \lambda_1(1 - \lambda_2)(1 - \lambda_3). \tag{3.30}$$

Note that from Lemma 3.14 we get $c_K^{-1} n < |Q| < c_K n$, hence

$$\operatorname{spt} \nu_A \subset \left\{ \xi \in \mathbb{R}^{2\times 2} : c_K^{-1} n < |\xi| < c_K n \right\}.$$

It remains to obtain the estimate (3.22).[2] By direct calculation we find

$$(1-\lambda_2)(1-\lambda_3) = \frac{n+(t-n)}{n+1} \frac{2n+1-k+(t-n)(1+k)}{2n+1+k+(t-n)(1-k)}. \tag{3.31}$$

Recall that $(t-n)$ is bounded in (3.26) by $r < 1$, which we treat as a small parameter. First we bound (3.31) from above:

$$\begin{aligned}
(1-\lambda_2)(1-\lambda_3) &= \left(1+\frac{t-n}{n}\right)\frac{n}{n+1}\left(1-\frac{2k(1-(t-n))}{2n+1+k+(t-n)(1-k)}\right) \\
&\le \left(1+\frac{r}{n}\right)\left(1-\frac{1}{n+1}\right)\left(1-\frac{k}{n+1}+k\frac{r}{n+1}\right) \\
&\le \left(1+c_K\frac{r}{n}\right)\left(1-\frac{1}{n+1}\right)\left(1-\frac{k}{n+1}\right),
\end{aligned}$$

where c_K is a constant so that

$$k\frac{r}{n+1}\left(1+\frac{r}{n}\right) \le (c_K-1)\frac{r}{n}\left(1-\frac{k}{n+1}\right).$$

Moreover

$$\left(1-\frac{1}{n+1}\right)\left(1-\frac{k}{n+1}\right) = 1-\frac{1+k}{n+1}+\frac{k}{(n+1)^2} \le 1-\frac{1+k}{n+2},$$

hence we deduce

$$(1-\lambda_2)(1-\lambda_3) \le \left(1+c_K\frac{r}{n}\right)\left(1-\frac{1+k}{n+2}\right).$$

The bound from below is very similar:

$$\begin{aligned}
(1-\lambda_2)(1-\lambda_3) &\ge \left(1-\frac{r}{n}\right)\left(1-\frac{1}{n+1}\right)\left(1-\frac{k}{n}-k\frac{r}{n}\right) \\
&\ge \left(1-c_K\frac{r}{n}\right)\left(1-\frac{1}{n+1}\right)\left(1-\frac{k}{n}\right) \\
&\ge \left(1-c_K\frac{r}{n}\right)\left(1-\frac{1+k}{n}\right).
\end{aligned}$$

[2] In fact from (3.26) we have that $\lambda_2 = \frac{1+k}{2n} + r\,O(\frac{1}{n})$ and $\lambda_3 = \frac{1+k}{2n} + r\,O(\frac{1}{n})$, and the correct asymptotics for $\nu_A\left(\mathcal{S}_{n+1}\right)$ can be deduced from this and (3.25). Nevertheless we give the precise calculation to obtain the estimate for all $n \in \mathbb{N}$ and all $0 < r < 1/2$.

Combining with (3.25) we finally obtain

$$\left(1 - c_K \frac{r}{n}\right) \beta_n \leq \nu_A \left(\left\{(n+1) J \tilde{R}\right\}\right) \leq \left(1 + c_K \frac{r}{n}\right) \beta_{n+2}. \qquad \square$$

Iterating Lemma 3.16 would yield the analogue of Proposition 3.10. Instead of doing this, however, it seems more convenient to construct directly approximate solutions to the differential inclusion $\nabla f(x) \in E_{\{k,-k\}}$ whose gradient distributions behave like the staircase laminate (see (3.33) below). It is important to note that the mapping we obtain is piecewise affine.

Proposition 3.17. *Let $K > 1$ and $k = \frac{K-1}{K+1}$. Let $\alpha \in (0,1)$, $\delta > 0$ and $\tau : [0,\infty) \to (0,1]$ a continuous, non-increasing function with*

$$\int_1^\infty \frac{\tau(t)}{t} dt < \infty. \tag{3.32}$$

Then, there exists $\delta_0 > 0$ depending on τ and K such that for any bounded open set $\Omega \subset \mathbb{R}^2$ and any nonzero matrix F with $\operatorname{dist}(F, E_\infty) < \delta_0 |F|$ there exists a piecewise affine mapping $f \in W^{1,1}(\Omega;\mathbb{R}^2) \cap C^\alpha(\overline{\Omega};\mathbb{R}^2)$ with the following properties:

(i) $f(x) = Fx$ *on* $\partial\Omega$,

(ii) $[f - F]_{C^\alpha(\overline{\Omega})} < \delta$,

(iii) $\operatorname{dist}\left(\nabla f(x), E_{\{k,-k\}}\right) < \tau(|\nabla f(x)|)$ *a.e. in* Ω,

and there exists a constant $c_{K,\tau} > 1$ so that for all $t > |F|$ we have

$$c_{K,\tau}^{-1} |F|^{\frac{2K}{K+1}} t^{-\frac{2K}{K+1}} < \frac{|\{x \in \Omega : |\nabla f(x)| > t\}|}{|\Omega|} < c_{K,\tau} |F|^{\frac{2K}{K+1}} t^{-\frac{2K}{K+1}}. \tag{3.33}$$

Proof. First of all, by considering $\tilde{F} = |F|^{-1} F$, $\tilde{\delta} = |F|^{-1}\delta$ and

$$\tilde{\tau}(t) = \begin{cases} |F|^{-1}\tau(|F|t) & \text{if } |F| \geq 1, \\ \tau(t) & \text{if } |F| < 1, \end{cases}$$

we can reduce to the case

$$\operatorname{dist}(F, \mathcal{S}_1) < \delta_0.$$

Indeed, if $\tilde{f}$ satisfies (i)-(iii) and (3.33) with $\tilde{F}$, $\tilde{\delta}$ and $\tilde{\tau}$, then $f = |F|\tilde{f}$ satisfies (i) and (ii) with F and δ,

$$\begin{aligned} \operatorname{dist}(\nabla f, E_{\{k,-k\}}) &= |F| \operatorname{dist}(\nabla \tilde{f}, E_{\{k,-k\}}) \\ &< |F| \tilde{\tau}(|F|^{-1}|\nabla f|) \leq \tau(|\nabla f|), \end{aligned}$$

and

$$|\{x \in \Omega : |\nabla f(x)| > t\}| = \left|\left\{x \in \Omega : |\nabla \tilde{f}(x)| > |F|^{-1} t\right\}\right|.$$

Also, in case $|F| \geq 1$, we have

$$\int_1^\infty \frac{\tilde{\tau}(t)}{t}\, dt = \frac{1}{|F|} \int_{|F|}^\infty \frac{\tau(t)}{t}\, dt \leq \int_1^\infty \frac{\tau(t)}{t}\, dt.$$

Furthermore if (iii) is fulfilled for some $\tau_0 \leq \tau$, then it is also fulfilled for τ. Therefore we may assume without loss of generality that

$$\tau(0) < 1/4 \min\left(c_K^{-1}, \operatorname{dist}(J, E_{\{k,-k\}})\right), \tag{3.34}$$

where c_K is the constant from Lemma 3.16.

We define a sequence of piecewise affine mappings $\{f_n\}$ inductively, using repeatedly Proposition 2.3 and Lemma 3.16. Let $f_1(x) = Fx$ in Ω. For the inductive step we assume the existence of a piecewise affine Lipschitz mapping $f_n : \Omega \to \mathbb{R}^2$ such that

(a) $f_n(x) = Fx$ on $\partial\Omega$,
(b) $[f_n - F]_{C^\alpha(\overline{\Omega})} < (1 - 2^{-n})\delta$,
(c) $\operatorname{dist}\left(\nabla f_n(x), E_{\{k,-k\}} \cup \mathcal{S}_n\right) < \tau(|\nabla f_n(x)|)$ a.e. in Ω,

and

$$\Omega_n \stackrel{\text{def}}{=} \left\{x \in \Omega : \operatorname{dist}\left(\nabla f_n(x), E_{\{k,-k\}}\right) \geq \tau(|\nabla f_n(x)|)\right\}$$

satisfies

$$\prod_{j=1}^{n-1} \left(1 - c_K \frac{\tau(j)}{j}\right) \beta_j \leq \frac{|\Omega_n|}{|\Omega|} \leq \prod_{j=1}^{n-1} \left(1 + c_K \frac{\tau(j)}{j}\right) \beta_{j+2}. \tag{3.35}$$

To show that f_1 satisfies the inductive hypothesis, we now choose δ_0 appropriately. Indeed, it suffices to ensure that $\operatorname{dist}(F, \mathcal{S}_1) < \delta_0$ implies

$$\operatorname{dist}(F, \mathcal{S}_1) < \tau(|F|) \text{ and } \operatorname{dist}(F, E_{\{k,-k\}}) > \tau(|F|), \tag{3.36}$$

since in this case (c) is satisfied, and $\Omega_1 = \Omega$ hence (3.35) is also satisfied. Using the monotonicity of τ and (3.34) we see that (3.36) will follow for sufficiently small $\delta_0 > 0$ (depending only on τ and K).

To obtain f_{n+1} we modify f_n on the set Ω_n. Because f_n is piecewise affine, we have a decomposition into pairwise disjoint open subsets $\Omega_{n,i}$ such that

$$\left|\Omega_n \setminus \bigcup_{i=1}^\infty \Omega_{n,i}\right| = 0,$$

and $f_n(x) = A_i x + b_i$ in $\Omega_{n,i}$ for some A_i with $\operatorname{dist}(A_i, \mathcal{S}_n) < \tau(|A_i|)$ and $b_i \in \mathbb{R}^2$. First of all we claim that

$$\operatorname{dist}(A_i, \mathcal{S}_n) < \tau(n). \tag{3.37}$$

Indeed, by assumption there exists $R \in SO(2)$ such that $|nJR - A_i| < \tau(|A_i|) \leq 1/4$. In particular $|A_i| \geq |nJR| - 1/4 = n\sqrt{2} - 1/4 \geq n$ for all $n \in \mathbb{N}$. But then (3.37) follows from the monotonicity of τ.

For each i we use Proposition 2.3 with the laminate ν_{A_i} from Lemma 3.16 to obtain a piecewise affine Lipschitz mapping $g_i : \Omega_{n,i} \to \mathbb{R}^2$ with

(d) $g_i(x) = A_i x + b_i$ on $\partial\Omega_{n,i}$,
(e) $[g_i - f_n]_{C^\alpha(\Omega_{n,i})} < 2^{-(n+1+i)}\delta$,
(f) $c_K^{-1} n < |\nabla g_i(x)| < c_K n$ a.e. in $\Omega_{n,i}$,
(g) $\operatorname{dist}\left(\nabla g_i(x), E_{\{k,-k\}} \cup \mathcal{S}_n\right) < \tau(c_K n)$ a.e. in $\Omega_{n,i}$,

and

$$\begin{aligned} &\left(1 - c_K \frac{\tau(n)}{n}\right)\beta_n \\ &\leq \frac{1}{|\Omega_{n,i}|}\left|\left\{x \in \Omega_{n,i} : \operatorname{dist}(\nabla g_i(x), \mathcal{S}_n) < \tau(c_K n)\right\}\right| \\ &\leq \left(1 + c_K \frac{\tau(n)}{n}\right)\beta_{n+2}. \end{aligned} \tag{3.38}$$

We then define

$$f_{n+1}(x) = \begin{cases} f_n(x) & \text{if } x \in \Omega \setminus \bigcup_{i=1}^\infty \Omega_{n,i} \\ g_i(x) & \text{if } x \in \Omega_{n,i} \end{cases}$$

It is clear that $f_{n+1}(x) = Fx$ on $\partial\Omega$, and from (e) we get $[f_{n+1} - f_n]_{C^\alpha(\overline{\Omega})} < 2^{-(n+1)}\delta$, hence (b) follows. Because τ is non increasing, (c) follows from (f) and (g). Finally (3.35) follows from (3.38).

Observe that on $\Omega \setminus \Omega_n$ we have that $\nabla f_{n+1} = \nabla f_n$ almost everywhere. Therefore $\Omega_{n+1} \subset \Omega_n$, so that the sequence $\{f_n\}$ is obtained by modification on a *nested* sequence of open sets Ω_n whose measure satisfies

$$|\Omega| \prod_{j=1}^{n-1}\left(1 - c_K \frac{\tau(j)}{j}\right)\beta_j \leq |\Omega_n| \leq |\Omega| \prod_{j=1}^{n-1}\left(1 + c_K \frac{\tau(j)}{j}\right)\beta_{j+2}.$$

By the condition (3.32) on τ and since $\tau(1) < 1/2c_K^{-1}$, the products

$$\prod_{j=1}^{\infty}\left(1 - c_K \frac{\tau(j)}{j}\right) = c_1, \text{ and } \prod_{j=1}^{\infty}\left(1 + c_K \frac{\tau(j)}{j}\right) = c_2$$

are finite and nonzero: $0 < c_1 < c_2 < \infty$. For the factor $\prod_{j=1}^n \beta_j$ we take logarithms, recalling that $k+1 = \frac{2K}{K+1}$, to obtain that

$$\left| \log\left(\prod_{j=1}^n \beta_j\right) + \frac{2K}{K+1}\log n \right| < c. \tag{3.39}$$

Therefore there exists a constant $c_{\tau,K} > 1$, depending only on K and on $\int_1^\infty \frac{\tau(t)}{t}dt$, such that

$$\frac{1}{c_{\tau,K}} n^{-\frac{2K}{K+1}} \leq |\Omega_n| \leq c_{\tau,K} n^{-\frac{2K}{K+1}}, \tag{3.40}$$

and in particular $|\Omega_n| \to 0$ as $n \to \infty$. Therefore the limit $f(x) = \lim_{n\to\infty} f_n(x)$ exists pointwise almost everywhere and f is piecewise affine. Moreover, it satisfies (i)-(iii) by our construction.

Finally, the distribution function of ∇f can be estimated as follows. Using (f), for $n \in \mathbb{N}$ we have

$$|\nabla f(x)| > \frac{n}{c_K} \text{ in } \Omega_n, \text{ and } |\nabla f(x)| < c_K n \text{ in } \Omega \setminus \Omega_n.$$

For given $t > c_K$ let n_1 be the integer part of $c_K t$ and n_2 the integer part of $c_K^{-1} t$. Then

$$\Omega_{n_1+1} \subset \{x \in \Omega : |\nabla f(x)| > t\} \subset \Omega_{n_2},$$

and therefore (3.33) follows from (3.40). Lastly, (3.33) implies that ∇f_n is uniformly bounded in L^1, hence $f \in W^{1,1}$ by dominated convergence. □

Concerning the constant $c_{K,\tau}$ in (3.33) we remark that, as the proof shows, it depends monotonically on τ, in other words if $\tau_1 \leq \tau_2$ then $c_{K,\tau_1} \leq c_{K,\tau_2}$.

Using Proposition 3.17 we can now construct a sequence of approximate solutions to the differential inclusion that converge to a real solution.

Theorem 3.18. *Let $K > 1$, $k = \frac{K-1}{K+1}$ and let F be any 2×2 matrix. For any $\alpha \in (0,1)$, $\delta > 0$ and for any bounded open set $\Omega \subset \mathbb{R}^2$ there exists a mapping $f \in W^{1,1}(\Omega; \mathbb{R}^2) \cap C^\alpha(\overline{\Omega}; \mathbb{R}^2)$ with the following properties:*

(i) *$f(x) = Fx$ on $\partial\Omega$,*

(ii) *$[f - F]_{C^\alpha(\overline{\Omega})} < \delta$,*

(iii) *$\nabla f(x) \in E_{\{k,-k\}}$ a.e. in Ω,*

(iv) *for any ball $B \subset \Omega$ there exists a constant $c_B > 1$ such that*

$$\frac{1}{c_B} t^{-\frac{2K}{K+1}} < |\{x \in B : |\nabla f(x)| > t\}| < c_B t^{-\frac{2K}{K+1}} \tag{3.41}$$

for all $t \geq 1$.

In particular $f \in W^{1,q}(\Omega)$ *for all* $q < \frac{2K}{K+1}$, *but for any ball* $B \subset \Omega$ *we have* $\int_B |\nabla f(x)|^{\frac{2K}{K+1}} dx = \infty$.

Proof. The proof is similar to the proof of Proposition 3.17 in that we construct a sequence of piecewise affine mappings converging to a solution by modifying affine pieces. An important difference is that at each step we modify our sequence *almost everywhere* in the domain Ω in order to obtain (3.41) in every ball. Therefore it becomes crucial to control the L^1-norm of the difference $\nabla f_{n+1} - \nabla f_n$. Let

$$\tau(t) = \min(1, t^{-k}).$$

After multiplication with a constant if necessary, we may assume without loss of generality that $|F| < 1$.

Let $f_0(x) = Fx$. Our plan is to construct a sequence of piecewise affine mappings $f_n \in W^{1,1}(\Omega) \cap C^\alpha(\overline{\Omega})$ such that

- $f_n(x) = Fx$ on $\partial\Omega$,
- $[f_{n+1} - f_n]_{C^\alpha(\overline{\Omega})} < 2^{-(n+1)}\delta$,
- $\|\nabla f_{n+1} - \nabla f_n\|_{L^1(\Omega)} < c_K 2^{-n}$,
- $\operatorname{dist}(\nabla f_n(x), E_{\{k,-k\}}) < 2^{-n}\tau(|\nabla f_n(x)|)$ a.e. $x \in \Omega$.

For such a sequence the limit f will exist in $W^{1,1}(\Omega) \cap C^\alpha(\overline{\Omega})$, and will hence satisfy (i),(ii) and (iii) in the theorem. The main emphasis will be on obtaining (3.41).

To obtain f_{n+1} from f_n, decompose Ω into a union of pairwise disjoint open sets of diameter no more than $\frac{1}{n}$ with

$$\left|\Omega \setminus \bigcup_i \Omega_i^n\right| = 0,$$

so that f_n is affine in each Ω_i^n, with $\nabla f_n = A_i^n$. In each Ω_i^n we replace f_n with a new mapping having the same affine boundary values on $\partial\Omega_i^n$. The induction hypothesis here is that $A_i^n \in \mathcal{O}_n$, where

$$\mathcal{O}_n = \left\{A \in \mathbb{R}^{2\times 2} : \operatorname{dist}(A, E_{\{k,-k\}}) < 2^{-n}\tau(|A|)\right\}.$$

Step 1. In order to keep the notation simple, we first show how to construct the new mapping with linear boundary values given by some $A \in \mathbb{R}^{2\times 2}$ which satisfies the induction hypothesis:

Claim: Let $A \in \mathcal{O}_n$ for some $n \in \mathbb{N}$, *i.e.*

$$\operatorname{dist}(A, E_k) < 2^{-n}\tau(|A|).$$

For any open subset $\omega \subset \Omega$ and any $\eta > 0$ there exists a piecewise affine mapping $h \in W^{1,1}(\Omega) \cap C^\alpha(\overline{\Omega})$ with the following properties:

(a) $h(x) = Ax$ on $\partial\omega$,
(b) $[h - A]_{C^\alpha(\overline{\omega})} < \eta$,
(c) $\operatorname{dist}(\nabla h(x), E_{\{k,-k\}}) < 2^{-(n+1)}\tau(|\nabla h(x)|)$ a.e. $x \in \omega$,
(d) $\int_\omega |\nabla h(x) - A|\,dx \le c_K 2^{-n}|\omega|$,
(e) there exists an open subset $\tilde{\omega} \subset \omega$ and $\tilde{\delta} > 0$, depending on n such that,
   (e_1) $|\nabla h(x) - A| < c_K 2^{-n}$ a.e. $x \in \omega \setminus \tilde{\omega}$,
   (e_2) $\tilde{\delta} t^{-\frac{2K}{K+1}} < |\{x \in \tilde{\omega} : |\nabla h| > t\}| < c_K 2^{-n} t^{-\frac{2K}{K+1}}|\omega|$ for $t > 1$,
   (e_3) if $|A| \ge c_K$, then $|\tilde{\omega}| \le 2^{-n}|\omega|$.

Here $c_K > 1$ is some fixed constant depending only on K.

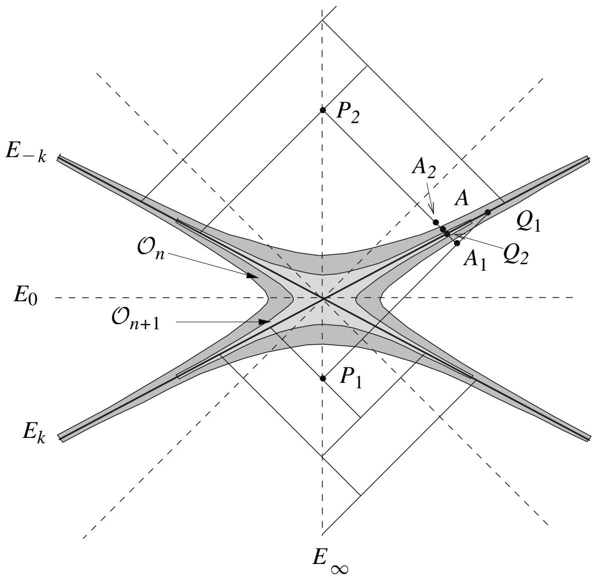

Figure 3.4. One step in the construction: first A is split using the measure μ_A in (3.45) into $\{P_1, P_2, Q_1, Q_2\}$, and then P_1 and P_2 are replaced by the staircase constructed in Proposition 3.17. In the figure, since $A \neq 0$ we can take $Q_2 = Q_0$ lying in the line between A_1 and P_2.

Proof of Claim: First of all there exists $Q_0 \in E_k$ with $\det(A - Q_0) = 0$. This follows for example by invoking Lemma 3.15. In turn Lemma 3.14 implies that

$$|A - Q_0| \le (1 + \sqrt{2}K)2^{-n}\tau(|A|).$$

Let $A_1, A_2 \in \mathbb{R}^{2\times 2}$ be matrices on the line going through A and Q_0 (if $A = Q_0$, then take any rank-one line through A) with the property that

$$Q_0 = \frac{1}{2}A_1 + \frac{1}{2}A_2 \quad \text{and} \quad |A_j - Q_0| = (1 + \sqrt{2}K)2^{-n}\tau(|A|) \text{ for } j = 1, 2.$$

Then, using Lemma 3.14 we deduce that

$$2^{-n}\tau(|A|) \leq \operatorname{dist}(A_j, E_k) \leq (1+\sqrt{2}K)2^{-n}\tau(|A|) \text{ for } j=1,2. \tag{3.42}$$

Moreover by construction

$$|A_j - A| \leq 2(1+\sqrt{2}K)2^{-n}\tau(|A|),$$

and there exists $\lambda \in [0,1]$ such that

$$A = \lambda A_1 + (1-\lambda)A_2.$$

Note also that $|A_1 - A_2| \leq c_K 2^{-n}\tau(|A|) \leq c_K|A_2|$ from (3.42), so that

$$\frac{1}{c_K}|A_2| \leq |A_1| \leq c_K|A_2|. \tag{3.43}$$

Now we invoke Lemma 3.15 (notice that A_1, A_2 are nonzero because of (3.42)) to find nonzero matrices $P_j \in E_\infty$ and $Q_j \in E_k$ and numbers $\lambda_j \in [0,1]$ for $j = 1, 2$ such that

$$A_j = \lambda_j P_j + (1-\lambda_j)Q_j,$$

where

$$\frac{1}{c_K}|A_j| \leq |P_j| \leq c_K|A_j|, \tag{3.44}$$

and, using Lemmas 3.15 and 3.14,

$$\lambda_j = \frac{|A_j - Q_j|}{|P_j - Q_j|}$$

satisfies

$$\frac{1}{c_K}\frac{2^{-n}\tau(|A|)}{|A_j|} \leq \lambda_j \leq c_K\frac{2^{-n}\tau(|A|)}{|A_j|}.$$

In summary, the measure

$$\mu_A = \lambda\left(\lambda_1\delta_{P_1} + (1-\lambda_1)\delta_{Q_1}\right) + (1-\lambda)\left(\lambda_2\delta_{P_2} + (1-\lambda_2)\delta_{Q_2}\right) \tag{3.45}$$

is a laminate (of second order) with barycenter $\bar{\mu}_A = A$. Let

$$\lambda_A = \lambda\lambda_1 + (1-\lambda)\lambda_2,$$

so that, using (3.43),

$$\frac{1}{c_K}\frac{2^{-n}\tau(|A|)}{|A_1|} \leq \lambda_A \leq c_K\frac{2^{-n}\tau(|A|)}{|A_1|}. \tag{3.46}$$

Using the laminate μ_A, for any $\eta > 0$ and any open subdomain $\omega \subset \Omega$ Proposition 2.3 provides a piecewise affine Lipschitz mapping $g : \omega \to \mathbb{R}^2$ with $g(x) = Ax$ if $x \in \partial\omega$, $[g - A]_{C^\alpha(\overline{\omega})} < \eta/2$, and for some $\varepsilon > 0$ to be chosen later

$$|\{x \in \omega : \operatorname{dist}(\nabla g(x), \{P_1, P_2\}) < \varepsilon\}| = \lambda_A|\omega|,$$
$$|\{x \in \omega : \operatorname{dist}(\nabla g(x), \{Q_1, Q_2\}) < \varepsilon\}| = (1 - \lambda_A)|\omega|.$$

Let $\tilde{\omega} = \{x \in \omega : \operatorname{dist}(\nabla g(x), \{P_1, P_2\}) < \varepsilon\}$. Since g i s piecewise affine, there exists a decomposition of $\tilde{\omega}$ into pairwise disjoint open subsets $\tilde{\omega}_i$, with $|\tilde{\omega} \setminus \bigcup_{i=1}^{\infty} \tilde{\omega}_i| = 0$, such that $g(x) = \tilde{P}_i x + b_i$ in $\tilde{\omega}_i$ for some $b_i \in \mathbb{R}^2$ and

$$\tilde{P}_i \in B_\varepsilon(P_1) \cup B_\varepsilon(P_2). \tag{3.47}$$

Let $\tilde{\tau} = 2^{-n}\tau$ and $\tilde{\delta_0}$ the corresponding parameter given by Proposition 3.17. We choose ϵ small enough so that $\operatorname{dist}(\tilde{P}_i, E_\infty) \leq \tilde{\delta_0}|\tilde{P}_i|$. Since $P_1, P_2 \in E_\infty \setminus \{0\}$ we can use Proposition 3.17 in each subset $\tilde{\omega}_i$. By replacing g in each subset $\tilde{\omega}_i$ with the mappings provided by Proposition 3.17 we obtain a piecewise affine mapping $h : \omega \to \mathbb{R}^2$ such that $h(x) = Ax$ on $\partial\omega$, $[h - A]_{C^\alpha(\overline{\omega})} < \eta$, and

$$|\nabla h(x) - A| < c_K 2^{-n} \text{a.e } x \in \omega \setminus \tilde{\omega}, \tag{3.48}$$

which is e_1. Furthermore, since $Q_j \in E_k$ for $j = 1, 2$ we can ensure that ε is sufficiently small so that

$$\operatorname{dist}(Q, E_{\{k,-k\}}) < 2^{-(n+1)}\tau(|Q|) \text{ for all } Q \in B_\varepsilon(Q_1) \cup B_\varepsilon(Q_2).$$

Combining this with Proposition 3.17 (iii) yields

$$\operatorname{dist}\left(\nabla h(x), E_{\{k,-k\}}\right) < 2^{-(n+1)}\tau(|\nabla h|) \text{ a.e. in } \omega.$$

Moreover, (3.33) in Proposition 3.17 together with (3.43) and (3.44) leads to

$$\frac{1}{c_K}|A_1|^{\frac{2K}{K+1}} t^{-\frac{2K}{K+1}} < \frac{|\{x \in \tilde{\omega} : |\nabla h(x)| > t\}|}{|\tilde{\omega}|} < c_K |A_1|^{\frac{2K}{K+1}} t^{-\frac{2K}{K+1}} \tag{3.49}$$

for all $t > |A_1|$. So far we have proved that h satisfies the properties (a),(b),(c) and (e_1) in the claim.

Recall that $|\tilde{\omega}| = \lambda_A|\omega|$, so that ($e_3$) follows directly from (3.46). We now prove that h is close to A in $W^{1,1}$ which is the key to obtain the strong convergence in $W^{1,1}$ of the sequence $\{f_n\}$. Note that $|A| \le c_K|A_1|$, hence

$$\begin{aligned}\int_\omega |\nabla h(x) - A|dx &\le \int_{\tilde\omega} |\nabla h(x)| + |A|dx + \int_{\omega\setminus\tilde\omega} |\nabla h(x) - A|dx \\ &\le \int_{|A_1|}^{\infty} |\{x \in \tilde\omega : |\nabla h(x)| > t\}|\, dt + c_K|A_1||\tilde\omega| + c_K 2^{-n}|\omega| \\ &\le c_K \int_{|A_1|}^{\infty} |A_1|^{\frac{2K}{K+1}} t^{-\frac{2K}{K+1}} dt|\tilde\omega| + c_K|A_1||\tilde\omega| + c_K 2^{-n}|\omega| \\ &\le 2c_K|A_1|\lambda_A|\omega| + c_K 2^{-n}|\omega| \\ &\le (2c_K^2 + c_K)2^{-n}|\omega|,\end{aligned}$$

where we have used (3.48), (3.49) and (3.46).

Finally, to prove (e_2) we use (3.46) and that $k = \frac{K-1}{K+1}$ to deduce from (3.49) that

$$|\{x \in \tilde\omega : |\nabla h(x)| > t\}| < c_K|A_1|^k \tau(|A|)2^{-n} t^{-\frac{2K}{K+1}}|\omega|.$$

On the other hand by our construction we have $|A_1| \le c_K \max(1, |A|)$, therefore

$$|\{x \in \tilde\omega : |\nabla h(x)| > t\}| < c_K 2^{-n} t^{-\frac{2K}{K+1}}|\omega|, \tag{3.50}$$

which is the upper bound in (e_2). The lower bound in e_2 follows directly from (3.49) since $A_1 \ne 0$ by (3.42). In fact by combining (61) and (65) we can put $\tilde\delta = \frac{1}{c_K}2^{-n}$ if $|A| \ge 1$ and $\tilde\delta = \frac{1}{c_K}2^{-2n}$ otherwise, so that $\tilde\delta$ only depends on n and not on A, though this is not needed in the proof.

Step 2. By applying the above construction in each Ω_i^n (with $\eta = 2^{-(n+i+1)}\delta$) we construct the sequence f_n such that $f_n(x) = Fx$ on $\partial\Omega$,

$$\operatorname{dist}(\nabla f_n(x), E_{\{k,-k\}}) < 2^{-n}\tau(|\nabla f_n(x)|) \text{ a.e. } x \in \Omega,$$

and

$$[f_{n+1} - f_n]_{C^\alpha(\overline{\Omega})} < 2^{-(n+1)}\delta, \tag{3.51}$$

$$\int_\Omega |\nabla f_{n+1} - \nabla f_n|dx < c_K 2^{-n}. \tag{3.52}$$

Thus the sequence converges strongly to some limit f in $W^{1,1}(\Omega)$ and $C^\alpha(\overline{\Omega})$, and this limit satisfies $f(x) = Fx$ on $\partial\Omega$, $[f - F]_{C^\alpha} < \delta$, and $\nabla f(x) \in E_{\{k,-k\}}$ almost everywhere in Ω. To conclude with the proof of the theorem, we need to provide estimates from above and below for the distribution function of the gradient ∇f.

To get an estimate from above, let $\tilde{\Omega}_i^n$ denote the open subset corresponding to Ω_i^n in (e) of the claim. For any $t > 1$ we have

$$
\begin{aligned}
|\{x \in \Omega : |\nabla f_{n+1}(x)| > t\}| &= \sum_{i=1}^{\infty} \left|\left\{x \in \Omega_i^n : |\nabla f_{n+1}(x)| > t\right\}\right| \\
&= \sum_{i=1}^{\infty} \left|\left\{x \in \tilde{\Omega}_i^n : |\nabla f_{n+1}(x)| > t\right\}\right| \\
&\quad + \left|\left\{x \in \Omega_i^n \setminus \tilde{\Omega}_i^n : |\nabla f_{n+1}(x)| > t\right\}\right|,
\end{aligned}
$$

which, by (e_1) and (e_2), is bounded by

$$
\begin{aligned}
&\le c_K \sum_{i=1}^{\infty} 2^{-n} \left|\Omega_i^n\right| t^{-\frac{2K}{K+1}} + \left|\left\{x \in \Omega : |\nabla f_n(x)| > t - c_K 2^{-n}\right\}\right| \\
&\le c_K 2^{-n} |\Omega| t^{-\frac{2K}{K+1}} + \left|\left\{x \in \Omega : |\nabla f_n(x)| > t - c_K 2^{-n}\right\}\right|.
\end{aligned}
$$

Since $f_0(x) = Fx$ with $|F| < 1$, we deduce that for any $t > 2c_K + 1$ and any $n \in \mathbb{N}$

$$
|\{x \in \Omega : |\nabla f_n(x)| > t\}| \le c_K |\Omega| t^{-\frac{2K}{K+1}}, \tag{3.53}
$$

which yields the upper bound in (3.41).

For the estimate from below, let $B \subset \Omega$ be an open ball. For large enough $n_0 \in \mathbb{N}$ there exists i such that $\Omega_i^{n_0} \subset B$ and $\nabla f_{n_0} = A_i^{n_0}$ with $|A_i^{n_0}| \ge c_K$. By (e_2) of the claim there exists a constant $\tilde{\delta} > 0$ (depending on n_0) such that

$$
\left|\left\{x \in \tilde{\Omega}_i^{n_0} : |\nabla f_{n_0+1}(x)| > t\right\}\right| > \tilde{\delta} t^{-\frac{2K}{K+1}} \quad \text{for all } t > 1.
$$

Recall that f_{n_0+2} was obtained by applying Step 1 in each subdomain of $\tilde{\Omega}_i^{n_0}$ where f_{n_0+1} is affine. Since $\left\{x \in \tilde{\Omega}_i^{n_0} : |\nabla f_{n_0+1}(x)| > t\right\}$ can be decomposed into a union of such subdomains, (e_1) and (e_3) yield that

$$
\begin{aligned}
&\left|\left\{x \in \tilde{\Omega}_i^{n_0} : |\nabla f_{n_0+2}(x)| > t\right\}\right| \\
&\qquad \ge (1 - 2^{-n_0-1}) \left|\left\{x \in \tilde{\Omega}_i^{n_0} : |\nabla f_{n_0+1}(x)| > t + 2^{-n_0-1} c_K\right\}\right|
\end{aligned}
$$

Iterating we obtain that

$$
\begin{aligned}
|\{x \in B : |\nabla f_n(x)| > t\}| &\geq \left|\left\{x \in \tilde{\Omega}_i^{n_0} : |\nabla f_n(x)| > t\right\}\right| \\
&\geq \prod_{j=n_0+2}^{n} (1-2^{-j}) \left|\left\{x \in \tilde{\Omega}_i^{n_0} : |\nabla f_{n_0+1}(x)| > t + c_K\right\}\right| \\
&> c\left|\left\{x \in \tilde{\Omega}_i^{n_0} :\ |\nabla f_{n_0+1}(x)| > 2t\right\}\right| \\
&> c\tilde{\delta}\, t^{-\frac{2K}{K+1}}
\end{aligned}
\tag{3.54}
$$

for all $t > c_K$, where $c = \prod_{j=1}^{\infty}(1-2^{-j}) > 0$. In turn (3.41) follows from (3.53) and (3.54) using the strong $W^{1,1}$ convergence $f_n \to f$. This concludes the proof of the theorem. □

Remark 3.19. Theorem 3.18 shows that there exist distributional solutions of equations of the form

$$\operatorname{div} \sigma\nabla u = 0, \ \text{ with } \sigma \in L^{\infty}\left(\Omega, \left\{K, \frac{1}{K}\right\}\right),$$

with $u \in C^{\alpha}(\overline{\Omega}) \cap W^{1,1}(\Omega)$ for which

$$|\{x \in B :\ |\nabla u(x)| > t\}| \sim t^{-\frac{2K}{K+1}}$$

for any ball $B \subset \Omega$. By appropriate modifications (using the laminates from Proposition 3.10 instead of Lemma 3.16) the same techniques can be used to show the existence of weak solutions $u \in W^{1,2}(\Omega)$ with

$$|\{x \in B :\ |\nabla u(x)| > t\}| \sim t^{-\frac{2K}{K-1}}$$

for any ball $B \subset \Omega$. We also remark that the positive result in Theorem 1.1 in fact holds in the stronger form that any weak solution with $\nabla u \in L^{\frac{2K}{K+1}}(\Omega)$ in fact satisfies $\nabla u \in L^{\frac{2K}{K-1}}_{\text{weak}}(\Omega)$ (see [1]).

Remark 3.20. In the papers [42, 43] B. Yan constructed very weak quasiregular mappings $f : \Omega \to \mathbb{R}^n$ for $n \geq 3$ such that $\nabla f(x)$ satisfies that $\|\nabla f\|^n = \rho \det \nabla f$ where $\rho(x) \in \{1, K\}$ and $f - Ax \in W_0^{1,p}(\Omega)$ for $p < \frac{nK}{K+1}$ where A is any $n \times n$ matrix. A question raised in [42] is whether such mappings exist fulfilling the more demanding condition

$$\|\nabla f\|^n = K \det \nabla f \ \text{ a.e.} \tag{3.55}$$

For $n = 2$ Theorem 3.18 answers this in positive and in fact the control on the range of the gradient is substantially more precise than (3.55). We also remark that although Yan also used convex integration to obtain very weak quasiregular mappings, he used radial stretchings as the basic building blocks (see [42, Proposition 4.2]), leading to mappings which are unbounded. In contrast our construction yields examples in Hölder spaces. It is an interesting question what happens in higher dimensions (see [17]).

Remark 3.21. It can be easily seen that by minor modifications Theorems 3.13 and 3.18 yield very weak solutions with the same properties to the classical Beltrami equation. We just need to replace the definition of E_Δ with

$$\{A = (a_+, a_-) : a_- = \mu a_+ \text{ for some } \mu \in \Delta\}$$

and observe that the geometric properties necessary for the proof, Lemmas 3.14, 3.15 and 3.16 still hold.

Remark 3.22. Very weak solutions which fail to be solutions are really false solutions in the sense that they do not enjoy any of the special properties of honest weak solutions, like openness and discreteness, unique continuation, maximum principles and so forth. The investigation of this type of pathological solutions to elliptic equations started with the classical example by Serrin [37], see also [20] for the concept of weak minimizer. Other types of very weak quasiregular mappings can be found in [19, Theorem 6.5.1,Theorem 11.6.1]. It is interesting to note however, that our mappings are Hölder continuous for any exponent $0 < \alpha < 1$. A different type of Hölder continuous very weak quasiregular mapping has been constructed by Jan Maly, [25], using radial functions.

Remark 3.23. We conclude the section by discussing why we were not able to use the Baire Category argument as for the upper exponent. The Baire category argument for the upper exponent was based in finding a set $\mathcal{U} \in \mathbb{R}^{2\times 2}$ (the quasiconvex hull of E) which would contain the range of gradients of weak limits in $W^{1,2}$ of exact solutions subject to fixed boundary datum. In turn, the set X of Theorem 1.2 is the set of solutions to the "relaxed problem" $\nabla f \in \mathcal{U}$, again subject to fixed boundary datum (*cf.* Definition 3.4). The crucial information was that with this definition the set X is bounded in $W^{1,2}$, and therefore the weak topology is metrizable. Therefore we could recover exact solutions as points of continuity of the gradient in X. To repeat the argument for the lower exponent we would like to do the same in the $W^{1,q}$ topology where $q < \frac{2K}{K+1}$. However, as Theorem 3.18 shows, for any $F \in \mathbb{R}^{2\times 2}$ we can find a sequence f_j such that $Df_j(x) \in E_{\{k,-k\}}$ a.e in Ω and Df_j converge to F weakly in $W^{1,q}$ for every $q < \frac{2K}{K+1}$. Thus, the corresponding hull $\mathcal{U}$ would be the entire $\mathbb{R}^{2\times 2}$, and hence X would have to be the entire $W^{1,q}$. A way to go around this problem is to work with subsets of $E_{\{k,-k\}}$, which still support the appropriate staircase laminate but have smaller quasiconvex hulls (i.e associated sets $\mathcal{U}$). Unfortunately, it seems that these sets $\mathcal{U}$ must still contain rank-one half-lines, which prevents us from having a bound for X in any $W^{1,q}$ for $q > 1$.

4. Equations in non-divergence form

We follow here basically the same lines of reasoning as in the case of isotropic equations, since from the point of view of differential inclusions the structure of both problems is very similar. The entire construction lies in the set of 2 by 2 symmetric matrices $\mathbb{R}^{2\times 2}_{\mathrm{sym}} \sim \mathbb{R}^3$. Also, note that

$$A(v) = a_+ v + a_- \overline{v}$$

is symmetric if and only if $a_+ \in \mathbb{R}$.

For a set $\Delta \subset \mathbb{C} \cup \{\infty\}$ we use the notation

$$\mathbb{E}_\Delta = \left\{ B = (b_+, b_-) : \ 2b_+ = \mu b_- + \overline{\mu b_-} \text{ for some } \mu \in \Delta \right\} \subset \mathbb{R}^{2\times 2}_{\mathrm{sym}}. \tag{4.1}$$

Then $\mathbb{E}_0$ consists of anticonformal matrices and $\mathbb{E}_\infty$ is the space of symmetric conformal matrices, that is, the one dimensional real subspace spanned by the identity. Observe that compared with the case of equations in divergence-form, the roles of conformal and anticonformal matrices are now reversed.

We start by reinterpreting the equation as a differential inclusion. Recall that for $A, B \in \mathbb{R}^{2\times 2}_{\mathrm{sym}}$, (2.1) implies

$$\mathrm{Tr}(AB) = 2\Re(a_+ b_+ + a_- \overline{b}_-) = 2a_+ b_+ + (a_- \overline{b}_- + \overline{a}_- b_-). \tag{4.2}$$

Lemma 4.1. *Let $K \geq 1$ and $k = \frac{K-1}{K+1}$ and $1 \leq p \leq \infty$. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain. Then $u \in W^{2,p}(\Omega, \mathbb{R})$ is a solution to*

$$\mathrm{Tr}\left(A(x) D^2 u(x)\right) = 0 \textit{ in } \Omega \tag{4.3}$$

for some $A \in L^\infty(\Omega, \mathbb{R}^{2\times 2}_{\mathrm{sym}})$ with

$$A(x) \in \left\{ \begin{pmatrix} \frac{1}{\sqrt{K}} & 0 \\ 0 & \sqrt{K} \end{pmatrix}, \begin{pmatrix} \sqrt{K} & 0 \\ 0 & \frac{1}{\sqrt{K}} \end{pmatrix} \right\} \textit{ a.e. } x \in \Omega \tag{4.4}$$

if and only if

$$D^2 u(x) \in \mathbb{E}_{\{k,-k\}} \textit{ a.e. } x \in \Omega.$$

Proof. Suppose first $A = (a_+, a_-)$, $B = (b_+, b_-) \in \mathbb{R}^{2\times 2}_{\mathrm{sym}}$ with A positive definite. Set $\mu_A = \overline{a_-}/a_+$. Using (4.2) we may write the equation

$$\mathrm{Tr}(AB) = 0 \tag{4.5}$$

in conformal coordinates and obtain $b_+ = -\Re(\mu_A b_-)$ or, equivalently,

$$B \in \mathbb{E}_{\{-\mu_A\}}. \tag{4.6}$$

Hence u solves (4.3) for a general $A(x)$ if and only if $D^2 u(x) \in \mathbb{E}_{\{-\mu_{A(x)}\}}$ for almost every $x \in \Omega$. On the other hand, (4.4) is equivalent to

$$\mu_{A(x)} = \pm \frac{K-1}{K+1} = \pm k$$

and this observation completes the proof. □

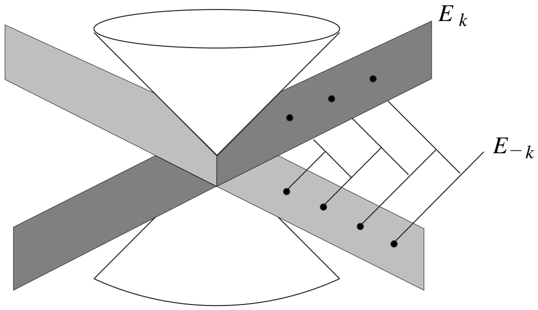

Figure 4.1. The rank-one cone in $\mathbb{R}^{2\times 2}_{\text{sym}}$, as axis the a_+-axis, and the symmetric strong staircase (*cf.* Figure 3.2).

4.1. Upper exponents

As in the isotropic case, also for Theorem 1.6 the first step is to define the appropriate complete metric space.

Definition 4.2. Let

$$\mathcal{U} = \left\{ A \in \mathbb{R}^{2\times 2}_{\text{sym}} : |a_+| < k|\Re a_-| \right\},$$

and let X be the closure in the weak topology of $W^{2,2}$ of the set

$$X_0 = \left\{ u \in W^{2,\infty}(\overline{\Omega}, \mathbb{R}^2) : \begin{array}{l} \bullet\, u \text{ piecewise quadratic} \\ \bullet\, D^2 u(x) \in \mathcal{U} \text{ a.e.} \\ \bullet\, u(x) = \frac{|x|^2}{2} \text{ on } \partial\Omega \\ \bullet\, \nabla u(x) = x \text{ on } \partial\Omega \end{array} \right\}.$$

Lemma 4.3. *With the above definitions,*

$$\mathbb{E}^{lc,1}_{\{k,-k\}} = \mathbb{E}^{pc}_{\{k,-k\}} = \overline{\mathcal{U}}$$

Proof. First we prove that $\mathbb{E}^{pc}_{\{k,-k\}} \subset \overline{\mathcal{U}}$. Let

$$f(A) = |a_+|^2 - k^2|\Re a_-|^2,$$

where $\Re z = (z + \bar{z})/2$ and $\Im z = (z - \bar{z})/(2i)$. Then for all $A \in \mathbb{R}^{2\times 2}_{\text{sym}}$

$$A \in \overline{\mathcal{U}} \text{ if and only if } f(A) \leq 0.$$

Let $A \in \mathbb{E}^{pc}_{\{k,-k\}}$, so that by Definition 2.4 there exists a probability measure ν with $\operatorname{spt} \nu \subset \mathbb{E}_{\{k,-k\}}$, $\overline{\nu} = A$, and $\det A = \int \det d\nu$. Note that f can be written as[3]

$$\begin{aligned} f(A) &= |a_+|^2 - |a_-|^2 + |\Im a_-|^2 + (1-k^2)|\Re a_-|^2 \\ &= \det A + |\Im a_-|^2 + (1-k^2)|\Re a_-|^2, \end{aligned}$$

[3] In technical terms f is polyconvex, see [28].

so that from Jensen's inequality we get

$$f(A) \le \int_{\mathbb{E}_{\{k,-k\}}} f(B)\, d\nu(B).$$

Since $b_+ = \pm k\Re b_-$ for every $B \in \mathbb{E}_{\{k,-k\}}$ we obtain $f(A) \le 0$ and hence $A \in \overline{\mathcal{U}}$.

On the other hand $\overline{\mathcal{U}} \subset \mathbb{E}^{lc,1}_{\{k,-k\}}$. To see this let $A \in \mathcal{U}$ so that $a_+ \in \mathbb{R}$ and $|a_+| < k|\Re a_-|$, and consider the rank-one line $l(t) = (a_+ + t, a_- + ti)$. Since $\Re l_-(t) = \Re a_-$ for all $t \in \mathbb{R}$, it is clear that there exist $t_1 < 0 < t_2$ such that $|l_+(t_j)| = k|\Re l_-(t_j)|$, and hence $l(t_1), l(t_2) \in \mathbb{E}_{\{k,-k\}}$. This means that $\mathcal{U} \subset \mathbb{E}^{lc,1}_{\{k,-k\}}$.

The proof is completed by observing that $\partial\mathcal{U} = \mathbb{E}_{\{k,-k\}}$ and that $\mathbb{E}^{lc,1}_{\{k,-k\}} \subset \mathbb{E}^{pc}_{\{k,-k\}}$. □

Remark 4.4. The above lemma is equivalent to characterizing the G-closure for equations of the type

$$2\partial_z f = \mu\partial_{\bar z} f + \overline{\mu\partial_{\bar z} f}, \tag{4.7}$$

where $\mu \in \{k, -k\}$.

Now we can repeat the arguments in Section 3 almost word for word. The only difference is that we need to use the part (ii) of Lemma 2.1 to stay in symmetric matrices.

Lemma 4.5. *The space* (X, w) *from Definition* 4.2 *is metrizable, with metric* d*, and for any* $f \in X$ *we have* $D^2u(x) \in \overline{\mathcal{U}}$ *a.e. in* Ω*. Furthermore the set of continuity points of the map* $D^2 : (X, w) \to L^2(\Omega, \mathbb{R}^{2\times 2}_{\mathrm{sym}})$ *is of second category in* (X, w).

Proof. We can use elliptic regularity here as well to obtain that X is metrizable. Indeed, by [2, Theorem 3.6][4] there exists an uniform constant $c = c(K, \Omega)$ such that

$$\int_\Omega |D^2u|^2 \le c$$

for every $u \in X$. Therefore the weak $W^{2,2}$ topology on X is metrizable.

Since $f_u = (u_x, -u_y)$ is K-quasiregular and affine on the boundary of Ω, we obtain that $Df_u \in W^{1,p}(\Omega)$ for $p > 2$. Thus we obtain continuity of the determinant with respect to the topology of X, and this implies, as in Lemma 3.7, that $D^2u(x) \in \mathcal{U}^{pc} = \overline{\mathcal{U}}$. The rest of the proof is exactly the same as in Lemma 3.7. □

Lemma 4.6. *The set of points of continuity in* (X, d) *of* D^2 *satisfy that* $D^2u(x) \in \mathbb{E}_{\{k,-k\}}$ *almost everywhere.*

[4] Alternatively this estimate can be obtained directly by noting that any $f \in X_0$ satisfies $|\triangle f| \le k|\partial_1^2 f - \partial_2^2 f|$ and using the Calderón-Zygmund theory of the Laplacian.

Proof. We can repeat line by line the proof of Lemma 3.8. The only difference is at (3.15) where we have to use part (ii) of Lemma 2.1 instead of part (i). □

Corollary 4.7. *The set of mappings in X such that $D^2u(x) \in \mathbb{E}_{\{k,-k\}}$ is of second category.*

4.1.1. *Laminates and Integrability*

Proposition 4.8. *Every $A \in \mathcal{U}$ is the center of mass of a sequence $\{\mu_n\}_{n=1}^{\infty}$ of laminates of finite order supported in $\mathcal{U}$ such that*

$$\lim_{n\to\infty} \int_{\mathbb{R}^{2\times 2}_{\mathrm{sym}}} |\lambda|^{\frac{2K}{K-1}} d\mu_n = \infty. \tag{4.8}$$

Proof. Recall that the laminates constructed in [15] are supported in the set $E_{\{k,-k\}} \cap \mathcal{D}$ where $\mathcal{D}$ denotes diagonal matrices, *cf.* the proof of Proposition 3.10. In other words we have a sequence of laminates ν_n satisfying $\overline{\nu}_n = I = (1, 0)$,

$$\operatorname{spt} \nu_n \subset \left\{ A \in \mathbb{R}^{2\times 2} : \, a_+, a_- \in \mathbb{R} \text{ and } a_- = \pm k a_+ \right\},$$

and

$$\lim_{n\to\infty} \int_{\mathbb{R}^{2\times 2}} |\lambda|^{\frac{2K}{K-1}} d\nu_n = \infty.$$

But then we just need to interchange the roles of a_+ and a_- to obtain the required laminates μ_n in the case the matrix $A = I$ (compare also Figures 3.2 and 4.1). More precisely, if T is the linear transformation $T((a_+, a_-)) = (a_-, a_+)$, then T preserves rank-one lines and hence $\mu_n = T_{\#}\nu_n$ are laminates such that $\overline{\mu}_n = (0, 1)$,

$$\operatorname{spt} \mu_n \subset \mathbb{E}_{\{k,-k\}},$$

and (4.8) holds.

For the general case note first that shifting the supports of ν_n by the matrix $R = (0, 1)$ keeps the support in $\mathcal{U}$. That is, the measures $\tilde{\nu_n}(\cdot) = \nu_n(\cdot + (0, 1))$ are supported in $\mathcal{U}$, satisfy (4.8) and have barycenter R.

Next, note that the set $\mathbb{E}_{\{k,-k\}}$ is invariant under multiplication with scalars and under addition of matrices with conformal coordinates $(0, ti)$ (with $t \in \mathbb{R}$). Using this invariance we obtain laminates with center of mass in the anticonformal plane $\mathbb{E}_0$. Finally, any $A \in \mathcal{U}$ is rank-one connected to $\mathbb{E}_0$ along translates of the rank one line $\{(t, t) : t \in \mathbb{R}\}$. Thus, we can argue as in the Proposition 3.10 to obtain laminates with center of mass $A \in \mathcal{U}$. □

Since the remaining arguments are exactly analogous to those in Section 3 we just quote the final outcome, proving Theorem 1.6.

Proposition 4.9. *The set of points in X such that $\int_{B(x,r)} |D^2u|^{\frac{2K}{K-1}} = \infty$ for all $B(x, r) \subset \Omega$ is second category in (X, d).*

Theorem 4.10. *Let $K > 1$ and $k = \frac{K-1}{K+1}$. For any bounded open set $\Omega \subset \mathbb{R}^2$ there exists a function $u \in W^{2,2}(\Omega; \mathbb{R}^2)$ with the following properties:*

(i) $u(x) = \frac{|x|^2}{2}$ *on* $\partial\Omega$,

(ii) $D^2u(x) \in \mathbb{E}_{\{k,-k\}}$ *a.e. in* Ω,

(iii) *for any ball* $B \subset \Omega$ *we have* $\int_B |D^2u(x)|^{\frac{2K}{K-1}} dx = \infty$.

4.2. Lower critical exponent

As in the case of the upper exponent the proofs here are very similar, we just indicate the main steps. First we need to deal with the geometry of rank-one connections to $\mathbb{E}_{\{k,-k\}}$.

Lemma 4.11. *For every $A \in \mathbb{R}^{2\times 2}_{\mathrm{sym}} \setminus \{0\}$ there exists $P \in E_\infty \setminus \{0\}$ and $Q \in \mathbb{E}_k \setminus \{0\}$ with rank $(P - Q) = 1$ such that $A \in [P, Q]$ and*

$$\frac{1}{c_K}|A| \leq |P - Q|, |P|, |Q| \leq c_K|A|. \tag{4.9}$$

The same holds if we replace $\mathbb{E}_k$ with $\mathbb{E}_{-k}$.

Proof. It suffices to prove the lemma for $\mathbb{E}_k$. Let $A = (a_+, a_-) \in \mathbb{R}^{2\times 2}_{\mathrm{sym}}$, and assume that $a_+ \geq k\Re a_-$. Then $P = (a_+ + |a_-|, 0) \in E_\infty$ and $\det(P - A) = 0$. Therefore the line

$$l(t) = P + t(-|a_-|, a_-)$$

is a rank-one line with $l(0) = P$, $l(1) = A$. Moreover

$$l_+(t) - k\Re l_-(t) = a_+ + (1-t)|a_-| - kt\Re a_- \geq 0$$

for $0 \leq t \leq 1$ and there exist $t_0 \geq 1$ so that $l_+(t_0) = k\Re l_-(t_0)$, in other words $l(t_0) \in \mathbb{E}_k$. Therefore we can take $Q = l(t_0)$.

If $a_+ \leq \Re a_-$, then we argue the same way, this time with $P = (a_+ - |a_-|, 0)$ and $l(t) = P + t(|a_-|, a_-)$.

The estimates (4.9) follow from Lemma 3.14 in the same way as in the proof of Lemma 3.15. □

Next, we find that laminates with the required integrability exists also in this setting. In fact the situation is simpler because the steps of the staircase consist of single (conformal) matrices this time (*cf.* (3.20)).

Lemma 4.12. *Let $A \in B_r(nI)$ for some $0 < r < 1/2$. There exists a laminate ν_A of third order with the following properties:*

- $\overline{\nu}_A = A$,
- $\mathrm{spt}\, \nu_A \subset \mathbb{E}_{\{k,-k\}} \cup \{(n+1)I\}$,

- $\operatorname{spt} \nu_A \subset \left\{\xi \in \mathbb{R}^{2\times 2} : c_K^{-1} n < |\xi| < c_K n\right\}$,
- $(1 - c_K \frac{r}{n})\beta_n \leq \nu_A\left(\{(n+1)I\}\right) \leq (1 + c_K \frac{r}{n})\beta_{n+2}$,

where

$$\beta_n = 1 - \frac{1+k}{n}, \tag{4.10}$$

and $c_K > 1$ is a constant only depending on K.

Proof. Let

$$C_k = \frac{1}{1+k}(k,1), \quad C_{-k} = \frac{1}{1+k}(k,-1) \tag{4.11}$$

in conformal coordinates. If $A = nI = (n,0)$, the claim follows by considering the laminate

$$\nu_A = \Big(\lambda_1 \delta_{nC_k} + (1-\lambda_1)\left(\lambda_2 \delta_{(n+1)C_{-k}} + (1-\lambda_2)\delta_{(n+1)I}\right)\Big),$$

with

$$\lambda_1 = \frac{1+k}{2n+1+k}, \tag{4.12}$$

$$\lambda_2 = \frac{(1+k)}{2(n+1)}. \tag{4.13}$$

It is not difficult to check that ν_A is indeed a laminate with center of mass $\overline{\nu}_A = nI$. Note again the similarity with the proof of Lemma 3.16 in the case $r = 0$. This is by definition a laminate, and

$$\nu_A\left(\{(n+1)I\}\right) = (1-\lambda_1)(1-\lambda_2) = \frac{n}{n+1}\frac{2n+1-k}{2n+1+k},$$

so that

$$\beta_n \leq \nu_A\left(\{(n+1)I\}\right) \leq \beta_{n+2}.$$

The argument for general $A \in B(nI, r)$ combines this with Lemma 4.11, just as in the case of isotropic equations in Lemma 3.16. □

Proposition 4.13 (The symmetric weak staircase). *Let $K > 1$ and $k = \frac{K-1}{K+1}$. Let $\alpha \in (0,1)$, $\delta > 0$ and $\tau : [0,\infty) \to (0,1]$ a continuous, non-increasing function with $\int_1^\infty \frac{\tau(t)}{t} dt < \infty$.*

There exists $\delta_0 > 0$ such that for any bounded open set $\Omega \subset \mathbb{R}^2$ and any nonzero matrix $F \in \mathbb{R}^{2\times 2}_{\text{sym}}$ with $\operatorname{dist}(F, \mathbb{E}_\infty) < \delta_0 |F|$ there exists a piecewise quadratic function $u \in W^{2,1}(\Omega, \mathbb{R}) \cap C^{1,\alpha}(\overline{\Omega}, \mathbb{R})$ with the following properties:

(i) $u(x) = \frac{1}{2}\langle Fx, x\rangle$ *on* $\partial\Omega$,

(ii) $[u - \frac{1}{2}\langle Fx, x\rangle]_{C^{1,\alpha}(\overline{\Omega})} < \delta$,

(iii) $\operatorname{dist}\left(D^2u(x), \mathbb{E}_{\{k,-k\}}\right) < \tau\left(|D^2u(x)|\right)$ *a.e. in* Ω,

and there exists a constant $c_{K,\tau} > 0$ *so that for all* $t > |F|$ *we have*

$$c_{K,\tau}^{-1}|F|^{\frac{2K}{K+1}}t^{-\frac{2K}{K+1}} < \frac{\left|\left\{x \in \Omega : |D^2u(x)| > t\right\}\right|}{|\Omega|} < c_{K,\tau}|F|^{\frac{2K}{K+1}}t^{-\frac{2K}{K+1}}. \tag{4.14}$$

Proof. The proof is a verbatim repetition of the proof of Proposition 3.17 for isotropic equations. The difference is that the step laminates are those from Lemma 4.12 and that since the laminates are supported in symmetric matrices we can approximate (in the sense of Proposition 2.3) the laminates by distributions of second derivatives of functions. □

Theorem 4.14. *Let* $K > 1$, $k = \frac{K-1}{K+1}$ *and let* F *be any symmetric* 2×2 *matrix. For any* $\alpha \in (0, 1)$, $\delta > 0$ *and for any bounded open set* $\Omega \subset \mathbb{R}^2$ *there exists a function* $u \in W^{2,1}(\Omega; \mathbb{R}) \cap C^\alpha(\overline{\Omega}; \mathbb{R})$ *with the following properties:*

(i) $u(x) = \frac{1}{2}\langle Fx, x\rangle$ *on* $\partial\Omega$,

(ii) $[u - \frac{1}{2}\langle Fx, x\rangle]_{C^{1,\alpha}(\overline{\Omega})} < \delta$,

(iii) $D^2u(x) \in \mathbb{E}_{\{k,-k\}}$ *a.e. in* Ω,

(iv) *for any ball* $B \subset \Omega$ *there exists a constant* $c_B > 1$ *such that*

$$\frac{1}{c_B}t^{-\frac{2K}{K+1}} < \left|\left\{x \in B : |D^2u(x)| > t\right\}\right| < c_B t^{-\frac{2K}{K+1}}$$

for all $t \geq 1$.

In particular $u \in W^{2,q}(\Omega)$ *for every* $q < \frac{2K}{K+1}$, *but for any ball* $B \subset \Omega$ *we have* $\int_B |D^2u(x)|^{\frac{2K}{K+1}}dx = \infty$.

Proof. The scheme of the proof for the corresponding theorem for isotropic equations, Theorem 3.18 can be followed line by line, replacing always Lemma 3.15 by Lemma 4.11 and part (i) of Lemma 2.1 by part (ii). □

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