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Sharp upper bounds for a singular perturbation problem related to micromagnetics

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Abstract. We construct an upper bound for the following family of functionals $\{E_\varepsilon\}_{\varepsilon>0}$, which arises in the study of micromagnetics:

$$E_\varepsilon(u) = \int_\Omega \varepsilon|\nabla u|^2 + \frac{1}{\varepsilon}\int_{\mathbb{R}^2} |H_u|^2.$$

Here Ω is a bounded domain in $\mathbb{R}^2$, $u \in H^1(\Omega, S^1)$ (corresponding to the magnetization) and H_u, the demagnetizing field created by u, is given by

$$\begin{cases} \mathrm{div}\,(\tilde{u} + H_u) = 0 & \text{in } \mathbb{R}^2\,, \\ \mathrm{curl}\, H_u = 0 & \text{in } \mathbb{R}^2\,, \end{cases}$$

where $\tilde{u}$ is the extension of u by 0 in $\mathbb{R}^2 \setminus \Omega$. Our upper bound coincides with the lower bound obtained by Rivière and Serfaty.

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1. Introduction

In this paper we study the following energy-functional, related to micromagnetics:

$$E_\varepsilon(u) := \int_\Omega \varepsilon|\nabla u|^2 + \frac{1}{\varepsilon}\int_{\mathbb{R}^2} |H_u|^2. \tag{1.1}$$

Here Ω is a bounded domain in $\mathbb{R}^2$ with Lipschitz boundary, u is a unit-valued vector-field (corresponding to the magnetization) in $H^1(\Omega, S^1)$ and H_u, the demagnetizing field created by u, is given by

$$\begin{cases} \mathrm{div}(\tilde{u} + H_u) = 0 & \text{in } \mathbb{R}^2 \\ \mathrm{curl}\, H_u = 0 & \text{in } \mathbb{R}^2\,, \end{cases} \tag{1.2}$$

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where $\tilde{u}$ is the extension of u by 0 in $\mathbb{R}^2 \setminus \Omega$. For the physical models related to E_ε, we refer to [18] and all the references therein.

We can rewrite (1.1) in the following form. Denoting by $\Delta^{-1}\tilde{u}$ the Newtonian potential of $\tilde{u}$, we observe that the vector-field $\bar{H}_u := -\nabla\big(\operatorname{div}(\Delta^{-1}\tilde{u})\big)$ belongs to $L^2(\mathbb{R}^2, \mathbb{R}^2)$. Moreover,

$$\begin{cases} \operatorname{div} \bar{H}_u = -\operatorname{div} \tilde{u} & \text{in } \mathbb{R}^2 \\ \operatorname{curl} \bar{H}_u = 0 & \text{in } \mathbb{R}^2 . \end{cases}$$

So $H_u = \bar{H}_u$ and we obtain

$$E_\varepsilon(u) = \int_\Omega \varepsilon|\nabla u|^2 + \frac{1}{\varepsilon}\int_{\mathbb{R}^2} \big|\nabla\big(\operatorname{div}(\Delta^{-1}\tilde{u})\big)\big|^2. \tag{1.3}$$

In [19] T. Rivière and S. Serfaty proved the following theorem, giving compactness and a lower bound for the energies E_ε.

Theorem 1.1. *Let Ω be a bounded simply connected domain in $\mathbb{R}^2$. Let $\varepsilon_n \to 0$ and $u_n \in H^1(\Omega, S^1)$ with a lifting $\varphi_n \in H^1(\Omega, \mathbb{R})$ i.e., $u_n = e^{i\varphi_n}$ a.e., and such that*

$$E_{\varepsilon_n}(u_n) \leq C \tag{1.4}$$

$$\|\varphi_n\|_{L^\infty(\Omega)} \leq N . \tag{1.5}$$

Then, up to extraction of a subsequence, there exists u and φ in $\cap_{q<\infty} L^q(\Omega)$ such that

$$\begin{aligned} &\varphi_n \to \varphi \text{ in } \cap_{q<\infty} L^q(\Omega) \\ &u_n \to u \text{ in } \cap_{q<\infty} L^q(\Omega) . \end{aligned}$$

Moreover, if we consider

$$\begin{cases} T^t\varphi(x) := \inf\big(\varphi(x), t\big) \\ T^t u(x) := e^{i\,T^t\varphi(x)} , \end{cases}$$

then $\operatorname{div}_x T^t u$ *is a bounded Radon measure on $\Omega\times\mathbb{R}$, with $t \mapsto$* $\operatorname{div}_x T^t u$ *continuous from $\mathbb{R}$ to $\mathcal{D}'(\Omega)$. In addition*

$$2\int_{\mathbb{R}}\int_\Omega \big|\operatorname{div}_x T^t u\big|\,dx\,dt \leq \varliminf_{n\to\infty} \int_\Omega 2|\nabla\varphi_n \cdot H_{u_n}| \leq \varliminf_{n\to\infty} E_{\varepsilon_n}(u_n) < \infty .$$

The main contribution of this paper is to establish the upper bound for E_ε in the case where u and its lifting φ belong to BV. First of all we observe that if $\varepsilon_n \to 0$,

$E_{\varepsilon_n}(u_n) \le C$, and $u_n \to u$ in L^q, where $|u_n| = 1$ and $u \in BV$ then clearly $\lim_{n\to\infty} \int_{\mathbb{R}^2} |H_{u_n}|^2 = 0$, which implies

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} \tilde{u}_n \cdot \nabla\delta = -\lim_{n\to\infty} \int_{\mathbb{R}^2} H_{u_n} \cdot \nabla\delta = 0 \qquad \forall \delta \in C_c^\infty(\mathbb{R}^2, \mathbb{R}) .$$

Therefore, div $\tilde{u} = 0$ as a distribution, *i.e.*,

$$\begin{cases} |u| = 1 & \text{a.e. in } \Omega\,, \\ \operatorname{div} u = 0 & \text{in } \Omega\,, \\ u \cdot \boldsymbol{n} = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{1.6}$$

The main result of this paper is the following theorem.

Theorem 1.2. *Let Ω be a bounded domain in $\mathbb{R}^2$ with Lipschitz boundary. Consider $u \in BV(\Omega, S^1)$, satisfying* div $u = 0$ *in Ω and $u \cdot \boldsymbol{n} = 0$ on $\partial\Omega$ and assume there exist $\varphi \in BV(\Omega, \mathbb{R})$, such that $u = e^{i\varphi}$ a e. in Ω. Then there exists a family of functions $\{v_\varepsilon\} \subset C^2(\mathbb{R}^N, \mathbb{R})$ satisfying*

$$\lim_{\varepsilon\to 0^+} v_\varepsilon(x) = \varphi(x) \quad \textit{in} \quad L^1(\Omega, \mathbb{R})$$

and

$$\lim_{\varepsilon\to 0} E_\varepsilon(e^{iv_\varepsilon}) = 2 \int_{\mathbb{R}} \int_\Omega \left|\operatorname{div}_x T^t u\right| dx\, dt\,.$$

Moreover, if $\varphi \in BV(\Omega, \mathbb{R}) \cap L^\infty$, then we have

$$\lim_{\varepsilon\to 0^+} v_\varepsilon(x) = \varphi(x) \quad \textit{in} \quad L^p(\Omega, \mathbb{R}) \quad \forall p \in [1, \infty)\,.$$

In order to construct $\{v_\varepsilon\}$ we take the convolution of φ with a varying smoothing kernel, *i.e.*, we set $v_\varepsilon(x) := \varepsilon^{-2} \int_{\mathbb{R}^2} \eta\big(\frac{y-x}{\varepsilon}, x\big)\varphi(y)dy$ where $\eta \in C_c^2(\mathbb{R}^2 \times \mathbb{R}^2)$ satisfies $\int_{\mathbb{R}^2} \eta(z, x)dx = 1$ for every $x \in \Omega$, and we optimize the choice of the kernel η. A similar approach was used in [16] and [17], but a new ingredient is required here, since the non-local term $\int_{\mathbb{R}^2} |H_u|^2$ gives certain difficulties.

1.1. The basic idea

We shall follow essentially the strategy of [16] and [17]. The main new ingredient here is the calculation of

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon} \int_{\mathbb{R}^2} \left|\nabla\big(\operatorname{div}(\Delta^{-1}\{\chi_\Omega e^{iv_\varepsilon}\})\big)\right|^2 . \tag{1.7}$$

We first calculate

$$L(l) := \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon} \int_{\mathbb{R}^2} \left|\nabla\big(\operatorname{div}(\Delta^{-1}\{\varphi_\varepsilon\})\big)\right|^2 ,$$

where $\varphi_\varepsilon(x) := \varepsilon^{-2}\int_{\mathbb{R}^2} l\big(\frac{y-x}{\varepsilon}, x\big)\varphi(y)dy$, with $l \in C_c^2(\mathbb{R}^2 \times \Omega, \mathbb{R}^2)$ satisfying $\int_{\mathbb{R}^2} l(z,x)dx = 0$ for every $x \in \Omega$. Since $\nabla\big(\mathrm{div}(\Delta^{-1}\{\phi_\varepsilon\})\big)$ has the same asymptotic behavior as $\varepsilon^{-2}\int_{\mathbb{R}^2} s\big(\frac{y-x}{\varepsilon}, x\big)\varphi(y)dy$, where $s(z,x) := \nabla_z\big(\mathrm{div}_z(\Delta_z^{-1} l(z,x))\big)$, we can calculate the limit $L(l)$ in a similar way to what was done in [16] and [17] (see Lemmas 3.1 and 3.2 for the details). Using the results of [17] (see Proposition 2.2 below) it is easy to calculate

$$D(l) := \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_\Omega \left|e^{iv_\varepsilon} - \varphi_\varepsilon - u\right|^2 dx\,.$$

Now, given a fixed η and a small $\delta > 0$, we choose $l = l_\delta$ in such a way that $D(l_\delta) < \delta$. Then, using the estimate

$$\int_{\mathbb{R}^2}\left|\nabla\big(\mathrm{div}(\Delta^{-1}\{\chi_\Omega f\})\big)\right|^2 \le C\int_\Omega |f|^2\,,$$

we deduce that

$$\begin{aligned}&\overline{\lim_{\varepsilon\to 0}}\left|\frac{1}{\varepsilon}\int_{\mathbb{R}^2}\left|\nabla\big(\mathrm{div}(\Delta^{-1}\{\chi_\Omega e^{iv_\varepsilon}\})\big)\right|^2 - \frac{1}{\varepsilon}\int_{\mathbb{R}^2}\left|\nabla\big(\mathrm{div}(\Delta^{-1}\{\chi_\Omega \varphi_\varepsilon\})\big)\right|^2\right|\\ &\le \overline{\lim_{\varepsilon\to 0}}\, C\left\{\frac{1}{\varepsilon}\int_{\mathbb{R}^2}\left|\nabla\big(\mathrm{div}(\Delta^{-1}\{\chi_\Omega(e^{iv_\varepsilon} - \varphi_\varepsilon - u)\})\big)\right|^2\right\}^{1/2} \le C\delta^{1/2}\,.\end{aligned}$$

Finally, tetting δ tend to 0, we conclude that the limit in (1.7) should be equal to $\lim_{\delta\to 0} L(l_\delta)$. We follow basically this strategy in the proof of Proposition 4.1.

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2. Preliminaries

Throughout this section we assume that Ω is a bounded domain in $\mathbb{R}^2$ with Lipschitz boundary. We begin by introducing some notation. For every $\boldsymbol{\nu} \in S^1$ (the unit sphere in $\mathbb{R}^2$) and $R > 0$ we denote

$$B_R^+(x,\boldsymbol{\nu}) = \{y \in \mathbb{R}^2 \,:\, |y-x| < R,\ (y-x)\cdot\boldsymbol{\nu} > 0\}\,, \tag{2.1}$$

$$B_R^-(x,\boldsymbol{\nu}) = \{y \in \mathbb{R}^2 : |y-x| < R,\ (y-x)\cdot\boldsymbol{\nu} < 0\}\,, \tag{2.2}$$

$$H_+(x,\boldsymbol{\nu}) = \{y \in \mathbb{R}^2 : (y-x)\cdot\boldsymbol{\nu} > 0\}\,, \tag{2.3}$$

$$H_-(x,\boldsymbol{\nu}) = \{y \in \mathbb{R}^2 : (y-x)\cdot\boldsymbol{\nu} < 0\} \tag{2.4}$$

and

$$H_{\boldsymbol{\nu}}^0 = \{y \in \mathbb{R}^2 \,:\, y \cdot \boldsymbol{\nu} = 0\}\,. \tag{2.5}$$

Definition 2.1. Consider a function $f \in BV(\Omega, \mathbb{R}^m)$ and a point $x \in \Omega$.

i) We say that x is a point of *approximate continuity* of f if there exists $z \in \mathbb{R}^m$ such that

$$\lim_{\rho \to 0^+} \frac{\int_{B_\rho(x)} |f(y) - z|\, dy}{\mathcal{L}^2\big(B_\rho(x)\big)} = 0\,.$$

In this case z is called an *approximate limit* of f at x and we denote z by $\tilde{f}(x)$. The set of points of approximate continuity of f is denoted by G_f.

ii) We say that x is an *approximate jump point* of f if there exist $a, b \in \mathbb{R}^m$ and $\boldsymbol{\nu} \in S^{N-1}$ such that $a \neq b$ and

$$\lim_{\rho \to 0^+} \frac{\int_{B_\rho^+(x,\boldsymbol{\nu})} |f(y) - a|\, dy}{\mathcal{L}^2\big(B_\rho(x)\big)} = 0, \qquad \lim_{\rho \to 0^+} \frac{\int_{B_\rho^-(x,\boldsymbol{\nu})} |f(y) - b|\, dy}{\mathcal{L}^2\big(B_\rho(x)\big)} = 0. \tag{2.6}$$

The triple $(a, b, \boldsymbol{\nu})$, uniquely determined by (2.6) up to a permutation of (a, b) and a change of sign of $\boldsymbol{\nu}$, is denoted by $(f^+(x), f^-(x), \boldsymbol{\nu}_f(x))$. We shall call $\boldsymbol{\nu}_f(x)$ the *approximate jump vector* and we shall sometimes write simply $\boldsymbol{\nu}(x)$ if the reference to the function f is clear. The set of approximate jump points is denoted by J_f. A choice of $\boldsymbol{\nu}(x)$ for every $x \in J_f$ (which is unique up to sign) determines an orientation of J_f. At a point of approximate continuity x, we shall use the convention $f^+(x) = f^-(x) = \tilde{f}(x)$.

We refer to [2] for the results on BV-functions that we shall use in the sequel.

Consider a function $\Phi = (\varphi_1, \varphi_2, \ldots, \varphi_d) \in BV(\Omega, \mathbb{R}^d)$. By [2, Proposition 3.21] we may extend Φ to a function $\bar{\Phi} \in BV(\mathbb{R}^2, \mathbb{R}^d)$, such that $\bar{\Phi} = \Phi$ a.e. in Ω, supp $\bar{\Phi}$ is compact and $\|D\bar{\Phi}\|(\partial\Omega) = 0$. From the proof of Proposition 3.21 in [2] it follows that if $\Phi \in BV(\Omega, \mathbb{R}^d) \cap L^\infty$ then its extension $\bar{\Phi}$ is also in $BV(\mathbb{R}^2, \mathbb{R}^d) \cap L^\infty$. Consider also a matrix valued function $\Xi \in C_c^2(\mathbb{R}^2 \times \mathbb{R}^2, \mathbb{R}^{l \times d})$ For every $\varepsilon > 0$ define a function $\Psi_\varepsilon(x) : \mathbb{R}^2 \to \mathbb{R}^l$ by

$$\begin{aligned} \Psi_\varepsilon(x) &:= \frac{1}{\varepsilon^2} \int_{\mathbb{R}^2} \Xi\Big(\frac{y - x}{\varepsilon}, x\Big) \cdot \bar{\Phi}(y) dy \\ &= \int_{\mathbb{R}^2} \Xi(z, x) \cdot \bar{\Phi}(x + \varepsilon z)\, dz, \quad \forall x \in \mathbb{R}^2\,. \end{aligned} \tag{2.7}$$

Thanks to [17, Proposition 3.2], we have the following result. It generalizes Proposition 3.2 from [16] and provides the key tool for the calculation of the upper bound, both in [17] and in the current paper. In the proof of Lemma 3.2 we shall also follow the general strategy of its proof in [17].

Proposition 2.2. *Let $W \in C^1(\mathbb{R}^l \times \mathbb{R}^q, \mathbb{R})$ satisfying*

$$\nabla_a W(a,b) = 0 \ \text{ whenever } W(a,b) = 0\,. \tag{2.8}$$

Consider $\Phi \in BV(\Omega, \mathbb{R}^d) \cap L^\infty$ and $u \in BV(\Omega, \mathbb{R}^q) \cap L^\infty$ satisfying

$$W\bigg(\bigg\{\int_{\mathbb{R}^2} \Xi(z,x)\,dz\bigg\} \cdot \Phi(x),\ u(x)\bigg) = 0 \quad \text{for a.e. } x \in \Omega\,,$$

where $\Xi \in C_c^2(\mathbb{R}^2 \times \mathbb{R}^2, \mathbb{R}^{l\times d})$, as above. Let Ψ_ε be as in (2.7). *Then,*

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_\Omega \frac{1}{\varepsilon} W\big(\Psi_\varepsilon(x), u(x)\big)dx\\ &= \int_{J_\Phi}\bigg\{\int_{-\infty}^0 W\big(\Gamma(t,x), u^+(x)\big)dt + \int_0^{+\infty} W\big(\Gamma(t,x), u^-(x)\big)dt\bigg\}d\mathcal{H}^1(x)\,,\end{aligned} \tag{2.9}$$

where

$$\Gamma(t,x) = \bigg(\int_{-\infty}^t P(s,x)\,ds\bigg) \cdot \Phi^-(x) + \bigg(\int_t^{+\infty} P(s,x)\,ds\bigg) \cdot \Phi^+(x), \tag{2.10}$$

with

$$P(t,x) = \int_{H^0_{\boldsymbol{\nu}(x)}} \Xi(t\boldsymbol{\nu}(x) + y, x)\,d\mathcal{H}^1(y)\,, \tag{2.11}$$

$\boldsymbol{\nu}(x)$ is the jump vector of Φ and it is assumed that the orientation of J_u coincides with the orientation of J_Φ $\mathcal{H}^1$ a.e. on $J_u \cap J_\Phi$.

Definition 2.3. Given $f \in L^\infty(\mathbb{R}^2, \mathbb{R}^k)$ with compact support, we define its Newtonian potential

$$\big(\Delta^{-1} f\big)(x) := \int_{\mathbb{R}^2} \frac{1}{2\pi} \ln|x-y|\, f(y)\,dy\,.$$

Than it is well known that

$$\int_{\mathbb{R}^2} \big|\nabla^2\big(\Delta^{-1} f\big)(x)\big|^2\,dx = \int_{\mathbb{R}^2} |f(x)|^2\,dx\,, \tag{2.12}$$

where given $v = (v_1, \ldots, v_k) : \mathbb{R}^2 \to \mathbb{R}^k$ we denote by $\nabla^2 v \in \mathbb{R}^{k\times 2\times 2}$ the tensor with lij-th component $\partial^2_{ij} v_l$.

Definition 2.4. Let $\mathcal{V}$ be the class of all functions $\eta \in C_c^2(\mathbb{R}^2 \times \mathbb{R}^2, \mathbb{R})$ such that

$$\int_{\mathbb{R}^2} \eta(z,x)\,dz = 1 \quad \forall x \in \Omega\,. \tag{2.13}$$

Let $\mathcal{U}$ be the class of all functions $l(z,x) \in C_c^2(\mathbb{R}^2 \times \Omega, \mathbb{R}^2)$ such that

$$\int_{\mathbb{R}^2} l(z,x)\,dz = 0 \quad \forall x \in \mathbb{R}^2\,. \tag{2.14}$$

In [17, Lemma 5.1], we proved the following statement. This statement generalize Claim 3 of Lemma 3.4 from [16] and was an essential tool in the optimizing the upper bound in [17].

Lemma 2.5. *Let μ be positive finite Borel measure on Ω and $\boldsymbol{\nu}_0(x) : \Omega \to \mathbb{R}^2$ a Borel measurable function with $|\boldsymbol{\nu}_0| = 1$. Let $\mathcal{W}_1$ denote the set of functions $p(t,x) : \mathbb{R} \times \Omega \to \mathbb{R}$ satisfying the following conditions*:

i) *p is Borel measurable and bounded,*
ii) *there exists $M > 0$ such that $p(t,x) = 0$ for $|t| > M$ and any $x \in \Omega$,*
iii) $\int_{\mathbb{R}} p(t,x)\,dt = 1,\ \forall x \in \Omega.$

Then for every $p(t,x) \in \mathcal{W}_1$, there exists a sequence of functions $\{\eta_n\} \subset \mathcal{V}$ (see Definition 2.4*), such that the sequence of functions $\{p_n(t,x)\}$ defined on $\mathbb{R} \times \Omega$ by*

$$p_n(t,x) = \int_{H^0_{\boldsymbol{\nu}_0(x)}} \eta_n(t\boldsymbol{\nu}_0(x) + y, x)\,d\mathcal{H}^1(y),$$

has the following properties:

i) *there exists C_0 such that $\|p_n\|_{L^\infty} \le C_0$ for every n,*
ii) *there exist $M > 0$ such that $p_n(t,x) = 0$ for $|t| > M$ and every $x \in \Omega$, for all n.*
iii) $\lim_{n\to\infty} \int_\Omega \int_{\mathbb{R}} |p_n(t,x) - p(t,x)|\,dt\,d\mu(x) = 0.$

With the same method it is not difficult to prove

Lemma 2.6. *Let μ be positive finite Borel measure on Ω and $\boldsymbol{\nu}_0(x) : \Omega \to \mathbb{R}^2$ a Borel measurable function with $|\boldsymbol{\nu}_0| = 1$. Let $\mathcal{W}_0$ denote the set of functions $q(t,x) : \mathbb{R} \times \Omega \to \mathbb{R}^2$ satisfying the following conditions*:

i) *q is Borel measurable and bounded,*
ii) *there exists $M > 0$ such that $q(t,x) = 0$ for $|t| > M$ and every $x \in \Omega$.*
iii) $\int_{\mathbb{R}} q(t,x)\,dt = 0,\ \forall x \in \Omega.$

Then for every $q(t,x) \in \mathcal{W}_0$, there exists a sequence of functions $\{l_n\} \subset \mathcal{U}$ (see Definition 2.4*), such that the sequence of functions $\{q_n(t,x)\}$ defined on $\mathbb{R} \times \Omega$ by*

$$q_n(t,x) = \int_{H^0_{\boldsymbol{\nu}_0(x)}} l_n(t\boldsymbol{\nu}_0(x) + y, x)\,d\mathcal{H}^1(y),$$

has the following properties:

i) *there exists C_0 such that $\|q_n\|_{L^\infty} \le C_0$ for every n,*
ii) *there exist $M > 0$ such that $q_n(t,x) = 0$ for $|t| > M$ and every $x \in \Omega$, for all n,*
iii) $\lim_{n\to\infty} \int_\Omega \int_{\mathbb{R}} |q_n(t,x) - q(t,x)|\,dt\,d\mu(x) = 0.$

3. First estimates

Throughout this section we assume that Ω is a bounded domain in $\mathbb{R}^2$ with Lipschitz boundary.

Let $l \in \mathcal{U}$ (see Definition 2.4). Consider $r(z,x) := \Delta_z^{-1} l(z,x)$. Then $r \in C^2(\mathbb{R}^2 \times \mathbb{R}^2, \mathbb{R}^2)$ with $\operatorname{supp} r \subset \mathbb{R}^2 \times K$, where $K \Subset \Omega$. Moreover, since $\int_{\mathbb{R}^2} l(z,x)dz = 0$, for every $k = 0, 1, 2 \dots$ we have the estimates

$$\begin{aligned} |\nabla_x^k r(z,x)| &\le \frac{C_k}{|z|+1}, \\ \big|\nabla_x^k\big(\nabla_z r(z,x)\big)\big| &\le \frac{C_k}{|z|^2+1}, \\ \big|\nabla_x^k\big(\nabla_z^2 r(z,x)\big)\big| &\le \frac{C_k}{|z|^3+1}, \end{aligned} \tag{3.1}$$

where $C_k > 0$ does not depend on z and x.

Lemma 3.1. *Let $\varphi \in BV(\Omega,\mathbb{R}) \cap L^\infty$ and $l \in \mathcal{U}$ (see Definition 2.4). For every $\varepsilon > 0$ consider the function $\varphi_\varepsilon \in C^1(\mathbb{R}^2,\mathbb{R}^2)$ by*

$$\varphi_\varepsilon(x) := \frac{1}{\varepsilon^2}\int_{\mathbb{R}^2} l\Big(\frac{y-x}{\varepsilon}, x\Big)\bar\varphi(y)\,dy = \int_{\mathbb{R}^2} l(z,x)\bar\varphi(x+\varepsilon z)\,dz\,, \tag{3.2}$$

where $\bar\varphi$ is some bounded BV extension of φ to $\mathbb{R}^2$ with compact support. Next consider $r(z,x) := \Delta_z^{-1} l(z,x)$ and set

$$\begin{aligned} \xi_\varepsilon(x) &:= \frac{1}{\varepsilon^2}\int_{\mathbb{R}^2} \nabla_1\big(\operatorname{div}_1 r\big)\Big(\frac{y-x}{\varepsilon}, x\Big)\bar\varphi(y)\,dy \\ &= \int_{\mathbb{R}^2} \nabla_z\big(\operatorname{div}_z r(z,x)\big)\bar\varphi(x+\varepsilon z)\,dz\,, \end{aligned} \tag{3.3}$$

where $\nabla_1(\operatorname{div}_1 r)$ is the gradient of divergence of $r(z,x)$ in its first variable, namely z. Then,

$$\int_{\mathbb{R}^2} \frac{1}{\varepsilon}\Big|\nabla\big(\operatorname{div}(\Delta^{-1}\varphi_\varepsilon)\big)(x)\Big|^2 dx = o_\varepsilon(1) + \int_\Omega \frac{1}{\varepsilon}\,\varphi_\varepsilon(x)\cdot\xi_\varepsilon(x)\,dx\,. \tag{3.4}$$

Proof. Since $l(z,x) = 0$ if $x \notin K$, where K is some compact subset of Ω, we have, in particular, $\varphi_\varepsilon(x) = 0$ for every $x \in \mathbb{R}^2 \setminus \Omega$. Then, integrating by part two times, we conclude

$$\begin{aligned} &\int_{\mathbb{R}^2} \frac{1}{\varepsilon}\Big|\nabla\big(\operatorname{div}(\Delta^{-1}\varphi_\varepsilon)\big)(x)\Big|^2 dx \\ &= -\int_{\mathbb{R}^2} \frac{1}{\varepsilon}\,\Delta\big(\operatorname{div}(\Delta^{-1}\varphi_\varepsilon)\big)(x)\cdot\big(\operatorname{div}(\Delta^{-1}\varphi_\varepsilon)\big)(x)\,dx \\ &= -\int_{\mathbb{R}^2} \frac{1}{\varepsilon}\operatorname{div}\varphi_\varepsilon(x)\cdot\big(\operatorname{div}(\Delta^{-1}\varphi_\varepsilon)\big)(x)\,dx \\ &= \int_\Omega \frac{1}{\varepsilon}\,\varphi_\varepsilon(x)\cdot\nabla\big(\operatorname{div}(\Delta^{-1}\varphi_\varepsilon)\big)(x)\,dx\,. \end{aligned} \tag{3.5}$$

Next consider the function $\zeta_\varepsilon \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ given by

$$\zeta_\varepsilon(x) := \frac{1}{\varepsilon^2}\int_{\mathbb{R}^2} r\Big(\frac{y-x}{\varepsilon}, x\Big)\bar{\varphi}(y)\,dy = \int_{\mathbb{R}^2} r(z,x)\bar{\varphi}(x+\varepsilon z)\,dz\,. \tag{3.6}$$

We will prove now that

$$\Big|\varepsilon^2\nabla^2\zeta_\varepsilon(x) - \int_{\mathbb{R}^2}\nabla_z^2 r(z,x)\bar{\varphi}(x+\varepsilon z)\,dz\Big| \le C\varepsilon^{2/3} \quad \forall x \in \Omega\,. \tag{3.7}$$

We shall denote by $\nabla_1 l$ and $\nabla_2 l$ the gradient of $l(z,x)$ with respect to the variables z and x respectively. We have,

$$\begin{aligned}
&\varepsilon^2\nabla^2\zeta_\varepsilon(x) - \int_{\mathbb{R}^2}\nabla_z^2 r(z,x)\bar{\varphi}(x+\varepsilon z)\,dz\\
&= \int_{\mathbb{R}^2}\nabla_x^2 r\Big(\frac{y-x}{\varepsilon}, x\Big)\bar{\varphi}(y)\,dy - \frac{1}{\varepsilon^2}\int_{\mathbb{R}^2}\nabla_1^2 r\Big(\frac{y-x}{\varepsilon}, x\Big)\bar{\varphi}(y)\,dy\\
&= -\frac{1}{\varepsilon}\int_{\mathbb{R}^2}\Big\{\nabla_1\nabla_2 r\Big(\frac{y-x}{\varepsilon}, x\Big) + \nabla_2\nabla_1 r\Big(\frac{y-x}{\varepsilon}, x\Big)\Big\}\bar{\varphi}(y)\,dy\\
&\quad + \int_{\mathbb{R}^2}\nabla_2^2 r\Big(\frac{y-x}{\varepsilon}, x\Big)\bar{\varphi}(y)\,dy\,.
\end{aligned} \tag{3.8}$$

Therefore, by the Hölder inequality and the estimates in (3.1), we obtain

$$\begin{aligned}
&\Big|\varepsilon^2\nabla^2\zeta_\varepsilon(x) - \int_{\mathbb{R}^2}\nabla_z^2 r(z,x)\bar{\varphi}(x+\varepsilon z)\,dz\Big|\\
&\le \varepsilon^{2/3}\Big(\frac{1}{\varepsilon^2}\int_{\mathbb{R}^2}\Big|\nabla_1\nabla_2 r\Big(\frac{y-x}{\varepsilon}, x\Big) + \nabla_2\nabla_1 r\Big(\frac{y-x}{\varepsilon}, x\Big)\Big|^{6/5} dy\Big)^{5/6}\Big(\int_{\mathbb{R}^2}|\bar{\varphi}(y)|^6 dy\Big)^{1/6}\\
&\quad + \varepsilon^{2/3}\Big(\frac{1}{\varepsilon^2}\int_{\mathbb{R}^2}\Big|\nabla_2^2 r\Big(\frac{y-x}{\varepsilon}, x\Big)\Big|^3 dy\Big)^{1/3}\Big(\int_{\mathbb{R}^2}|\bar{\varphi}(y)|^{3/2} dy\Big)^{2/3}\\
&= \varepsilon^{2/3}\Big(\int_{\mathbb{R}^2}\Big|\nabla_1\nabla_2 r(z,x) + \nabla_2\nabla_1 r(z,x)\Big|^{6/5} dz\Big)^{5/6}\Big(\int_{\mathbb{R}^2}|\bar{\varphi}(y)|^6 dy\Big)^{1/6}\\
&\quad + \varepsilon^{2/3}\Big(\int_{\mathbb{R}^2}\Big|\nabla_2^2 r(z,x)\Big|^3 dz\Big)^{1/3}\Big(\int_{\mathbb{R}^2}|\bar{\varphi}(y)|^{3/2} dy\Big)^{2/3} \le C\varepsilon^{2/3}
\end{aligned}$$

which gives (3.7). In particular,

$$\begin{aligned}
\Big|\varepsilon^2\Delta\zeta_\varepsilon(x) - \varphi_\varepsilon(x)\Big| &= \Big|\varepsilon^2\Delta\zeta_\varepsilon(x) - \int_{\mathbb{R}^2}\Delta_z r(z,x)\bar{\varphi}(x+\varepsilon z)\,dz\Big|\\
&\le C_0\varepsilon^{2/3}\,.
\end{aligned} \tag{3.9}$$

Next by (3.5),

$$\begin{aligned}\int_{\mathbb{R}^2}\frac{1}{\varepsilon}\Big|\nabla\big(\mathrm{div}(\Delta^{-1}\varphi_\varepsilon)\big)(x)\Big|^2dx=&\int_\Omega\frac{1}{\varepsilon}\varphi_\varepsilon(x)\cdot\nabla\big(\mathrm{div}(\Delta^{-1}\varphi_\varepsilon)\big)(x)dx\\ =&\int_\Omega\frac{1}{\varepsilon}\varphi_\varepsilon(x)\cdot\nabla\Big(\mathrm{div}\big(\varepsilon^2\zeta_\varepsilon(x)\big)\Big)dx\\ &-\int_\Omega\frac{1}{\varepsilon}\varphi_\varepsilon(x)\cdot\nabla\Big(\mathrm{div}\Big(\Delta^{-1}\big(\varepsilon^2\Delta\zeta_\varepsilon-\varphi_\varepsilon\big)\Big)\Big)(x)dx.\end{aligned}\tag{3.10}$$

The last integral can be estimated by

$$\begin{aligned}&\left|\int_\Omega\frac{1}{\varepsilon}\varphi_\varepsilon(x)\cdot\nabla\Big(\mathrm{div}\Big(\Delta^{-1}\big(\varepsilon^2\Delta\zeta_\varepsilon-\varphi_\varepsilon\big)\Big)\Big)(x)\,dx\right|\\ &\le\Big(\int_\Omega\frac{1}{\varepsilon}|\varphi_\varepsilon(x)|^2\Big)^{1/2}\Big(\frac{1}{\varepsilon}\int_{\mathbb{R}^2}\Big|\nabla\Big(\mathrm{div}\Big(\Delta^{-1}\big(\varepsilon^2\Delta\zeta_\varepsilon-\varphi_\varepsilon\big)\Big)\Big)(x)\Big|^2dx\Big)^{1/2}\\ &\le\Big(\int_\Omega\frac{1}{\varepsilon}|\varphi_\varepsilon(x)|^2\Big)^{1/2}\Big(\frac{2}{\varepsilon}\int_{\mathbb{R}^2}\Big|\nabla^2\Big(\Delta^{-1}\big(\varepsilon^2\Delta\zeta_\varepsilon-\varphi_\varepsilon\big)\Big)(x)\Big|^2dx\Big)^{1/2}\\ &=\Big(\int_\Omega\frac{1}{\varepsilon}|\varphi_\varepsilon(x)|^2\Big)^{1/2}\Big(\int_\Omega\frac{2}{\varepsilon}\big|\varepsilon^2\Delta\zeta_\varepsilon(x)-\varphi_\varepsilon(x)\big|^2dx\Big)^{1/2}.\end{aligned}$$

Then, since

$$\begin{aligned}\int_\Omega\frac{1}{\varepsilon}|\varphi_\varepsilon(x)|^2&\le C\int_\Omega\frac{1}{\varepsilon}|\varphi_\varepsilon(x)|\\ &\le\bar{C}\int_\Omega\frac{1}{\varepsilon}\left|\int_{B_R(0)}l(z,x)\big(\bar\varphi(x+\varepsilon z)-\varphi(x)\big)dz\right|dx\\ &\le\bar{C}\int_{B_R(0)}\frac{1}{\varepsilon}|l(z,x)|\Big(\int_\Omega\big|\bar\varphi(x+\varepsilon z)-\varphi(x)\big|dx\Big)dz\\ &\le\bar{C}\|D\bar\varphi\|(\mathbb{R}^2)\int_{B_R(0)}|l(z,x)|\cdot|z|\,dz=O(1)\,,\end{aligned}\tag{3.11}$$

using (3.9), from (3.10) we infer

$$\int_{\mathbb{R}^2}\frac{1}{\varepsilon}\Big|\nabla\big(\mathrm{div}(\Delta^{-1}\varphi_\varepsilon)\big)(x)\Big|^2dx=o_\varepsilon(1)+\int_\Omega\frac{1}{\varepsilon}\varphi_\varepsilon(x)\cdot\nabla\Big(\mathrm{div}\big(\varepsilon^2\zeta_\varepsilon(x)\big)\Big)dx.\tag{3.12}$$

Next we remind that ξ_ε is defined by (3.3). By (3.7), we have,

$$\Big|\nabla\Big(\mathrm{div}\big(\varepsilon^2\zeta_\varepsilon(x)\big)\Big)-\xi_\varepsilon(x)\Big|\le\bar{C}\varepsilon^{2/3}\quad\forall x\in\Omega\,.\tag{3.13}$$

Then as before,

$$\begin{aligned}&\left|\int_\Omega \frac{1}{\varepsilon}\,\varphi_\varepsilon(x)\cdot\Big(\nabla\big(\mathrm{div}\big(\varepsilon^2\zeta_\varepsilon(x)\big)\big)-\xi_\varepsilon(x)\Big)\,dx\right|\\ &\le\left(\int_\Omega\frac{1}{\varepsilon}|\varphi_\varepsilon(x)|^2\right)^{1/2}\left(\int_\Omega\frac{1}{\varepsilon}\Big|\nabla\big(\mathrm{div}\big(\varepsilon^2\zeta_\varepsilon(x)\big)\big)-\xi_\varepsilon(x)\Big|^2dx\right)^{1/2}\\ &\le C\varepsilon^{1/6}\,.\end{aligned}$$

Therefore, from (3.12) we infer (3.4). □

Lemma 3.2. *Let $\varphi\in BV(\Omega,\mathbb{R})\cap L^\infty$ and $l\in\mathcal{U}$ (see Definition 2.4). For every $\varepsilon>0$ and every $x\in\mathbb{R}^2$ consider the function $\varphi_\varepsilon\in C^1(\mathbb{R}^2,\mathbb{R}^2)$ as in (3.2). Then,*

$$\begin{aligned}&\lim_{\varepsilon\to0}\int_{\mathbb{R}^2}\frac{1}{\varepsilon}\Big|\nabla\big(\mathrm{div}(\Delta^{-1}\varphi_\varepsilon)\big)(x)\Big|^2dx\\ &=\int_{J_\varphi}\left\{\int_{-\infty}^{+\infty}|\varphi^+(x)-\varphi^-(x)|^2\cdot\left|\boldsymbol{\nu}(x)\cdot\int_t^{+\infty}q(s,x)ds\right|^2dt\right\}d\mathcal{H}^1(x)\,,\end{aligned}\tag{3.14}$$

where

$$q(t,x)=\int_{H^0_{\boldsymbol{\nu}(x)}}l(t\boldsymbol{\nu}(x)+y,x)\,d\mathcal{H}^1(y)\,,\tag{3.15}$$

and $\boldsymbol{\nu}(x)$ is the jump vector of φ.

Proof. By Lemma 3.1 we have

$$\int_{\mathbb{R}^2}\frac{1}{\varepsilon}\Big|\nabla\big(\mathrm{div}(\Delta^{-1}\varphi_\varepsilon)\big)(x)\Big|^2dx=o_\varepsilon(1)+\int_\Omega\frac{1}{\varepsilon}\,\varphi_\varepsilon(x)\cdot\xi_\varepsilon(x)\,dx\,.\tag{3.16}$$

From this point the strategy of the proof is similar to that in [16] and [17] (see also Proposition 2.2). The only difference is that here ξ_ε is defined by a convolution of $\bar\varphi$ with a kernel whose support is not compact. However, it turns out that this difference is not crucial and we can use almost the same approach.

Step 1. We prove a useful expression,

$$\begin{aligned}&\int_\Omega\frac{1}{\varepsilon}\varphi_\varepsilon(x)\cdot\xi_\varepsilon(x)\,dx\\ &=\int_0^1\left\{\int\limits_{\mathbb{R}^2}\left(\frac{1}{t^2\varepsilon^2}\int\limits_{\Omega\cap B_{Rt\varepsilon}(y)}\left\{\xi_\varepsilon(x)\cdot l\Big(\frac{y-x}{t\varepsilon},x\Big)\right\}\frac{y-x}{t\varepsilon}\,dx\right)\cdot d\,[D\bar\varphi](y)\right\}dt\,,\end{aligned}\tag{3.17}$$

where ξ_ε is as in (3.3) and $R > 0$ is such that $l(z,x) = 0$ whenever $|z| \geq R$. As before, we shall denote by $\nabla_1 l$ and $\nabla_2 l$ the gradient of l with respect to the first and second variables respectively. Denote $\big(\varphi_{t\varepsilon,1}(x), \varphi_{t\varepsilon,2}(x)\big) := \varphi_{t\varepsilon}(x)$ and $\big(l_1(z,x), l_2(z,x)\big) := l(z,x)$. Then for every $t \in (0,1]$, every $j \in \{1,2\}$ and every $x \in \mathbb{R}^2$ we have

$$\begin{aligned}\frac{d\big(\varphi_{t\varepsilon,j}(x)\big)}{dt} &= \frac{d}{dt}\bigg(\frac{1}{t^2\varepsilon^2}\int_{\mathbb{R}^2} l_j\Big(\frac{y-x}{t\varepsilon},x\Big)\bar{\varphi}(y)dy\bigg)\\ &= -\frac{1}{t^3\varepsilon^2}\int_{\mathbb{R}^2}\bigg\{\nabla_1 l_j\Big(\frac{y-x}{t\varepsilon},x\Big)\cdot\frac{y-x}{t\varepsilon} + 2l_j\Big(\frac{y-x}{t\varepsilon},x\Big)\bigg\}\bar{\varphi}(y)dy\\ &= -\frac{1}{t^2\varepsilon}\int_{\mathbb{R}^2}\mathrm{div}_y\bigg\{l_j\Big(\frac{y-x}{t\varepsilon},x\Big)\frac{y-x}{t\varepsilon}\bigg\}\bar{\varphi}(y)dy\\ &= \frac{1}{t^2\varepsilon}\int_{\mathbb{R}^2} l_j\Big(\frac{y-x}{t\varepsilon},x\Big)\frac{y-x}{t\varepsilon}\cdot d\,[D\bar{\varphi}](y)\,.\end{aligned}\tag{3.18}$$

Therefore, for any $\rho \in (0,1)$ we have,

$$\begin{aligned}&\int_\Omega \frac{1}{\varepsilon}\big(\varphi_\varepsilon(x) - \varphi_{\rho\varepsilon}(x)\big)\cdot\xi_\varepsilon(x)\,dx\\ &= \int_\Omega \frac{1}{\varepsilon}\xi_\varepsilon(x)\cdot\bigg(\int_\rho^1 \frac{d\big(\varphi_{t\varepsilon}(x)\big)}{dt}\bigg)dx\\ &= \int_\Omega\bigg\{\int_\rho^1 \xi_\varepsilon(x)\cdot\bigg(\frac{1}{t^2\varepsilon^2}\int_{\mathbb{R}^2} l\Big(\frac{y-x}{t\varepsilon},x\Big)\Big\{\frac{y-x}{t\varepsilon}\cdot d\,[D\bar{\varphi}](y)\Big\}\bigg)dt\bigg\}dx\\ &= \int_\rho^1\bigg\{\int_\Omega \xi_\varepsilon(x)\cdot\bigg(\frac{1}{t^2\varepsilon^2}\int_{\mathbb{R}^2} l\Big(\frac{y-x}{t\varepsilon},x\Big)\Big\{\frac{y-x}{t\varepsilon}\cdot d\,[D\bar{\varphi}](y)\Big\}\bigg)dx\bigg\}dt\\ &= \int_\rho^1\bigg\{\int_{\mathbb{R}^2}\bigg(\frac{1}{t^2\varepsilon^2}\int_{\Omega\cap B_{Rt\varepsilon}(y)}\Big\{\xi_\varepsilon(x)\cdot l\Big(\frac{y-x}{t\varepsilon},x\Big)\Big\}\frac{y-x}{t\varepsilon}\,dx\bigg)\cdot d\,[D\bar{\varphi}](y)\bigg\}dt.\end{aligned}\tag{3.19}$$

From our assumptions on φ, by (3.1), it follows that there exists a constant $C > 0$, independent of ρ, such that $\big|\xi_\rho(x)\big| \leq C$ for every $\rho > 0$ and every $x \in \Omega$. Therefore, letting ρ tend to zero in (3.19), using the fact that $\lim_{\rho\to 0}\|\varphi_\rho(x)\|_{L^1(\Omega)} = 0$ (see (3.11)), we get (3.17).

Step 2. We prove the identity

$$
\begin{aligned}
&\int_{\mathbb{R}^2} \frac{1}{\varepsilon}\Big|\nabla\big(\operatorname{div}(\Delta^{-1}\varphi_\varepsilon)\big)(x)\Big|^2 dx \\
&= o_\varepsilon(1) + \int_0^1 \bigg(\int_{J_\varphi} \Big\{ \int_{B_R(0)} \big\{ l(z,x)\cdot \xi_\varepsilon(x-\varepsilon t z)\big\}\, z\, dz \Big\} \cdot d\big[D\varphi\big](x)\bigg) dt \\
&\quad + \int_0^1 \bigg(\int_{G_\varphi} \Big\{ \int_{B_R(0)} \big\{ l(z,x)\cdot \xi_\varepsilon(x-\varepsilon t z)\big\}\, z\, dz \Big\} \cdot d\big[D\varphi\big](x)\bigg) dt\,,
\end{aligned}
\tag{3.20}
$$

where G_φ is the set of approximate continuity of φ. By (3.16) and (3.17) we deduce

$$
\begin{aligned}
&\int_{\mathbb{R}^2} \frac{1}{\varepsilon}\Big|\nabla\big(\operatorname{div}(\Delta^{-1}\varphi_\varepsilon)\big)(x)\Big|^2 dx \\
&= o_\varepsilon(1) + \int_0^1 \bigg\{ \int_{\mathbb{R}^2} \bigg(\frac{1}{t^2\varepsilon^2} \int_{\Omega\cap B_{Rt\varepsilon}(y)} \Big\{ \xi_\varepsilon(x)\cdot l\Big(\frac{y-x}{t\varepsilon}, x\Big)\Big\} \frac{y-x}{t\varepsilon}\, dx \bigg) \cdot d\,[D\bar\varphi](y) \bigg\} dt \\
&= o_\varepsilon(1) + \int_0^1 \bigg\{ \int_{\mathbb{R}^2} \bigg(\frac{1}{t^2\varepsilon^2} \int_{K\cap B_{Rt\varepsilon}(y)} \Big\{ \xi_\varepsilon(x)\cdot l\Big(\frac{y-x}{t\varepsilon}, x\Big)\Big\} \frac{y-x}{t\varepsilon}\, dx \bigg) \cdot d\,[D\bar\varphi](y) \bigg\} dt,
\end{aligned}
\tag{3.21}
$$

where $K \Subset \Omega$ is a compact set (see Definition 2.4). But, for every $\varepsilon < \frac{1}{R}\operatorname{dist}(K, \partial\Omega)$ we have

$$
\begin{aligned}
&\int_{\mathbb{R}^2} \bigg(\frac{1}{t^2\varepsilon^2} \int_{K\cap B_{Rt\varepsilon}(y)} \Big\{ \xi_\varepsilon(x)\cdot l\Big(\frac{y-x}{t\varepsilon}, x\Big)\Big\} \frac{y-x}{t\varepsilon}\, dx \bigg) \cdot d\,[D\bar\varphi](y) \\
&= \int_{\Omega} \bigg(\frac{1}{t^2\varepsilon^2} \int_{K\cap B_{Rt\varepsilon}(y)} \Big\{ \xi_\varepsilon(x)\cdot l\Big(\frac{y-x}{t\varepsilon}, x\Big)\Big\} \frac{y-x}{t\varepsilon}\, dx \bigg) \cdot d\,[D\bar\varphi](y) \\
&= \int_{\Omega} \bigg(\frac{1}{t^2\varepsilon^2} \int_{\mathbb{R}^2} \Big\{ \xi_\varepsilon(x)\cdot l\Big(\frac{y-x}{t\varepsilon}, x\Big)\Big\} \frac{y-x}{t\varepsilon}\, dx \bigg) \cdot d\,[D\bar\varphi](y)\,.
\end{aligned}
$$

Therefore, by (3.21), we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^2} \frac{1}{\varepsilon}\Big|\nabla\big(\operatorname{div}(\Delta^{-1}\varphi_\varepsilon)\big)(x)\Big|^2 dx \\
&= o_\varepsilon(1) + \int_0^1 \int_\Omega \bigg(\frac{1}{t^2\varepsilon^2}\int_{\mathbb{R}^2}\Big\{\xi_\varepsilon(x)\cdot l\Big(\frac{y-x}{t\varepsilon}, x\Big)\Big\}\,\frac{y-x}{t\varepsilon}\,dx\bigg)\cdot d\,[D\bar{\varphi}](y)\,dt \\
&= o_\varepsilon(1) + \int_0^1 \bigg(\int_\Omega \Big\{\int_{B_R(0)} \big\{l(z, y-\varepsilon tz)\cdot\xi_\varepsilon(y-\varepsilon tz)\big\}\,z\,dz\Big\}\cdot d\big[D\varphi\big](y)\bigg)dt \qquad (3.22)\\
&= o_\varepsilon(1) + \int_0^1 \bigg(\int_\Omega \Big\{\int_{B_R(0)} \big\{l(z, x)\cdot\xi_\varepsilon(x-\varepsilon tz)\big\}\,z\,dz\Big\}\cdot d\big[D\varphi\big](x)\bigg)dt\,,
\end{aligned}$$

where in the last equality we used the estimate $|l(z, x-\varepsilon tz) - l(z, x)| \le C\varepsilon t|z|$. Therefore we obtain (3.20).

Step 3. We will prove that the second integral in the r.h.s of (3.20) vanishes as $\varepsilon \to 0$. For every x in G_φ we have,

$$\lim_{\rho\to 0^+} \frac{1}{\rho^2}\int_{B_\rho(x)} |\bar{\varphi}(y) - \tilde{\bar{\varphi}}(x)|\,dy = 0\,.$$

Taking $\rho = L\varepsilon$, for every $L > 0$, gives

$$\lim_{\varepsilon\to 0^+} \int_{B_L(0)} |\bar{\varphi}(x+\varepsilon z) - \tilde{\bar{\varphi}}(x)|\,dz = 0, \quad \text{for } x \text{ in } G_\varphi\,. \qquad (3.23)$$

Using (3.1), since $\int_{\mathbb{R}^2} \nabla_z\big(\operatorname{div}_z r(z, x-\varepsilon ty)\big)dz = 0$, for every x in G_φ, $y \in B_R(0)$, $t \in [0, 1]$ and $L > 0$ we have,

$$\begin{aligned}
|\xi_\varepsilon(x-\varepsilon ty)| &= \bigg|\int_{\mathbb{R}^2} \nabla_z\big(\operatorname{div}_z r(z, x-\varepsilon ty)\big)\bar{\varphi}(x+\varepsilon z-\varepsilon ty)\,dz\bigg| \\
&= \bigg|\int_{\mathbb{R}^2} \nabla_z\big(\operatorname{div}_z r(z, x-\varepsilon ty)\big)\big(\bar{\varphi}(x+\varepsilon z-\varepsilon ty) - \tilde{\bar{\varphi}}(x)\big)\,dz\bigg| \\
&\le \int_{B_L(0)} \Big|\nabla_z\big(\operatorname{div}_z r(z, x-\varepsilon ty)\big)\Big|\cdot\Big|\bar{\varphi}(x+\varepsilon z-\varepsilon ty) - \tilde{\bar{\varphi}}(x)\Big|\,dz \\
&\quad + \int_{\mathbb{R}^2\setminus B_L(0)} \Big|\nabla_z\big(\operatorname{div}_z r(z, x-\varepsilon ty)\big)\Big|\cdot\Big|\bar{\varphi}(x+\varepsilon z-\varepsilon ty) - \tilde{\bar{\varphi}}(x)\Big|\,dz \qquad (3.24)\\
&\le A_L \int_{B_L(0)} |\bar{\varphi}(x+\varepsilon z-\varepsilon ty) - \tilde{\bar{\varphi}}(x)|\,dz + B\int_{\mathbb{R}^2\setminus B_L(0)} \frac{1}{|z|^3+1}dz \\
&\le A_L \int_{B_{(L+R)}(0)} |\bar{\varphi}(x+\varepsilon z) - \tilde{\bar{\varphi}}(x)|\,dz + B\int_{\mathbb{R}^2\setminus B_L(0)} \frac{1}{|z|^3+1}dz\,,
\end{aligned}$$

where $B > 0$ is a constant and $A_L > 0$ depends only on L. Given $\delta > 0$ we can take $L > 0$ such that

$$B \int_{\mathbb{R}^2 \setminus B_L(0)} \frac{1}{|z|^3 + 1} dz < \delta \,,$$

Then, using (3.24) and (3.23), we infer $\overline{\lim}_{\varepsilon \to 0^+} |\xi_\varepsilon(x - \varepsilon t y)| < \delta$ and since δ was arbitrary,

$$\lim_{\varepsilon \to 0^+} \xi_\varepsilon(x - \varepsilon t y) = 0 \quad \forall x \in G_\varphi,\ y \in B_R(0),\ t \in [0, 1]. \tag{3.25}$$

Using (3.1), we also have $|\xi_\varepsilon(x - \varepsilon t y)| \le C$, and therefore, plugging (3.25) into (3.20), we obtain

$$\begin{aligned} &\int_{\mathbb{R}^2} \frac{1}{\varepsilon} \Big| \nabla \big(\operatorname{div}(\Delta^{-1} \varphi_\varepsilon) \big)(x) \Big|^2 dx \\ &= o_\varepsilon(1) + \int_0^1 \left(\int_{J_\varphi} \left\{ \int_{B_R(0)} \big\{ l(z, x) \cdot \xi_\varepsilon(x - \varepsilon t z) \big\}\, z \, dz \right\} \cdot d\big[D\varphi\big](x) \right) dt \,. \end{aligned} \tag{3.26}$$

Step 4. Consider $\bar{l}(z, x) := \nabla_z \big(\operatorname{div}_z r(z, x) \big)$. For every $\varepsilon, t \in (0, 1)$, $x \in J_\varphi$ and $z \in B_R(0)$, we have

$$\begin{aligned} \xi_\varepsilon(x - \varepsilon t z) &= \int_{\mathbb{R}^2} \bar{l}(y, x - \varepsilon t z) \bar{\varphi}\big(x + \varepsilon(y - t z)\big)\, dy \\ &= \int_{\mathbb{R}^2} \bar{l}(y + t z, x - \varepsilon t z) \bar{\varphi}\big(x + \varepsilon y\big)\, dy \\ &= \int_{H_+(0, \boldsymbol{\nu}(x))} \bar{l}(y + t z, x - \varepsilon t z) \bar{\varphi}\big(x + \varepsilon y\big)\, dy \\ &\quad + \int_{H_-(0, \boldsymbol{\nu}(x))} \bar{l}(y + t z, x - \varepsilon t z) \bar{\varphi}\big(x + \varepsilon y\big)\, dy \\ &= \int_{H_+(0, \boldsymbol{\nu}(x))} \bar{l}(y + t z, x - \varepsilon t z) \varphi^+(x)\, dy \\ &\quad + \int_{H_-(0, \boldsymbol{\nu}(x))} \bar{l}(y + t z, x - \varepsilon t z) \varphi^-(x)\, dy \\ &\quad + \int_{H_+(0, \boldsymbol{\nu}(x))} \bar{l}(y + t z, x - \varepsilon t z) \big(\bar{\varphi}(x + \varepsilon y) - \varphi^+(x) \big)\, dy \\ &\quad + \int_{H_-(0, \boldsymbol{\nu}(x))} \bar{l}(y + t z, x - \varepsilon t z) \big(\bar{\varphi}(x + \varepsilon y) - \varphi^-(x) \big)\, dy \,. \end{aligned} \tag{3.27}$$

By the definition of J_φ, for every $L > 0$ we obtain,

$$\begin{aligned} &\lim_{\varepsilon\to 0^+}\int_{B_L^+(0,\boldsymbol{\nu}(x))} |\varphi(x+\varepsilon z)-\varphi^+(x)|\,dz=0,\\ &\lim_{\varepsilon\to 0^+}\int_{B_L^-(0,\boldsymbol{\nu}(x))} |\varphi(x+\varepsilon z)-\varphi^-(x)|\,dz=0. \end{aligned} \qquad \text{for } x\in J_\varphi\,. \tag{3.28}$$

Then, by (3.1), for every $L > R$ we have,

$$\begin{aligned} &\left|\int_{H_+(0,\boldsymbol{\nu}(x))} \bar{l}(y+tz, x-\varepsilon tz)\big(\bar\varphi(x+\varepsilon y)-\varphi^+(x)\big)\,dy\right|\\ &\leq \int_{B_L^+(0,\boldsymbol{\nu}(x))} |\bar{l}(y+tz, x-\varepsilon tz)|\cdot\big|\bar\varphi(x+\varepsilon y)-\varphi^+(x)\big|\,dy\\ &\quad+\int_{H_+(0,\boldsymbol{\nu}(x))\setminus B_L^+(0,\boldsymbol{\nu}(x))} |\bar{l}(y+tz, x-\varepsilon tz)|\cdot\big|\bar\varphi(x+\varepsilon y)-\varphi^+(x)\big|\,dy\\ &\leq A_L\int_{B_L^+(0,\boldsymbol{\nu}(x))}\big|\bar\varphi(x+\varepsilon y)-\varphi^+(x)\big|\,dy + B\int_{\mathbb{R}^2\setminus B_L(0)}\frac{1}{(|y|-R)^3+1}dy\,, \end{aligned} \tag{3.29}$$

where $B > 0$ is a constant and $A_L > 0$ depends only on L. Given $\delta > 0$ we can take $L > 0$ such that

$$B\int_{\mathbb{R}^2\setminus B_L(0)}\frac{1}{(|y|-R)^3+1}dy < \delta\,,$$

Then, using (3.29) and (3.28), we infer

$$\overline{\lim_{\varepsilon\to 0^+}}\left|\int_{H_+(0,\boldsymbol{\nu}(x))} \bar{l}(y+tz, x-\varepsilon tz)\big(\bar\varphi(x+\varepsilon y)-\varphi^+(x)\big)\,dy\right| < \delta\,,$$

and since δ was arbitrary,

$$\begin{aligned} &\lim_{\varepsilon\to 0^+}\int_{H_+(0,\boldsymbol{\nu}(x))} \bar{l}(y+tz, x-\varepsilon tz)\big(\bar\varphi(x+\varepsilon y)-\varphi^+(x)\big)\,dy=0\\ &\qquad\forall x\in J_\varphi,\ z\in B_R(0),\ t\in[0,1]. \end{aligned} \tag{3.30}$$

By the same method,

$$\begin{aligned} &\lim_{\varepsilon\to 0^+}\int_{H_-(0,\boldsymbol{\nu}(x))} \bar{l}(y+tz, x-\varepsilon tz)\big(\bar\varphi(x+\varepsilon y)-\varphi^-(x)\big)\,dy=0\\ &\qquad\forall x\in J_\varphi,\ z\in B_R(0),\ t\in[0,1]. \end{aligned} \tag{3.31}$$

Therefore, by (3.27) for every $\varepsilon, t \in (0,1)$, $x \in J_\varphi$ and $z \in B_R(0)$, we have

$$\begin{aligned}\xi_\varepsilon(x-\varepsilon tz) &= o_\varepsilon(1) + \varphi^+(x) \\ &\quad\times \int\limits_{H_+(0,\boldsymbol{\nu}(x))} \bar{l}(y+tz, x-\varepsilon tz)dy + \varphi^-(x) \int\limits_{H_-(0,\boldsymbol{\nu}(x))} \bar{l}(y+tz, x-\varepsilon tz)dy \\ &= o_\varepsilon(1) + \big(\varphi^+(x) - \varphi^-(x)\big) \int\limits_{H_+(0,\boldsymbol{\nu}(x))} \bar{l}(y+tz, x-\varepsilon tz)\, dy\,,\end{aligned} \tag{3.32}$$

where we used the equality $\int_{\mathbb{R}^2} \bar{l}(y+tz, x-\varepsilon tz)dy = 0$. Using (3.1), gives

$$\lim_{\varepsilon\to 0^+} \int_{H_+(0,\boldsymbol{\nu}(x))} \bar{l}(y+tz, x-\varepsilon tz)\, dy = \int_{H_+(0,\boldsymbol{\nu}(x))} \bar{l}(y+tz, x)\, dy.$$

Therefore, by (3.32), for every $x \in J_\varphi$, every $t \in (0,1)$ and every $z \in B_R(0)$, we obtain,

$$\lim_{\varepsilon\to 0^+} \xi_\varepsilon(x-\varepsilon tz) = \big(\varphi^+(x) - \varphi^-(x)\big) \int_{H_+(0,\boldsymbol{\nu}(x))} \bar{l}(y+tz, x)\, dy\,. \tag{3.33}$$

Note that

$$\begin{aligned}\int_{H_+(0,\boldsymbol{\nu}(x))} \bar{l}(y+tz, x)\, dy &= \int_{H_+(tz,\boldsymbol{\nu}(x))} \bar{l}(y, x)\, dy \\ &= \int_{t\boldsymbol{\nu}(x)\cdot z}^{+\infty} \Big(\int_{H^0_{\boldsymbol{\nu}(x)}} \bar{l}(t\boldsymbol{\nu}(x)+y, x)\, d\mathcal{H}^1(y)\Big)dt \\ &= \int_{t\boldsymbol{\nu}(x)\cdot z}^{+\infty} \bar{q}(\tau, x)\, d\tau,\end{aligned} \tag{3.34}$$

where

$$\bar{q}(t,x) = \int_{H^0_{\boldsymbol{\nu}(x)}} \bar{l}(t\boldsymbol{\nu}(x)+y, x)\, d\mathcal{H}^1(y)\,. \tag{3.35}$$

Combining (3.33) and (3.34), for every $x \in J_\varphi$, every $t \in (0,1)$ and every $z \in B_R(0)$ we obtain,

$$\lim_{\varepsilon\to 0+} \xi_\varepsilon(x-\varepsilon tz) = \big(\varphi^+(x) - \varphi^-(x)\big) \int_{t\boldsymbol{\nu}(x)\cdot z}^{+\infty} \bar{q}(\tau, x)\, d\tau. \tag{3.36}$$

Using (3.36) in (3.26), we obtain,

$$\begin{aligned}\int_{\mathbb{R}^2} \frac{1}{\varepsilon}\Big|\nabla\big(\mathrm{div}(\Delta^{-1}\varphi_\varepsilon)\big)(x)\Big|^2 dx &= o_\varepsilon(1)\\ &+\int_0^1\bigg(\int_{J_\varphi}\big(\varphi^+(x)-\varphi^-(x)\big)\bigg\{\int_{B_R(0)}\bigg(l(z,x)\cdot\int_{t\boldsymbol{\nu}(x)z}^{+\infty}\bar{q}(\tau,x)d\tau\bigg)z\,dz\bigg\}\cdot d\big[D\varphi\big](x)\bigg)dt\\ &= o_\varepsilon(1)\\ &+\int_{J_\varphi}\big|\varphi^+(x)-\varphi^-(x)\big|^2\bigg\{\int_{B_R(0)}\bigg(l(z,x)\cdot\int_0^1\int_{t\boldsymbol{\nu}(x)\cdot z}^{+\infty}\bar{q}(\tau,x)d\tau dt\bigg)\big(\nu(x)\cdot z\big)\,dz\bigg\}d\mathcal{H}^1(x).\end{aligned} \tag{3.37}$$

Step 5. We prove that

$$\bar{q}(t,x)=\big(q(t,x)\cdot\boldsymbol{\nu}(x)\big)\boldsymbol{\nu}(x)\,. \tag{3.38}$$

Consider $\big(r_1(z,x), r_2(z,x)\big) := r(z,x)$. Then, by (3.1), for every $k = 1, 2$, we obtain,

$$\begin{aligned}\int_{H^0_{\boldsymbol{\nu}(x)}}\nabla_z^2 r_k(t\boldsymbol{\nu}(x)+y,x)\,d\mathcal{H}^1(y) &= \int_{H^0_{\boldsymbol{\nu}(x)}}\boldsymbol{\nu}(x)\otimes\boldsymbol{\nu}(x)\,\frac{\partial^2 r_k}{\partial\big(\boldsymbol{\nu}(x)\big)^2}(t\boldsymbol{\nu}(x)+y,x)\,d\mathcal{H}^1(y)\\ &+\int_{H^0_{\boldsymbol{\nu}(x)}}\big(\boldsymbol{\nu}(x)\otimes\boldsymbol{\nu}^\perp(x)+\boldsymbol{\nu}^\perp(x)\otimes\boldsymbol{\nu}(x)\big)\frac{\partial^2 r_k}{\partial\big(\boldsymbol{\nu}^\perp(x)\big)\partial\big(\boldsymbol{\nu}(x)\big)}(t\boldsymbol{\nu}(x)+y,x)\,d\mathcal{H}^1(y)\\ &+\int_{H^0_{\boldsymbol{\nu}(x)}}\boldsymbol{\nu}^\perp(x)\otimes\boldsymbol{\nu}^\perp(x)\,\frac{\partial^2 r_k}{\partial\big(\boldsymbol{\nu}^\perp(x)\big)^2}(t\boldsymbol{\nu}(x)+y,x)\,d\mathcal{H}^1(y)\\ &=\int_{H^0_{\boldsymbol{\nu}(x)}}\boldsymbol{\nu}(x)\otimes\boldsymbol{\nu}(x)\,\frac{\partial^2 r_k}{\partial\big(\boldsymbol{\nu}(x)\big)^2}(t\boldsymbol{\nu}(x)+y,x)\,d\mathcal{H}^1(y)\,,\end{aligned} \tag{3.39}$$

where $\boldsymbol{\nu}^\perp(x)$ is the vector orthogonal to $\boldsymbol{\nu}(x)$ in $\mathbb{R}^2$ and all derivatives are taken in the first argument-z of $r(z,x)$. In particular

$$\begin{aligned}q(t,x) = \int_{H^0_{\boldsymbol{\nu}(x)}} l(t\boldsymbol{\nu}(x)+y,x)\,d\mathcal{H}^1(y) &= \int_{H^0_{\boldsymbol{\nu}(x)}}\Delta r(t\boldsymbol{\nu}(x)+y,x)\,d\mathcal{H}^1(y)\\ &=\int_{H^0_{\boldsymbol{\nu}(x)}}\frac{\partial^2 r}{\partial\big(\boldsymbol{\nu}(x)\big)^2}(t\boldsymbol{\nu}(x)+y,x)\,d\mathcal{H}^1(y)\,,\end{aligned}$$

and

$$\bar{q}(t,x) = \int\limits_{H^0_{\boldsymbol{\nu}(x)}} \bar{l}(t\boldsymbol{\nu}(x)+y,x)\,d\mathcal{H}^1(y) = \int\limits_{H^0_{\boldsymbol{\nu}(x)}} \nabla_z(\mathrm{div}_z\, r)(t\boldsymbol{\nu}(x)+y,x)\,d\mathcal{H}^1(y)$$

$$= \Bigg(\boldsymbol{\nu}(x)\cdot \int\limits_{H^0_{\boldsymbol{\nu}(x)}} \frac{\partial^2 r}{\partial\big(\boldsymbol{\nu}(x)\big)^2}(t\boldsymbol{\nu}(x)+y,x)\,d\mathcal{H}^1(y)\Bigg)\boldsymbol{\nu}(x)\,.$$

So, we obtain (3.38).

Step 6. Completing the proof. Plugging (3.38) into (3.37) gives

$$\int_{\mathbb{R}^2} \frac{1}{\varepsilon}\Big|\nabla\big(\mathrm{div}(\Delta^{-1}\varphi_\varepsilon)\big)(x)\Big|^2 dx = o_\varepsilon(1) \;+\; \int\limits_{J_\varphi} \big|\varphi^+(x)-\varphi^-(x)\big|^2$$

$$\times\Bigg\{\int\limits_{B_R(0)}\Bigg(\big(l(z,x)\cdot\boldsymbol{\nu}(x)\big)\int_0^1\!\!\int_{t\boldsymbol{\nu}(x)\cdot z}^{+\infty}\big(q(\tau,x)\cdot\boldsymbol{\nu}(x)\big)\,d\tau dt\Bigg)\big(\nu(x)\cdot z\big)\,dz\Bigg\}d\mathcal{H}^1(x)\,. \tag{3.40}$$

Next we have

$$\int\limits_{B_R(0)}\Bigg(\big(l(z,x)\cdot\boldsymbol{\nu}(x)\big)\int\limits_0^1\int\limits_{t\boldsymbol{\nu}(x)\cdot z}^{+\infty}\big(q(\tau,x)\cdot\boldsymbol{\nu}(x)\big)\,d\tau dt\Bigg)\big(\nu(x)\cdot z\big)\,dz$$

$$=\int\limits_{\mathbb{R}^2}\Bigg(\big(l(z,x)\cdot\boldsymbol{\nu}(x)\big)\int\limits_0^1\int\limits_{t\boldsymbol{\nu}(x)\cdot z}^{+\infty}\big(q(\tau,x)\cdot\boldsymbol{\nu}(x)\big)\,d\tau dt\Bigg)\big(\nu(x)\cdot z\big)\,dz$$

$$=\int\limits_{-\infty}^{+\infty} s\Bigg(\int\limits_0^1\int\limits_{ts}^{+\infty}\big(q(\tau,x)\cdot\boldsymbol{\nu}(x)\big)\,d\tau dt\Bigg)\Bigg(\int\limits_{H^0_{\boldsymbol{\nu}(x)}}\Big(l(s\boldsymbol{\nu}(x)+y,x)\cdot\boldsymbol{\nu}(x)\Big)\,d\mathcal{H}^1(y)\Bigg)ds$$

$$=\int\limits_{-\infty}^{+\infty} s\big(q(s,x)\cdot\boldsymbol{\nu}(x)\big)\Bigg(\int\limits_0^1\int\limits_{ts}^{+\infty}\big(q(\tau,x)\cdot\boldsymbol{\nu}(x)\big)\,d\tau dt\Bigg)ds$$

$$=\int\limits_{-\infty}^{+\infty}\big(q(s,x)\cdot\boldsymbol{\nu}(x)\big)\Bigg(\int\limits_0^s\int\limits_{t}^{+\infty}\big(q(\tau,x)\cdot\boldsymbol{\nu}(x)\big)\,d\tau dt\Bigg)ds\,. \tag{3.41}$$

Using the fact that $\int_{\mathbb{R}} q(\tau,x)d\tau = 0$ and integrating by path, we obtain,

$$\begin{aligned}\int_{-\infty}^{+\infty} \big(q(s,x)\cdot\boldsymbol{\nu}(x)\big)\bigg(\int_0^s\int_t^{+\infty}\big(q(\tau,x)\cdot\boldsymbol{\nu}(x)\big)\,d\tau dt\bigg)\,ds\\ =\int_{-\infty}^{+\infty}\bigg(\int_s^{+\infty}\big(q(\tau,x)\cdot\boldsymbol{\nu}(x)\big)\,d\tau\bigg)^2 ds\,.\end{aligned} \tag{3.42}$$

Therefore, returning to (3.41) we infer

$$\begin{aligned}\int\limits_{B_R(0)}\bigg(\big(l(z,x)\cdot\boldsymbol{\nu}(x)\big)\int_0^1\int_{t\boldsymbol{\nu}(x)\cdot z}^{+\infty}\big(q(\tau,x)\cdot\boldsymbol{\nu}(x)\big)\,d\tau dt\bigg)\big(\nu(x)\cdot z\big)\,dz\\ =\int_{-\infty}^{+\infty}\bigg(\int_t^{+\infty}\big(q(s,x)\cdot\boldsymbol{\nu}(x)\big)\,ds\bigg)^2 dt\,.\end{aligned} \tag{3.43}$$

Plugging (3.43) in (3.40) gives the desired result (3.14). □

4. Proof of the main result

As before, throughout this section we assume that Ω is a bounded domain in $\mathbb{R}^2$ with Lipschitz boundary. Next consider $u \in BV(\Omega, S^1)$, satisfying $\operatorname{div} u = 0$ in Ω and $u\cdot\boldsymbol{n} = 0$ on $\partial\Omega$ ($\boldsymbol{n}$ is the unit normal to $\partial\Omega$). Let $\varphi \in BV(\Omega,\mathbb{R})\cap L^\infty(\Omega,\mathbb{R})$, satisfying $u = e^{i\varphi}$ a e. in Ω. By [2, Proposition 3.21] we may extend φ to a function $\bar\varphi \in BV(\mathbb{R}^2,\mathbb{R})\cap L^\infty(\mathbb{R}^2,\mathbb{R})$ satisfying $\bar\varphi = \varphi$ a.e. in Ω, $\operatorname{supp}\bar\varphi$ is compact and $\|D\bar\varphi\|(\partial\Omega) = 0$ (from the proof of Proposition 3.21 in [2] it follows that if φ is bounded then its extension is also bounded). We also denote by $\bar u := e^{i\bar\varphi}$. Then $\bar u \in BV(\Omega'',\mathbb{R}^2)\cap L^\infty(\Omega'',\mathbb{R}^2)$ for some $\Omega''\Supset\Omega$, satisfying $\bar u = u$ a.e. in Ω and, by Volpert's chain rule, $\|D\bar u\|(\partial\Omega) = 0$. Consider $\eta\in\mathcal{V}$. For any $\varepsilon > 0$ define a function $\psi_\varepsilon(x) : \mathbb{R}^2\to\mathbb{R}$ by

$$\psi_\varepsilon(x) := \frac{1}{\varepsilon^2}\int_{\mathbb{R}^2}\eta\Big(\frac{y-x}{\varepsilon},x\Big)\bar\varphi(y)dy = \int_{\mathbb{R}^2}\eta(z,x)\bar\varphi(x+\varepsilon z)\,dz,\quad \forall x\in\mathbb{R}^2\,. \tag{4.1}$$

Proposition 4.1. *Let u, φ, $\bar u$, $\bar\varphi$ and η be as above. Consider $\psi_\varepsilon(x)$ defined by* (4.1)*. Then,*

$$\begin{aligned}\lim_{\varepsilon\to0}\int_{\mathbb{R}^2}\Big|\nabla\operatorname{div}\Delta^{-1}\big(\chi_\Omega(x)e^{i\psi_\varepsilon(x)}\big)\Big|^2dx\\ =\int_{J_\varphi}\bigg\{\int_{-\infty}^{+\infty}\Big|\boldsymbol{\nu}(x)\cdot\Big(e^{i\gamma(t,x)}-e^{i\varphi^-(x)}\Big)\Big|^2dt\bigg\}d\mathcal{H}^1(x)\,,\end{aligned} \tag{4.2}$$

where

$$\gamma(t,x) = \varphi^-(x)\int_{-\infty}^t p(s,x)ds + \varphi^+(x)\int_t^{+\infty}p(s,x)\,ds\,, \tag{4.3}$$

with

$$p(t,x) = \int_{H^0_{\boldsymbol{\nu}(x)}} \eta(t\boldsymbol{\nu}(x)+y,x)\,d\mathcal{H}^1(y)\,, \tag{4.4}$$

and χ_Ω *is the characteristic function of* Ω.

Proof. We follow basically the strategy that was described in Subsection 1.1.

Since $(u^+ - u^-)\cdot\boldsymbol{\nu} = 0$, the right hand side in (4.2) does not depend on the orientation of J_φ, we may assume that $\boldsymbol{\nu}(x)$ is Borel measurable. Together with $\eta \in \mathcal{V}$ we consider a second kernel $\bar\eta \in \mathcal{V}$. Let

$$\bar p(t,x) = \int_{H^0_{\boldsymbol{\nu}(x)}} \bar\eta(t\boldsymbol{\nu}(x)+y,x)\,d\mathcal{H}^1(y)\,. \tag{4.5}$$

For any $\varepsilon > 0$ define a function $u_\varepsilon(x) : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$u_\varepsilon(x) := \frac{1}{\varepsilon^2}\int_{\mathbb{R}^2} \bar\eta\Big(\frac{y-x}{\varepsilon},x\Big)\bar u(y)\,dy = \int_{\mathbb{R}^2} \bar\eta(z,x)\,e^{i\bar\varphi(x+\varepsilon z)}\,dz, \quad \forall x \in \mathbb{R}^2\,. \tag{4.6}$$

Define $Q : \mathbb{R}\times J_\varphi \to \mathbb{R}^2$ by

$$Q(t,x) := e^{i\gamma(t,x)} - \bigg(\Big\{\int_{-\infty}^{t} \bar p(s,x)ds\Big\}e^{i\varphi^-(x)} + \Big\{\int_{t}^{+\infty} \bar p(s,x)\,ds\Big\}e^{i\varphi^+(x)}\bigg), \tag{4.7}$$

where $\gamma(t,x)$ is defined by (4.3). Then define $q : \mathbb{R}\times\Omega \to \mathbb{R}^2$ by

$$q(t,x) = \begin{cases} -\dfrac{1}{(\varphi^+(x)-\varphi^-(x))}\,\dfrac{dQ(t,x)}{dt} & x \in J_\varphi\,, \\ 0 & x \in \Omega\setminus J_\varphi\,. \end{cases} \tag{4.8}$$

Then $q(t,x)$ is Borel measurable, q is bounded on $\mathbb{R}\times\Omega$, there exists $M > 0$ such that $\operatorname{supp} q \subset [-M,M]\times\Omega$ and $\int_{\mathbb{R}} q(t,x)\,dt = 0$ $\forall x \in \Omega$. Moreover

$$\big(\varphi^+(x)-\varphi^-(x)\big)\int_t^{+\infty} q(s,x)ds = Q(t,x)\,. \tag{4.9}$$

Then by Lemma 2.6, there exists a sequence of functions $l_n \in \mathcal{U}$ (see Definition 2.4), such that the sequence of functions $\{q_n\}$ defined on $\mathbb{R}\times\Omega$ by

$$q_n(t,x) = \int_{H^0_{\boldsymbol{\nu}_0(x)}} l_n(t\boldsymbol{\nu}_0(x)+y,x)d\mathcal{H}^1(y),$$

has the following properties:

$$\text{there exists } C_0 \text{ such that } \|q_n\|_{L^\infty} \le C_0, \tag{4.10}$$

$$\text{there exists } M > 0 \text{ such that } q_n(t,x) = 0 \text{ for } |t| > M, \text{ and every } x \in \Omega, \tag{4.11}$$

$$\lim_{n\to\infty}\int_\Omega\int_{\mathbb{R}} |q_n(t,x)-q(t,x)|\,dt\,d\|D\varphi\|(x) = 0. \tag{4.12}$$

In particular,

$$\lim_{n\to\infty}\int_{J_\varphi}\int_{\mathbb{R}}\left|\varphi^+(x)-\varphi^-(x)\right|\cdot|q_n(t,x)-q(t,x)|\,dt\,d\mathcal{H}^1(x)=0\,. \tag{4.13}$$

For every positive integer n and for every $\varepsilon>0$ consider the function $\varphi_{n,\varepsilon}\in C^1(\mathbb{R}^2,\mathbb{R}^2)$ given by

$$\varphi_{n,\varepsilon}(x):=\frac{1}{\varepsilon^2}\int_{\mathbb{R}^2}l_n\Big(\frac{y-x}{\varepsilon},x\Big)\bar{\varphi}(y)\,dy=\int_{\mathbb{R}^2}l_n(z,x)\bar{\varphi}(x+\varepsilon z)\,dz\,, \tag{4.14}$$

Next, we will use the following inequality, valid $\forall\, f(x),\, g(x),\, \lambda(x)\in L^2(\mathbb{R}^2,\mathbb{R}^2)$,

$$\begin{aligned}&\left|\int_{\mathbb{R}^2}|f(x)|^2dx-\int_{\mathbb{R}^2}|g(x)|^2dx\right|\\&\quad\le\Big(\|f-g-\lambda\|_{L^2}+\|\lambda\|_{L^2}\Big)\sqrt{2\Big(\int_{\mathbb{R}^2}|f(x)|^2dx+\int_{\mathbb{R}^2}|g(x)|^2dx\Big)}\,.\end{aligned} \tag{4.15}$$

Therefore, since $\varphi_{n,\varepsilon}(x)=0$ for $x\notin\Omega$ and since $\operatorname{div}\big(\chi_\Omega\bar{u}\big)=0$ as a distribution, we obtain,

$$\begin{aligned}&\left|\int_{\mathbb{R}^2}\Big|\nabla\operatorname{div}\Delta^{-1}\big(\chi_\Omega(x)e^{i\psi_\varepsilon(x)}\big)\Big|^2dx-\int_{\mathbb{R}^2}\Big|\nabla\operatorname{div}\Delta^{-1}\big(\varphi_{n,\varepsilon}(x)\big)\Big|^2dx\right|\\
&\le 2\bigg(\Big\|\nabla\operatorname{div}\Delta^{-1}\Big(\chi_\Omega\big(e^{i\psi_\varepsilon}-\varphi_{n,\varepsilon}-u_\varepsilon\big)\Big)\Big\|_{L^2}+\big\|\nabla\operatorname{div}\Delta^{-1}(\chi_\Omega u_\varepsilon)\big\|_{L^2}\bigg)\\
&\times\sqrt{\int_{\mathbb{R}^2}\Big|\nabla\operatorname{div}\Delta^{-1}\big(\chi_\Omega e^{i\psi_\varepsilon}\big)\Big|^2dx+\int_{\mathbb{R}^2}\Big|\nabla\operatorname{div}\Delta^{-1}\big(\varphi_{n,\varepsilon}\big)\Big|^2dx}\\
&=2\bigg(\Big\|\nabla\operatorname{div}\Delta^{-1}\Big(\chi_\Omega\big(e^{i\psi_\varepsilon}-\varphi_{n,\varepsilon}-u_\varepsilon\big)\Big)\Big\|_{L^2}+\Big\|\nabla\operatorname{div}\Delta^{-1}\big(\chi_\Omega(u_\varepsilon-\bar{u})\big)\Big\|_{L^2}\bigg)\\
&\times\sqrt{\int_{\mathbb{R}^2}\Big|\nabla\operatorname{div}\Delta^{-1}\big(\chi_\Omega(e^{i\psi_\varepsilon}-e^{i\bar{\varphi}})\big)\Big|^2dx+\int_{\mathbb{R}^2}\Big|\nabla\operatorname{div}\Delta^{-1}\big(\varphi_{n,\varepsilon}\big)\Big|^2dx}\,.\end{aligned} \tag{4.16}$$

But since for every $f\in L^\infty(\mathbb{R}^2,\mathbb{R}^2)$ with compact support we have

$$\int_{\mathbb{R}^2}\big|\nabla\operatorname{div}\Delta^{-1}f\big|^2dx\le 2\int_{\mathbb{R}^2}\big|\nabla^2\Delta^{-1}f\big|^2dx=2\int_{\mathbb{R}^2}|f|^2dx\,,$$

by (4.16), we obtain

$$\left|\frac{1}{\varepsilon}\int_{\mathbb{R}^2}\left|\nabla\operatorname{div}\Delta^{-1}\big(\chi_\Omega(x)e^{i\psi_\varepsilon(x)}\big)\right|^2dx-\frac{1}{\varepsilon}\int_{\mathbb{R}^2}\left|\nabla\operatorname{div}\Delta^{-1}\big(\varphi_{n,\varepsilon}(x)\big)\right|^2dx\right|$$

$$\le 4\left(\sqrt{\frac{1}{\varepsilon}\int_\Omega\left|e^{i\psi_\varepsilon}-\varphi_{n,\varepsilon}-u_\varepsilon\right|^2dx}+\sqrt{\frac{1}{\varepsilon}\int_\Omega|u_\varepsilon-u|^2dx}\right) \tag{4.17}$$

$$\times\sqrt{\frac{1}{\varepsilon}\int_\Omega\left|e^{i\psi_\varepsilon}-e^{i\varphi}\right|^2dx+\frac{1}{\varepsilon}\int_{\mathbb{R}^2}\left|\nabla\operatorname{div}\Delta^{-1}\big(\varphi_{n,\varepsilon}\big)\right|^2dx}\,.$$

Therefore, setting

$$L_0:=\int_{J_\varphi}\left\{\int_{-\infty}^{+\infty}\left|\,\boldsymbol{\nu}(x)\cdot\left(e^{i\gamma(t,x)}-e^{i\varphi^-(x)}\right)\right|^2dt\right\}d\mathcal{H}^1(x)\,,$$

we have

$$\left|\frac{1}{\varepsilon}\int_{\mathbb{R}^2}\left|\nabla\operatorname{div}\Delta^{-1}\big(\chi_\Omega e^{i\psi_\varepsilon}\big)\right|^2dx-L_0\right|\le\left|L_0-\frac{1}{\varepsilon}\int_{\mathbb{R}^2}\left|\nabla\operatorname{div}\Delta^{-1}(\varphi_{n,\varepsilon})\right|^2dx\right|$$

$$+4\left(\sqrt{\frac{1}{\varepsilon}\int_\Omega\left|e^{i\psi_\varepsilon}-\varphi_{n,\varepsilon}-u_\varepsilon\right|^2dx}+\sqrt{\frac{1}{\varepsilon}\int_\Omega|u_\varepsilon-u|^2dx}\right)$$

$$\times\sqrt{\frac{1}{\varepsilon}\int_\Omega\left|e^{i\psi_\varepsilon}-e^{i\varphi}\right|^2dx+\frac{1}{\varepsilon}\int_{\mathbb{R}^2}\left|\nabla\operatorname{div}\Delta^{-1}\big(\varphi_{n,\varepsilon}\big)\right|^2dx}\,. \tag{4.18}$$

Using Proposition 2.2 with $W(a,b)=W\big((a_1,a_2,a_3),b\big):(\mathbb{R}\times\mathbb{R}^2\times\mathbb{R}^2)\times\mathbb{R}\to\mathbb{R}$, defined by $W(a,b):=\left|e^{ia_1}-a_2-a_3\right|^2$, we obtain,

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int\limits_\Omega\left|e^{i\psi_\varepsilon}-\varphi_{n,\varepsilon}-u_\varepsilon\right|^2dx=D_n$$

$$:=\int\limits_{J_\varphi}\left\{\int\limits_{-\infty}^{+\infty}\left|e^{i\gamma(t,x)}-\big(\varphi^+(x)-\varphi^-(x)\big)\int\limits_t^{+\infty}q_n(s,x)ds-\Gamma(t,x)\right|^2dt\right\}d\mathcal{H}^1(x)\,, \tag{4.19}$$

where $\gamma(t,x)$ is defined by (4.3), and

$$\Gamma(t,x):=\left\{\int_{-\infty}^t\bar{p}(s,x)ds\right\}e^{i\varphi^-(x)}+\left\{\int_t^{+\infty}\bar{p}(s,x)\,ds\right\}e^{i\varphi^+(x)}\,.$$

Using Proposition 2.2 with $W(a,b) = W : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$, defined by $W(a,b) := \big|a-b\big|^2$, we also infer that

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int\limits_{\Omega}|u_\varepsilon - u|^2 dx = T(\bar\eta)\\
&:= \int\limits_{J_\varphi}\Bigg\{\int\limits_{-\infty}^{0}\big|\Gamma(t,x)-u^+(x)\big|^2 dt + \int\limits_{0}^{+\infty}\big|\Gamma(t,x)-u^-(x)\big|^2 dt\Bigg\}d\mathcal{H}^1(x)\\
&= \int\limits_{J_\varphi}\Bigg\{\int\limits_{-\infty}^{0}\Bigg|(u^+-u^-)\int\limits_{-\infty}^{t}\bar p(s,\cdot)ds\Bigg|^2 dt\\
&\quad + \int\limits_{0}^{+\infty}\Bigg|(u^+-u^-)\int\limits_{t}^{+\infty}\bar p(s,\cdot)\,ds\Bigg|^2 dt\Bigg\}d\mathcal{H}^1\,,
\end{aligned} \tag{4.20}$$

and

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{\Omega}\big|e^{i\psi_\varepsilon}-e^{i\varphi}\big|^2 dx\\
&= M := \int_{J_\varphi}\Bigg\{\int_{-\infty}^{0}\big|e^{i\gamma(t,x)}-e^{i\varphi^+(x)}\big|^2 dt + \int_{0}^{+\infty}\big|e^{i\gamma(t,x)}-e^{i\varphi^-(x)}\big|^2 dt\Bigg\}d\mathcal{H}^1(x).
\end{aligned} \tag{4.21}$$

By Lemma 3.2 we obtain

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{\mathbb{R}^2}\frac{1}{\varepsilon}\Big|\nabla\big(\mathrm{div}(\Delta^{-1}\varphi_{n,\varepsilon})\big)(x)\Big|^2 dx\\
&= L_n := \int_{J_\varphi}\Bigg\{\int_{-\infty}^{+\infty}|\varphi^+(x)-\varphi^-(x)|^2\cdot\Bigg|\boldsymbol{\nu}(x)\cdot\int_{t}^{+\infty}q_n(s,x)ds\Bigg|^2 dt\Bigg\}d\mathcal{H}^1(x).
\end{aligned} \tag{4.22}$$

Therefore, letting ε tend to 0 in (4.18), we get,

$$\begin{aligned}
&\overline{\lim_{\varepsilon\to 0^+}}\left|\frac{1}{\varepsilon}\int_{\mathbb{R}^2}\Big|\nabla\,\mathrm{div}\,\Delta^{-1}\big(\chi_\Omega e^{i\psi_\varepsilon}\big)\Big|^2 dx - L_0\right|\\
&\le |L_0 - L_n| + 4\Big(\sqrt{D_n}+\sqrt{T(\bar\eta)}\Big)\sqrt{M+L_n}\,.
\end{aligned} \tag{4.23}$$

Using (4.7), (4.9), (4.13), (4.10) and (4.11) we obtain

$$\lim_{n\to\infty} D_n = 0\,, \tag{4.24}$$

and since $\big(u^+(x) - u^-(x)\big)\perp \boldsymbol{\nu}(x)$ (by $\operatorname{div} u = 0$), we also infer

$$\lim_{n\to\infty} L_n = L_0 := \int_{J_\varphi} \left\{ \int_{-\infty}^{+\infty} \left| \boldsymbol{\nu}(x) \cdot \left(e^{i\gamma(t,x)} - e^{i\varphi^-(x)} \right) \right|^2 dt \right\} d\mathcal{H}^1(x) . \quad (4.25)$$

Therefore, letting n tend to $+\infty$ in (4.23), we obtain,

$$\overline{\lim_{\varepsilon\to 0^+}} \left| \frac{1}{\varepsilon} \int_{\mathbb{R}^2} \left| \nabla \operatorname{div} \Delta^{-1} \big(\chi_\Omega e^{i\psi_\varepsilon}\big) \right|^2 dx - \int_{J_\varphi} \left\{ \int_{-\infty}^{+\infty} \left| \boldsymbol{\nu} \cdot \left(e^{i\gamma(t,\cdot)} - e^{i\varphi^-} \right) \right|^2 dt \right\} d\mathcal{H}^1 \right|$$
$$\leq 4\sqrt{T(\bar\eta)}\sqrt{M + L_0} \,. \quad (4.26)$$

This equation is valid for any $\bar\eta \in \mathcal{V}$, and the constants M and L_0 do not depend on $\bar\eta$. For every $\delta > 0$ we always can choose $\bar\eta_\delta \in C^2(\mathbb{R}^2 \times \mathbb{R}^2, \mathbb{R})$, satisfying $\bar\eta_\delta \geq 0$, $supp\, \bar\eta_\delta \Subset B_\delta(0) \times \mathbb{R}^2$ and $\int_{\mathbb{R}^2} \bar\eta_\delta(z, x) dz = 1$ for any $x \in \Omega$. Then, as before, define $\bar p_\delta(t, x) : \mathbb{R} \times J_\varphi \to \mathbb{R}$ by

$$\bar p_\delta(t, x) = \int_{H^0_{\boldsymbol{\nu}(x)}} \bar\eta_\delta(t\boldsymbol{\nu}(x) + y, x)\, d\mathcal{H}^1(y) \,.$$

Since $\bar p_\delta \geq 0$ and $\operatorname{supp} \bar p_\delta(t, x) \subset [-\delta, \delta] \times J_\varphi$ and $\int_{-\infty}^{\infty} \bar p_\delta(t, x) dt = 1$, by (4.20) we infer

$$T(\bar\eta_\delta)$$
$$\leq \int_{J_\varphi} \left\{ \int_{-\delta}^{0} \left| (u^+ - u^-) \int_{-\infty}^{t} \bar p_\delta(s, \cdot) ds \right|^2 dt + \int_0^\delta \left| (u^+ - u^-) \int_{t}^{+\infty} \bar p_\delta(s, \cdot)\, ds \right|^2 dt \right\} d\mathcal{H}^1$$
$$\leq 2\delta \int_{J_\varphi} |u^+ - u^-|^2 d\mathcal{H}^1 \leq 4\delta \int_{J_\varphi} |u^+ - u^-| d\mathcal{H}^1 \leq 4\delta \|Du\|(\Omega) \,.$$

Therefore, by (4.26) we obtain

$$\overline{\lim_{\varepsilon\to 0^+}} \left| \frac{1}{\varepsilon} \int_{\mathbb{R}^2} \left| \nabla \operatorname{div} \Delta^{-1} \big(\chi_\Omega e^{i\psi_\varepsilon}\big) \right|^2 dx - \int_{J_\varphi} \left\{ \int_{-\infty}^{+\infty} \left| \boldsymbol{\nu} \cdot \left(e^{i\gamma(t,\cdot)} - e^{i\varphi^-} \right) \right|^2 dt \right\} d\mathcal{H}^1 \right|$$
$$\leq 8\sqrt{\delta}\sqrt{\|Du\|(\Omega)}\sqrt{M + L_0} \,. \quad (4.27)$$

For $\delta \to 0$, (4.27) gives (4.2). □

Let φ, $\bar\varphi$ and η be as in Proposition 4.1 and ψ_ε be defined by (4.1). Then using [16, Proposition 3.1], we obtain,

$$\lim_{\varepsilon\to 0}\int_\Omega \varepsilon|\nabla\psi_\varepsilon(x)|^2\,dx=\int_{J_\varphi}\left|\varphi^+(x)-\varphi^-(x)\right|^2\cdot\left(\int_{\mathbb{R}} p^2(t,x)dt\right)d\mathcal{H}^1(x), \quad (4.28)$$

where $p(t,x)$ is defined by (4.4). As in [16] and [17] we also easily deduce that

$$\lim_{\varepsilon\to 0^+}\psi_\varepsilon(x)=\varphi(x) \quad \text{in} \quad L^p(\Omega,\mathbb{R}) \quad \forall p\in[1,\infty)\,.$$

Combining these facts with the result of Proposition 4.1, we infer the following.

Corollary 4.2. *Let $u\in BV(\Omega,S^1)$, satisfying* $\operatorname{div} u=0$ *in Ω and $u\cdot\boldsymbol{n}=0$ on $\partial\Omega$ ($\boldsymbol{n}$ is the unit normal to $\partial\Omega$). Let $\varphi\in BV(\Omega,\mathbb{R})\cap L^\infty$ such that $u=e^{i\varphi}$ a.e. in Ω. Consider a function $\bar\varphi\in BV(\mathbb{R}^2,\mathbb{R})\cap L^\infty$ such that $\bar\varphi=\varphi$ a.e. in Ω,* $\operatorname{supp}\bar\varphi$ *is compact and $\|D\bar\varphi\|(\partial\Omega)=0$. Given $\eta\in\mathcal{V}$, for every $\varepsilon>0$ let ψ_ε be defined by* (4.1)*. Then,*

$$\lim_{\varepsilon\to 0}\left(\int_\Omega \varepsilon\left|\nabla e^{i\psi_\varepsilon(x)}\right|^2dx+\int_{\mathbb{R}^2}\left|\nabla \operatorname{div}\Delta^{-1}\left(\chi_\Omega(x)e^{i\psi_\varepsilon(x)}\right)\right|^2dx\right)$$

$$=Y_\varphi(\eta):=\int_{J_\varphi}\left|\varphi^+(x)-\varphi^-(x)\right|^2\cdot\left(\int_{-\infty}^{+\infty}p^2(t,x)dt\right)d\mathcal{H}^1(x) \quad (4.29)$$

$$+\int_{J_\varphi}\left\{\int_{-\infty}^{+\infty}\left|\boldsymbol{\nu}(x)\cdot\left(e^{i\gamma(t,x)}-e^{i\varphi^-(x)}\right)\right|^2dt\right\}d\mathcal{H}^1(x)\,,$$

where γ and p defined by (4.3) *and* (4.4) *respectively. Moreover,*

$$\lim_{\varepsilon\to 0^+}\psi_\varepsilon(x)=\varphi(x) \quad \textit{in} \quad L^p(\Omega,\mathbb{R}) \quad \forall p\in[1,\infty)\,.$$

Next we turn to the minimization problem of the term on the right hand side of (4.29), over all kernels $\eta\in\mathcal{V}$, analogously to that was done in [16] and [17]. By the same method, as there, we can obtain the following.

Lemma 4.3. *Let $Y_\varphi(\eta):\mathcal{V}\to\mathbb{R}$ be defined as the right hand side of* (4.29)*. Then,*

$$\inf_{\eta\in\mathcal{V}}Y_\varphi(\eta)=\mathcal{J}_0(\varphi)$$

$$:=\int_{J_\varphi}2\left|\varphi^+(x)-\varphi^-(x)\right|\left\{\int_0^1\left|\boldsymbol{\nu}(x)\cdot\left(e^{i\left(s\varphi^-(x)+(1-s)\varphi^+(x)\right)}-e^{i\varphi^-(x)}\right)\right|ds\right\}d\mathcal{H}^1(x)$$

$$=\int_{J_\varphi}2\left|\int_{\varphi^-(x)}^{\varphi^+(x)}\left|\boldsymbol{\nu}(x)\cdot\left(e^{it}-e^{i\varphi^-(x)}\right)\right|dt\right|d\mathcal{H}^1(x)\,. \quad (4.30)$$

By [19, (II.36)] we infer that

$$\begin{aligned}\mathcal{J}_0(\varphi) &= \int_{J_\varphi} 2\left|\int_{\varphi^-(x)}^{\varphi^+(x)} \left|\,\boldsymbol{\nu}(x)\cdot\left(e^{it}-e^{i\varphi^-(x)}\right)\right| dt\right| d\mathcal{H}^1(x)\\ &= 2\int_{\mathbb{R}}\int_{\Omega}\left|\operatorname{div}_x T^t u\right| dx\,dt\,,\end{aligned} \tag{4.31}$$

where we (as in [19]), consider $T^t\varphi := \inf(\varphi, t)$ and $T^t u := e^{i\,T^t\varphi}$.

Proof of Theorem 1.2. The case of $\varphi \in BV(\Omega,\mathbb{R})\cap L^\infty$ follows easily from Corollary 4.2 and Lemma 4.3 by using a standard diagonal argument as in the proofs of [17, Theorem 1.1 and Theorem 1.2].

It remains to consider the case of an unbounded $\varphi \in BV(\Omega,\mathbb{R})$, such that $e^{i\varphi(x)} = u(x)$ a.e. in Ω. First recall that by [6] there exists $\varphi_0 \in BV(\Omega,\mathbb{R})\cap L^\infty(\Omega,\mathbb{R})$ satisfying $e^{i\varphi_0(x)} = u(x)$ a.e. in Ω. Then $\varphi(x) = \varphi_0(x) + 2\pi l(x)$ where $l \in BV(\Omega,\mathbb{Z})$. For each integer $n \geq 1$ define,

$$l_n(x) := \begin{cases} l(x) & x\in\Omega,\ |l(x)|\leq n,\\ n & x\in\Omega,\ l(x)>n,\\ -n & x\in\Omega,\ l(x)<-n\,,\end{cases} \qquad \varphi_n(x) := \varphi_0(x) + 2\pi l_n(x)\,.$$

Clearly $\varphi_n \in BV(\Omega)\cap L^\infty(\Omega)$ and $e^{i\varphi_n(x)} = u(x)$ a.e. in Ω. From the case of a bounded φ, considered above, it follows that for each n there exists a family $\{v_{n,\varepsilon}\}_{\varepsilon>0} \subset C^2(\Omega,\mathbb{R})$ satisfying $\lim_{\varepsilon\to 0} v_{n,\varepsilon}(x) = \varphi_n(x)$ in $L^1(\Omega,\mathbb{R})$ and

$$\begin{aligned}&\lim_{\varepsilon\to 0}\left(\int_\Omega \varepsilon\left|\nabla e^{iv_{n,\varepsilon}(x)}\right|^2 dx + \int_{\mathbb{R}^2}\left|\nabla\operatorname{div}\Delta^{-1}\left(\chi_\Omega(x)e^{iv_{n,\varepsilon}(x)}\right)\right|^2 dx\right)\\ &= \mathcal{J}_0(\varphi_n) = \int_{J_{\varphi_n}} 2\left|\int_{\varphi_n^-(x)}^{\varphi_n^+(x)}\left|\,\boldsymbol{\nu}_n(x)\cdot\left(e^{it}-e^{i\varphi_n^-(x)}\right)\right| dt\right| d\mathcal{H}^1(x)\,.\end{aligned}$$

Since for any $x\in\Omega$ we have $|\varphi_n(x)| \leq |\varphi_0(x)| + 2\pi|l(x)|$ while $\varphi_n(x) = \varphi(x)$ for n sufficiently large, we deduce by dominated convergence that

$$\lim_{n\to\infty}\varphi_n(x) = \varphi(x) \quad \text{in } L^1(\Omega,\mathbb{R})\,.$$

Put $\lambda_n(x) := |\varphi_n^+(x) - \varphi_n^-(x)|$. For $\mathcal{H}^{N-1}$-almost every $x \in J_{\varphi_0}\cup J_l$ we have $\lambda_n(x) \leq \left|\varphi_0^+(x) - \varphi_0^-(x)\right| + 2\pi\left|l^+(x) - l^-(x)\right|$, while $\lambda_n(x) = |\varphi^+(x) - \varphi^-(x)|$ for sufficiently large n. Moreover, $\mathcal{H}^{N-1}\left(J_{\varphi_n}\setminus(J_{\varphi_0}\cup J_l)\right) = 0$ and $\boldsymbol{\nu}_n(x) = \boldsymbol{\nu}(x)$

for $\mathcal{H}^{N-1}$-a.e. $x \in J_{\varphi_n} \cap J_\varphi$, for each n. Therefore, by dominated convergence,

$$\lim_{n\to\infty} \int_{J_{\varphi_n}} 2\left| \int_{\varphi_n^-(x)}^{\varphi_n^+(x)} \left| \boldsymbol{\nu}_n(x) \cdot \left(e^{it} - e^{i\varphi_n^-(x)}\right) \right| dt \right| d\mathcal{H}^1(x)$$

$$= \int_{J_\varphi} 2\left| \int_{\varphi^-(x)}^{\varphi^+(x)} \left| \boldsymbol{\nu}(x) \cdot \left(e^{it} - e^{i\varphi^-(x)}\right) \right| dt \right| d\mathcal{H}^1(x) \,.$$

To complete the proof, we apply to $\{v_{n,\varepsilon}\}$ a standard diagonal argument. □

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