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The asymptotic behaviour of surfaces with prescribed mean curvature near meeting points of fixed and free boundaries

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Abstract. We study the shape of stationary surfaces with prescribed mean curvature in the Euclidean 3-space near boundary points where Plateau boundaries meet free boundaries. In deriving asymptotic expansions at such points, we generalize known results about minimal surfaces due to G. Dziuk. The main difficulties arise from the fact that, contrary to minimal surfaces, surfaces with prescribed mean curvature do not meet the support manifold perpendicularly along their free boundary, in general.

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1. Introduction

A typical boundary value problem for minimal surfaces or, more generally, surfaces with prescribed mean curvature is the following *partially free boundary value problem*: Let $\Gamma \subset \mathbb{R}^3$ denote a closed Jordan arc with endpoints $\mathbf{p}_1 \neq \mathbf{p}_2$, which lie on a two-dimensional, differentiable submanifold $\mathcal{S} \subset \mathbb{R}^3$ without boundary such that $\Gamma \cap \mathcal{S} = \{\mathbf{p}_1, \mathbf{p}_2\}$. Construct a parametrized surface $\mathbf{x} = \mathbf{x}(w) : B^+ \to \mathbb{R}^3$ with prescribed mean curvature $H = H(\mathbf{x}) \in C^0(\mathbb{R}^3, \mathbb{R})$ over the parameter domain

$$B^+ := \Big\{ w = (u, v) = u + iv \; : \; |w| < 1, \; v > 0 \Big\} \subset \mathbb{R}^2 \simeq \mathbb{C}$$

and subject to the boundary conditions

$$\begin{aligned} &\mathbf{x}(w) \in \mathcal{S} \quad \text{for all } w \in I, \\ &\mathbf{x}|_C : C \to \Gamma \text{ continuously and monotonic,} \\ &\mathbf{x}(-1) = \mathbf{p}_1, \quad \mathbf{x}(+1) = \mathbf{p}_2. \end{aligned} \tag{1.1}$$

Here we abbreviated

$$I := (-1, +1) \subset \partial B^+, \quad C := \partial B^+ \setminus I.$$

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(The parameter domain B^+ is chosen appropriately and can be replaced by any simply connected domain $\Omega \subset \mathbb{R}^2$.) Such a *partially free H-surface* can be constructed as a minimizer or, more generally, stationary point of an associated energy functional. In particular, one obtains a partially free minimal surface (that is $H \equiv 0$), in case this functional is Dirichlet's integral.

As a disadvantage of the variational method, the corresponding minimizing or stationary surfaces are *weak* (conformally parametrized) H-surfaces in the analytical and geometrical sense: One has to study their smoothness and immersed character a posteriori. For instance, one can prove that a minimizer $\mathbf{x}$ belongs to $C^\mu(\overline{B^+}, \mathbb{R}^3)$ for some $\mu \in (0, 1)$, whenever $\{\Gamma, \mathcal{S}\}$ satisfies a chord-arc condition; see [4, 13]. And if $\Gamma, \mathcal{S} \in C^{2,\alpha}$, $H \in C^\alpha(\mathbb{R}^3)$ holds true for some $\alpha \in (0, 1)$, we even have

$$\begin{aligned} &\mathbf{x} = \mathbf{x}(w) \in C^2(\overline{B^+} \setminus \{-1, +1\}, \mathbb{R}^3) \cap C^\mu(\overline{B^+}, \mathbb{R}^3) \cap H_2^1(B^+, \mathbb{R}^3), \\ &\Delta \mathbf{x}(w) = 2H(\mathbf{x}(w))\, \mathbf{x}_u \times \mathbf{x}_v(w), \quad w \in B^+, \\ &|\mathbf{x}_u(w)|^2 = |\mathbf{x}_v(w)|^2, \ \mathbf{x}_u \cdot \mathbf{x}_v(w) = 0, \quad w \in B^+; \end{aligned} \tag{1.2}$$

compare [6, 11]. ($H_2^1(B^+, \mathbb{R}^3)$ denotes the Sobolev space of componentially measurable mappings, which are quadratically integrable together with their first distributional derivatives.)

The aim of the present paper is to investigate the behaviour of stationary, partially free H-surfaces $\mathbf{x} \in C^2(\overline{B^+} \setminus \{-1, +1\}, \mathbb{R}^3) \cap C^\mu(\overline{B^+}, \mathbb{R}^3)$ near the corner points $w = \pm 1$, which are mapped onto $\Gamma \cap \mathcal{S}$. In 1981, G. Dziuk [3] could derive asymptotic expansions for $\mathbf{x}$ near these points, provided that $\mathbf{x}$ is a minimial surface or, more exactly, a stationary point of Dirichlet's integral. Besides the exact asymptotic form of $\mathbf{x}$, these expansions provide further information, for instance, the best possible Hölder exponent and the isolated character of $w = \pm 1$ as (geometrically) singular points.

Below we extend G. Dziuk's result to stationary, partially free H-surfaces. The main new difficulty is that H-surfaces do not have to meet the support surface $\mathcal{S}$ perpendicularly along the free trace $\mathbf{x}(I)$, in contrast to minimal surfaces, which are stationary points of the Dirichlet integral. Therefore, we use a generalized reflection across I, which was introduced already in [9, 11]. Following Dziuk's approach, we then have to generalize the two main ingredients in his proof: A growth estimate for the gradient of the stationary H-surface $\mathbf{x}$, which was already done in [12], and a theorem of Hartman-Wintner type, which turns out to be a perturbed version of a previous result by Dziuk [2].

The paper is organized as follows: In Section 2 we specify our notations and assumptions, and we state the main results. Section 3 is addressed to the proof of the Hartman-Wintner type result mentioned just above. In Section 4 we describe our reflection technique. Sections 5 and 6 contain the proofs of the main results and some corollaries, distinguished between the regular and the singular case (see Section 2). Finally, we discuss the branch points of $\mathbf{x}$ in Section 7, and we state a Gauß-Bonnet formula.

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2. Notations and main results

We always write $B_\varrho(w_0) := \{w \in \mathbb{R}^2 \ : \ |w - w_0| < \varrho\}$ for the disc with radius $\varrho > 0$ around $w_0 \in \mathbb{R}^2$, and we abbreviate $B := B_1(0) = B_1(0, 0)$. Moreover, we set $S_\varrho(w_0) := B^+ \cap B_\varrho(w_0)$ and $B^- := B \setminus \overline{B^+}$. Throughout we identify $\mathbb{R}^2 \simeq \mathbb{C}$ via $w = (u, v) = u + iv$, and we often use the *Wirtinger operators*

$$\frac{\partial}{\partial w} := \frac{1}{2}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right), \quad \frac{\partial}{\partial \overline{w}} := \frac{1}{2}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right).$$

Points in $\mathbb{R}^3$ are denoted by $\mathbf{p} = (p^1, p^2, p^3)$ and identified with column vectors, whereas $\mathbf{p}^t$ stands for the corresponding row vector. A ball in $\mathbb{R}^3$ is denoted by $\mathcal{B}_r(\mathbf{p}_0) := \{\mathbf{p} \in \mathbb{R}^3 \ : \ |\mathbf{p} - \mathbf{p}_0| < r\}$ for $\mathbf{p}_0 \in \mathbb{R}^3$, $r > 0$. Finally, all constants c appearing in our estimates are understood to be positive and independent of the particular point w.

Now let $\mathbf{x} = (x^1(w), x^2(w), x^3(w)) : B^+ \to \mathbb{R}^3$ be a partially free H-surface, that means $\mathbf{x}$ solves (1.1), (1.2). We call $\mathbf{x}$ *stationary (with respect to* $E_\mathbf{Q}$*)*, if it satisfies the inequality

$$\delta E_\mathbf{Q}(\mathbf{x}, \boldsymbol{\phi}) := \lim_{\varepsilon \to 0+} \frac{1}{\varepsilon}\big\{E_\mathbf{Q}(\mathbf{x}_\varepsilon) - E_\mathbf{Q}(\mathbf{x})\big\} \geq 0.$$

Here $\mathbf{x}_\varepsilon = \mathbf{x}(\cdot, \varepsilon)$, $\varepsilon \in [0, \varepsilon_0)$, denotes an outer variation of $\mathbf{x}$ with the direction $\frac{\partial}{\partial\varepsilon}\mathbf{x}(\cdot, \varepsilon)\big|_{\varepsilon=0} = \boldsymbol{\phi}$ (confer in [1, Definition 2, Section 5.4]). And $E_\mathbf{Q}$ is the *associated energy functional*

$$E_\mathbf{Q}(\mathbf{x}) := \iint\limits_{B^+} \left\{\frac{1}{2}|\nabla\mathbf{x}(w)|^2 + \mathbf{Q}(\mathbf{x}) \cdot \mathbf{x}_u \times \mathbf{x}_v(w)\right\} du\, dv$$

with a vector-field $\mathbf{Q} = \mathbf{Q}(\mathbf{x}) \in C^1(\mathbb{R}^3, \mathbb{R}^3)$, which satisfies

$$\begin{aligned} &\operatorname{div}\mathbf{Q}(\mathbf{x}) \equiv 2H(\mathbf{x}) \quad \text{in } \mathbb{R}^3, \\ &|\mathbf{Q}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x})| < 1 \quad \text{on } \mathcal{S} \end{aligned} \tag{2.1}$$

for a unit normal field $\mathbf{n} = \mathbf{n}(\mathbf{x})$ of $\mathcal{S}$. The natural boundary condition on I, arising from the stationarity of $\mathbf{x} = \mathbf{x}(w)$, can be written as

$$\mathbf{x}_v(w) + \mathbf{Q}(\mathbf{x}(w)) \times \mathbf{x}_u(w) \ \perp \ T_{\mathbf{x}(w)}\mathcal{S}, \quad w \in I, \tag{2.2}$$

where $T_\mathbf{x}\mathcal{S}$ denotes the tangential plane of $\mathcal{S}$ at $\mathbf{x} \in \mathcal{S}$; see [11].

Remark 2.1. The boundary condition (2.2) is equivalent to the possibly more common relation

$$\mathbf{Q}(\mathbf{x}(w)) \cdot \mathbf{n}(\mathbf{x}(w)) = -\mathbf{N}(w) \cdot \mathbf{n}(\mathbf{x}(w)), \quad w \in I. \tag{2.3}$$

Here $\mathbf{N} = \mathbf{N}(w) := |\mathbf{x}_u \times \mathbf{x}_v|^{-1} \mathbf{x}_u \times \mathbf{x}_v(w)$ denotes the surface normal of $\mathbf{x}$. Clearly, $\mathbf{N}$ is defined only at regular points $w \in \overline{B^+} \setminus \{-1, +1\}$ with $|\nabla \mathbf{x}(w)| \neq 0$, where the equivalence follows by taking the cross product with $\mathbf{x}_u$. On the other hand, we can extend $\mathbf{N}$ continuously to singular points $w \in I$ with $|\nabla \mathbf{x}(w)| \neq 0$, called *branch points*, in virtue of Theorem 2 in [11].

Relation (2.3) says that the contact angle between the stationary H-surface $\mathbf{x}$ (with respect to $E_{\mathbf{Q}}$) and the support surface $\mathcal{S}$ is prescribed by the normal component of the vector-field $\mathbf{Q}$.

In order to state our main results, which examine the behaviour of stationary, partially free H-surfaces near the corner points $w = \pm 1$, we may localize around one of these points; note the conformal invariance of the problem and the continuity of $\mathbf{x}(w)$. Concerning the boundary configuration we then assume (substitute Γ, $\mathcal{S}$ by $\Gamma \cap \mathcal{U}$, $\mathcal{S} \cap \mathcal{U}$ with some neighbourhood $\mathcal{U}$ of either $\mathbf{p}_1 = \mathbf{x}(-1)$ or $\mathbf{p}_2 = \mathbf{x}(+1)$ and apply a suitable rotation and translation):

Assumption (A). *Let $\Gamma, \mathcal{S} \in C^2$ be true and suppose $\Gamma \cap \mathcal{S} = \{\mathbf{0}\}$, where $\mathbf{0} \in \Gamma$ is an endpoint of Γ. Furthermore, we have the represention*

$$\Gamma = \{\mathbf{x} = (\gamma_1(x^3), \gamma_2(x^3), x^3) \; : \; x^3 \in [0, r]\}$$

with $\gamma_1, \gamma_2 \in C^2([0, r], [0, +\infty))$ for some $r > 0$ and with $\gamma_1(0) = \gamma_2(0) = 0$ as well as $\gamma_1'(0) = \gamma_2'(0) = 0$. The support surface $\mathcal{S}$ can be written as

$$\mathcal{S} = \{\mathbf{x} = (x^1, x^2, \psi(x^1, x^2)) \; : \; (x^1, x^2) \in B_r(0, 0)\}$$

with $\psi \in C^2(\overline{B_r(0, 0)}, \mathbb{R})$, and we assume

$$\mathbf{n}(x^1, x^2) = \frac{\big(-\psi_{x^1}(x^1, x^2), -\psi_{x^2}(x^1, x^2), 1\big)}{\sqrt{1 + \psi_{x^1}^2 + \psi_{x^2}^2}}, \quad (x^1, x^2) \in B_r(0, 0),$$

for the unit normal field $\mathbf{n} = \mathbf{n}(x^1, x^2)$ of $\mathcal{S}$, which we trivially extend to $\mathbf{n}(\mathbf{x}) := \mathbf{n}(x^1, x^2)$, $\mathbf{x} = (x^1, x^2, x^3) \in \mathcal{B}_r(\mathbf{0})$. Finally, let $\psi(0, 0) = 0$ as well as $\psi_{x^1}(0, 0) = a$ and $\psi_{x^2}(0, 0) = 0$ hold true with some number $a \in [0, +\infty)$.

The angle $\alpha \in (0, \frac{\pi}{2}]$ between Γ and $\mathcal{S}$ at $\mathbf{0}$ is then calculated by

$$a = \cot \alpha. \tag{2.4}$$

We can (and will do occasionally) diminish $r > 0$ in order to gain additional properties of the boundary configuration. For the stationary, partially free H-surface we may suppose (compare the proof of Corollary 7.1 for the necessary conformal reparametrization):

Assumption (B). *The mapping* $\mathbf{x} = \mathbf{x}(w) \in C^2(\overline{B^+} \setminus \{0\}) \cap C^\mu(\overline{B^+}) \cap H_2^1(B^+)$ *satisfies the differential system* (1.2) *as well as* $|\mathbf{x}(w)| < r$ *for all* $w \in \overline{B^+}$. *Setting* $I^- := (-1, 0)$, $I^+ := (0, +1)$, *we have the boundary conditions*

(i) $\mathbf{x}|_{I^-} : I^- \to \Gamma$, $\mathbf{x}|_{I^+} : I^+ \to \mathcal{S}$, $\mathbf{x}(0) = \mathbf{0}$
(ii) $\mathbf{x}_v + \mathbf{Q}(\mathbf{x}) \times \mathbf{x}_u \perp T_{\mathbf{x}}\mathcal{S}$ *on* I^+

with a vector-field $\mathbf{Q} = \mathbf{Q}(\mathbf{x}) \in C^1(\mathbb{R}^3, \mathbb{R}^3)$ *satisfying* (2.1).

Theorem 2.2. *Let* $\mathbf{x} = \mathbf{x}(w) : B^+ \to \mathbb{R}^3$ *be a stationary, partially free* H*-surface satisfying assumption (B) with a vector-field* $\mathbf{Q} \in C^1(\mathbb{R}^3, \mathbb{R}^3)$ *as in* (2.1). *Assume that the boundary configuration* $\{\Gamma, \mathcal{S}\}$ *fulfils assumption (A) and that*

$$|\mathbf{Q}(\mathbf{0}) \cdot \mathbf{n}(\mathbf{0})| < \cos\alpha \tag{2.5}$$

holds true with the angle $\alpha \in (0, \frac{\pi}{2})$ *between* Γ *and* $\mathcal{S}$*; compare* (2.4).
Then there exists a mapping $\mathbf{\Phi} = \mathbf{\Phi}(w) \in C^0(\overline{B^+}, \mathbb{C}^3)$ *and a number* $m \in \mathbb{N} \cup \{0\}$ *such that the relation*

$$\mathbf{x}_w(w) = w^{m-\kappa}\mathbf{\Phi}(w), \quad w \in \overline{B^+} \setminus \{0\}, \tag{2.6}$$

holds true with either $\kappa = 1 - \gamma_0$ *or* $\kappa = \gamma_0$*; here* $\gamma_0 \in (0, \frac{1}{2})$ *is given by* (5.12) *below. Writing* $\mathbf{\Phi} = (\Phi^1, \Phi^2, \Phi^3)$ *we have* $e^{-i\pi\kappa}\Phi^3(0) \in \mathbb{R} \setminus \{0\}$ *and moreover*

$$\Phi^1(0) = \pm i\sqrt{1 - \frac{[\mathbf{Q}(\mathbf{0}) \cdot \mathbf{n}(\mathbf{0})]^2}{\cos^2\alpha}}\,\Phi^3(0), \quad \Phi^2(0) = i\,\frac{\mathbf{Q}(\mathbf{0}) \cdot \mathbf{n}(\mathbf{0})}{\cos\alpha}\,\Phi^3(0). \tag{2.7}$$

The continuity of $\mathbf{\Phi}$ implies that the surface normal of $\mathbf{x} = \mathbf{x}(w)$ can be extended to $w = 0$ continuously, see Theorem 5.4 in Section 5. This property depends on the assumption (2.5) and is refered to as the *regular case*. The contrary situation is named *irregular case* and included in the following

Theorem 2.3. *Let the assumptions of Theorem* 2.2 *be satisfied except for the relation* (2.5)*, which has to be replaced by*

$$|\mathbf{Q}(\mathbf{0}) \cdot \mathbf{n}(\mathbf{0})| > \cos\alpha > 0. \tag{2.8}$$

Then there exist an integer $m \in \mathbb{N} \cup \{0\}$ *and a function* $\mathbf{\Phi} \in C^0(\overline{B^+} \setminus \{0\}, \mathbb{C}^3)$ *such that the relation*

$$\mathbf{x}_w(w) = w^{m-\frac{1}{2}}\mathbf{\Phi}(w), \quad w \in \overline{B^+} \setminus \{0\}, \tag{2.9}$$

is valid. Furthermore, the function $\mathbf{\Phi}$ *behaves discontinuously for* $w \to 0$*, and there exist numbers* $\delta \in (0, 1)$ *and* $c \geq 1$ *such that*

$$c^{-1} \leq |\mathbf{\Phi}(w)| \leq c \quad \text{for all } w \in \overline{S_\delta(0)} \setminus \{0\}$$

holds true.

A remark on the borderline case $|\mathbf{Q}(\mathbf{0})\cdot\mathbf{n}(\mathbf{0})| = \cos\alpha$ is included at the end of Section 5. Let us note that the irregular case will not appear, whenever the relation $|\mathbf{Q}\cdot\mathbf{n}| \equiv 0$ is satisfied on $\mathcal{S}$. In particular, this holds true in the minimal surface case $\mathbf{Q}\equiv\mathbf{0}$ on $\mathbb{R}^3$ studied by G. Dziuk [3]; see also [1, Section 8.4]. Now Theorem 2.2 proves (compare Remark 5.3 in Section 5):

Corollary 2.4. *Under the assumptions of Theorem 2.2, a stationary, partially free H-surface behaves near $w=\pm1$ asymptotically like a stationary, partially free minimal surface, if and only if $\mathbf{Q}(\mathbf{0})\cdot\mathbf{n}(\mathbf{0})=0$ holds true.*

Further applications of Theorems 2.2 and 2.3 appear in Sections 5-7.

As mentioned in the introduction, we will follow the arguments of G. Dziuk [3]. The basic idea is as follows: Reflect $\mathbf{x}_w$ in an appropriate manner across I, such that the resulting quantity remains continuous, say, on I^+. The unavoidable jump on I^- will be smoothed by multiplication with some matrix-valued function, which is primarily built by the eigenvalues and eigenvectors of the matrix $R_\Gamma(0)\circ R_{\mathcal{S}}(\mathbf{x})$; see the Sections 4-6. Then a theorem of Hartman-Wintner type proved in Section 3 can be applied, according to the gradient estimates of [12].

3. A result of Hartman-Wintner type

In this section we shall prove a perturbed version of Hilfssätze 5-7 in [2] or Theorem 3 in [1, Section 8.1], namely

Lemma 3.1. *Let $\alpha\in(0,\frac12]$ and $\nu\in(0,1)$ be given. Then there exists $\varepsilon_0=\varepsilon_0(\alpha,\nu)>0$ such that any two functions $F,G\in C^{0,1}(B\setminus\{0\},\mathbb{C})$ satisfying*

$$|F(w)|\le c|w|^{\nu-\alpha},\quad |G(w)|\le c|w|^{\nu-\beta}\quad \text{on } B\setminus\{0\} \tag{3.1}$$

as well as

$$\begin{aligned}|F_{\overline{w}}(w)| &\le c\Big\{|w|^{-\beta}|F(w)|^2+|w|^{1-3\alpha}|G(w)|^2\Big\},\\ |G_{\overline{w}}(w)| &\le c\Big\{|w|^{1-3\beta}|F(w)|^2+|w|^{-\alpha}|G(w)|^2\Big\}\quad \text{a.e. on } B\setminus\{0\}\end{aligned} \tag{3.2}$$

with $\beta:=1-\alpha+\varepsilon$ and $\varepsilon\in[0,\varepsilon_0)$ have the following properties: There is a number $m\in\mathbb{N}\cup\{0\}$ such that the functions

$$f^m(w):=w^{-m}F(w),\quad g^m(w):=w^{-m}G(w)$$

fulfil $f^m\in C^\mu(B,\mathbb{C})$ for all $0<\mu<\min\{1,m+\alpha-\varepsilon\}$ as well as the alternative:

(i) *In the case $f^m(0)\ne0$ we obtain*

$$|f^m_{\overline{w}}(w)|\le c|w|^{m-\beta},\quad |g^m_{\overline{w}}(w)|\le c|w|^{m+1-3\beta}\quad \text{a.e. on } B\setminus\{0\}. \tag{3.3}$$

Furthermore, g^m is continuous in B provided $m>0$ holds true.

(ii) *If $f^m(0) = 0$ is valid, then we find $g^m \in C^\mu(B, \mathbb{C})$ for all $0 < \mu < \min\{1, m + \beta - \varepsilon\}$ and $g^m(0) \neq 0$. In addition, we have the estimates*

$$|f^m_{\overline{w}}(w)| \le c|w|^{m+1-3\alpha}, \quad |g^m_{\overline{w}}(w)| \le c|w|^{m-\alpha} \quad \textit{a.e. on } B \setminus \{0\}. \tag{3.4}$$

Remark 3.2. Lemma 3.1 clearly remains true if we replace the unit disc B by some disc $B_\delta(0)$ of radius $\delta \in (0, 1)$.

The proof of Lemma 3.1 will be based on the following:

Proposition 3.3. *Let $\alpha \in (0, \frac{1}{2}]$, $\nu \in (0, 1)$ and two functions F, G be given as in Lemma 3.1. Then there exists $\varepsilon_0 = \varepsilon_0(\alpha, \nu) > 0$ such that $F \in C^\mu(B, \mathbb{C})$ is satisfied for all $\mu \in (0, \alpha - \varepsilon)$. Moreover, the estimates*

$$|F_{\overline{w}}(w)| \le c|w|^{-\beta}, \quad |G_{\overline{w}}(w)| \le c|w|^{1-3\beta} \quad \textit{a.e. in } B \setminus \{0\} \tag{3.5}$$

hold true with $\beta = 1 - \alpha + \varepsilon$ and $\varepsilon \in [0, \varepsilon_0)$.

In the case $F(0) = 0$, we additionally have the regularity $G \in C^\mu(B, \mathbb{C})$ for all $\mu \in (0, \beta - \varepsilon)$, and the improved estimates

$$|F_{\overline{w}}(w)| \le c|w|^{1-3\alpha}, \quad |G_{\overline{w}}(w)| \le c|w|^{-\alpha} \quad \textit{a.e. in } B \setminus \{0\} \tag{3.6}$$

are satisfied.

In the proof of Proposition 3.3 we will frequently use the following:

Proposition 3.4. (Lemma 6 in [1, Section 8.1])
Let $f = f(w) \in C^{0,1}(B \setminus \{0\}, \mathbb{C})$ satisfy

$$|f(w)| \le c|w|^{\lambda'} \quad \textit{and} \quad |f_{\overline{w}}(w)| \le c|w|^{\lambda}$$

with $\lambda' > -1$, $\lambda > -2$. Then the following estimates are valid:

(i) $|f(w)| \le c|w|^{1+\lambda}$, *if $\lambda < -1$,*
(ii) $|f(w)| \le c|w|^{-\delta}$ *for all $\delta > 0$, if $\lambda = -1$,*
(iii) $f \in C^\mu(B, \mathbb{C})$ *for all $0 < \mu < \min(1, 1 + \lambda)$, if $\lambda > -1$.*

Proof of Proposition 3.3.

1. Because (3.1) remains true if we diminish $\nu \in (0, 1)$, we may assume $2^{k_0}\nu < \alpha < 2^{k_0+1}\nu$ for some $k_0 \in \mathbb{N} \cup \{0\}$. Additionally, let $\alpha \neq 1 - 2^{k_0+1}\nu$ be fulfilled. Suppose that F, G satisfy the inequalities

$$\begin{aligned} &|F(w)| \le c|w|^{2^k\nu - \alpha - 2(2^k - 1)\varepsilon}, \\ &|G(w)| \le c|w|^{2^k\nu - \beta - 2(2^k - 1)\varepsilon} \quad \text{in } B \setminus \{0\} \end{aligned} \tag{3.7}$$

for some nonnegative integer $k < k_0$. According to (3.1), formula (3.7) is valid for $k = 0$. If we insert (3.7) into (3.2), then we infer

$$\begin{aligned} |F_{\overline{w}}(w)| &\le c\Big\{|w|^{2^{k+1}\nu-2\alpha-\beta-4(2^k-1)\varepsilon} + |w|^{2^{k+1}\nu+1-3\alpha-2\beta-4(2^k-1)\varepsilon}\Big\} \\ &\le c|w|^{2^{k+1}\nu-1-\alpha-2(2^{k+1}-1)\varepsilon} \end{aligned} \tag{3.8}$$

as well as

$$\begin{aligned} |G_{\overline{w}}(w)| &\le c\Big\{|w|^{2^{k+1}\nu+1-2\alpha-3\beta-4(2^k-1)\varepsilon} + |w|^{2^{k+1}\nu-\alpha-2\beta-4(2^k-1)\varepsilon}\Big\} \\ &\le c|w|^{2^{k+1}\nu-1-\beta-2(2^{k+1}-1)\varepsilon}. \end{aligned} \tag{3.9}$$

Note that $2^k\nu - \beta - 2(2^k - 1)\varepsilon > -1$ holds true for all $k \le k_0$ and $\varepsilon \in [0, \varepsilon_0)$ with a sufficiently small $\varepsilon_0 = \varepsilon_0(\alpha, \nu)$. For $k < k_0$ we also have $2^{k+1}\nu - 1 - \alpha - 2(2^{k+1} - 1)\varepsilon < -1$. By virtue of $\beta \ge \alpha$, we thus obtain

$$\begin{aligned} |F(w)| &\le c|w|^{2^{k+1}\nu-\alpha-2(2^{k+1}-1)\varepsilon}, \\ |G(w)| &\le c|w|^{2^{k+1}\nu-\beta-2(2^{k+1}-1)\varepsilon} \quad \text{in } B \setminus \{0\} \end{aligned}$$

from Proposition 3.4, *i.e.* (3.7) holds true for $k + 1 \le k_0$, too. Iterating over $k = 0, \dots, k_0$, we then find (3.7) and also (3.8)-(3.9) for $k = k_0$.
Next we may choose $\varepsilon_0 = \varepsilon_0(\alpha, \nu)$ sufficiently small to ensure $2^{k_0+1}\nu - 1 - \alpha - 2(2^{k_0+1} - 1)\varepsilon - \varepsilon > -1$ as well as $2^{k_0+1}\nu - 1 - \beta - 2(2^{k_0+1} - 1)\varepsilon \ne -1$ for all $\varepsilon \in [0, \varepsilon_0)$. From Proposition 3.4 we thus infer

$$|F(w)| \le c, \quad |G(w)| \le c\big(1 + |w|^{2^{k_0+1}\nu-\beta-2(2^{k_0+1}-1)\varepsilon}\big) \quad \text{in } B \setminus \{0\}.$$

Consequently, we deduce

$$\begin{aligned} |F_{\overline{w}}| &\le c\Big\{|w|^{-\beta} + |w|^{1-3\alpha} + |w|^{2[2^{k_0+1}\nu-\alpha-2(2^{k_0+1}-1)\varepsilon-\varepsilon]+\alpha-1}\Big\} \\ &\le c|w|^{-\beta} \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} |G_{\overline{w}}| &\le c\Big\{|w|^{1-3\beta} + |w|^{-\alpha} + |w|^{2[2^{k_0+1}\nu-\alpha-2(2^{k_0+1}-1)\varepsilon]+\alpha-2\beta}\Big\} \\ &\le c|w|^{1-3\beta}, \end{aligned} \tag{3.11}$$

in virtue of (3.2) and $\alpha \le \frac{1}{2}$. Finally, Proposition 3.4 implies $F \in C^\mu(B, \mathbb{C})$ for all $\mu \in (0, \alpha - \varepsilon)$, if we suppose $\varepsilon < \varepsilon_0 \le \alpha$.

2. Now let $F(0) = 0$ hold true. Then we have $|F(w)| \le |w|^\mu$ for $0 < \mu < \alpha - \varepsilon$ and the function $f(w) := w^{-1}F(w)$ satisfies

$$|f(w)| \le c|w|^{\mu-1} \quad \text{in } B \setminus \{0\}. \tag{3.12}$$

Because of $f_{\overline{w}}(w) = w^{-1} F_{\overline{w}}(w)$, we obtain

$$\begin{aligned} |f_{\overline{w}}(w)| &\le c\Big\{|w|^{1-\beta}|f(w)|^2 + |w|^{-3\alpha}|G(w)|^2\Big\}, \\ |G_{\overline{w}}(w)| &\le c\Big\{|w|^{3(1-\beta)}|f(w)|^2 + |w|^{-\alpha}|G(w)|^2\Big\} \quad \text{a.e. in } B \setminus \{0\} \end{aligned} \tag{3.13}$$

from the assumption (3.2). We now have to distinguish between three cases:

(a) $1 - 3\beta > -1$, that means $\alpha > \frac{1}{3} + \varepsilon$. With the aid of Proposition 3.4 we find $|G(w)| \le c$ according to (3.11). Inserting this and (3.12) into (3.13), we infer

$$\begin{aligned} |f_{\overline{w}}(w)| &\le c\big\{|w|^{-1-\beta+2\mu} + |w|^{-3\alpha}\big\} \le c|w|^{-3\alpha}, \\ |G_{\overline{w}}(w)| &\le c\big\{|w|^{1-3\beta+2\mu} + |w|^{-\alpha}\big\} \le c|w|^{-\alpha} \quad \text{a.e. in } B \setminus \{0\}, \end{aligned}$$

since we may choose $\mu = \frac{1}{3} - \frac{1}{2}\varepsilon < \alpha - \varepsilon$, *e.g.* This implies (3.6) as well as $G \in C^\mu(B, \mathbb{C})$ for $\mu \in (0, \beta - \varepsilon)$.

(b) $1 - 3\beta = -1$, that is $\alpha = \frac{1}{3} + \varepsilon$. From (3.11) and Proposition 3.4 we infer $|G(w)| \le c|w|^{-\delta}$ for arbitrary $\delta > 0$. Therefore, formulas (3.12) and (3.13) yield

$$\begin{aligned} |f_{\overline{w}}(w)| &\le c\big\{|w|^{-1-\beta+2\mu} + |w|^{-3\alpha-2\delta}\big\} \le c|w|^{-3\alpha-2\delta}, \\ |G_{\overline{w}}(w)| &\le c\big\{|w|^{1-3\beta+2\mu} + |w|^{-\alpha-2\delta}\big\} \le c|w|^{-\alpha-2\delta} \end{aligned}$$

a.e. in $B \setminus \{0\}$, in case we choose $\mu = \frac{1}{3} - \frac{1}{2}\varepsilon - \delta < \alpha - \varepsilon$ with sufficiently small $\delta > 0$. Now Proposition 3.4 provides the estimates $|f(w)| \le c|w|^{1-3\alpha-2\delta}$ and $|G(w)| \le c$, which in turn imply

$$\begin{aligned} |f_{\overline{w}}(w)| &\le c\big\{|w|^{1-\beta+2(1-3\alpha-2\delta)} + |w|^{-3\alpha}\big\} \le c|w|^{-3\alpha}, \\ |G_{\overline{w}}(w)| &\le c\big\{|w|^{3(1-\beta)+2(1-3\alpha-2\delta)} + |w|^{-\alpha}\big\} \le c|w|^{-\alpha} \end{aligned}$$

a.e. in $B \setminus \{0\}$. Here we have taken $\varepsilon \in [0, \varepsilon_0)$ and $\varepsilon_0 = \varepsilon_0(\alpha) > 0$, $\delta > 0$ sufficiently small. Thus we find $G \in C^\mu(B, \mathbb{C})$ for any $\mu \in (0, \beta - \varepsilon)$, and the estimates (3.6) are satisfied.

(c) $1 - 3\beta < -1$, hence $\alpha < \frac{1}{3} + \varepsilon$. By virtue of (3.11) and Proposition 3.4, we conclude

$$|G(w)| \le c|w|^{2-3\beta} \le c|w|^{\mu+\alpha-\beta} \quad \text{in } B \setminus \{0\} \tag{3.14}$$

for any $\mu \in (0, \alpha - 2\varepsilon]$. If $\mu > 0$ is small enough, then there exists $k_0 \in \mathbb{N} \cup \{0\}$ such that $2^{k_0}(\mu+\alpha) < 1-\alpha < 2^{k_0+1}(\mu+\alpha)$ as well as $2^{k_0}(\mu+\alpha) \ne 1+\alpha$ hold true (again, we may exclude the equalities $2^{k_0}(\mu + \alpha) = 1 \pm \alpha$, taking $\mu > 0$ even smaller). Now let us suppose

$$\begin{aligned} |f(w)| &\le c|w|^{2^k(\mu+\alpha)-1-\alpha-2(2^k-1)\varepsilon}, \\ |G(w)| &\le c|w|^{2^k(\mu+\alpha)-\beta-2(2^k-1)\varepsilon} \quad \text{in } B \setminus \{0\} \end{aligned} \tag{3.15}$$

for some $k \in \{0, 1, \dots, k_0 - 1\}$. According to (3.12) and (3.14), this is already true for $k = 0$. Formulas (3.13) and (3.15) imply

$$\begin{aligned} |f_{\overline{w}}(w)| &\leq c|w|^{2^{k+1}(\mu+\alpha)-1-\beta-2\alpha-4(2^k-1)\varepsilon} \\ &\quad + c|w|^{2^{k+1}(\mu+\alpha)-3\alpha-2\beta-4(2^k-1)\varepsilon} \\ &\leq c|w|^{2^{k+1}(\mu+\alpha)-2-\alpha-2(2^{k+1}-1)\varepsilon} \quad \text{a.e. in } B \setminus \{0\} \end{aligned} \tag{3.16}$$

and also

$$\begin{aligned} |G_{\overline{w}}(w)| &\leq c|w|^{2^{k+1}(\mu+\alpha)+1-2\alpha-3\beta-4(2^k-1)\varepsilon} \\ &\quad + c|w|^{2^{k+1}(\mu+\alpha)-\alpha-2\beta-4(2^k-1)\varepsilon} \\ &\leq c|w|^{2^{k+1}(\mu+\alpha)-1-\beta-2(2^{k+1}-1)\varepsilon} \quad \text{a.e. in } B \setminus \{0\}. \end{aligned} \tag{3.17}$$

Taking $\varepsilon < \varepsilon_0(\mu, \alpha)$ sufficiently small, we obtain

$$\begin{aligned} &|f(w)| \leq c|w|^{2^{k+1}(\mu+\alpha)-1-\alpha-2(2^{k+1}-1)\varepsilon}, \\ &|G(w)| \leq c|w|^{2^{k+1}(\mu+\alpha)-\beta-2(2^{k+1}-1)\varepsilon} \quad \text{in } B \setminus \{0\} \end{aligned}$$

from Proposition 3.4. Hence the estimates (3.15) hold true for $k + 1 \leq k_0$, as well. Iteration over $k = 0, \dots, k_0$ yields (3.15)-(3.17) with $k = k_0$, and for small $\varepsilon < \varepsilon_0(\mu, \alpha)$ we infer

$$|f(w)| \leq c\big(1 + |w|^{2^{k_0+1}(\mu+\alpha)-1-\alpha-2(2^{k_0+1}-1)\varepsilon}\big), \quad |G(w)| \leq c \tag{3.18}$$

in $B \setminus \{0\}$ as a consequence of Proposition 3.4. Finally, we insert (3.18) into (3.13) and conclude

$$|f_{\overline{w}}(w)| \leq c|w|^{-3\alpha}, \quad |G_{\overline{w}}(w)| \leq c|w|^{-\alpha} \quad \text{a.e. } B \setminus \{0\}$$

for sufficiently small $\varepsilon < \varepsilon_0(\mu, \alpha)$. This completes the proof of (3.6), and the property $G \in C^\mu(B, \mathbb{C})$ for all $\mu \in (0, \beta - \varepsilon)$ follows again from Proposition 3.4. □

Now we are able to give the

Proof of Lemma 3.1. Let $F(0) \neq 0$ be valid. Then case (i) of Lemma 3.1 is satisfied with $m = 0$, due to the first assertion of Proposition 3.3. On the other hand, in accordance with the second assertion of Proposition 3.3, the case (ii) is fulfilled with $m = 0$, whenever we have $F(0) = 0$ and $G(0) \neq 0$. Thus the case $F(0) = G(0) = 0$ remains to be considered.

Setting $f(w) := w^{-1}F(w)$ and $g(w) := w^{-1}G(w)$, we infer

$$|f_{\overline{w}}(w)| \leq c|w|^{-3\alpha}, \quad |g_{\overline{w}}(w)| \leq c|w|^{-1-\alpha} \quad \text{a.e. in } B \setminus \{0\}$$

from Proposition 3.3, and Proposition 3.4 implies

$$|f(w)| \leq \begin{cases} c, & \text{if } \alpha < \frac{1}{3} \\ c|w|^{-\delta} \text{ for all } \delta > 0, & \text{if } \alpha = \frac{1}{3} \\ c|w|^{1-3\alpha}, & \text{if } \alpha > \frac{1}{3} \end{cases}, \qquad |g(w)| \leq c|w|^{-\alpha}$$

in $B \setminus \{0\}$. If we distingiush between the three cases $\alpha <, =, > \frac{1}{3}$, we find $\lambda, \lambda' \in (0, 1)$ such that the estimates

$$|\mathbf{h}(w)| \leq c|w|^{-\lambda'} \quad \text{in } B \setminus \{0\},$$
$$|\mathbf{h}_{\overline{w}}(w)| \leq c|w|^{-\lambda}|\mathbf{h}(w)| \quad \text{a.e. in } B \setminus \{0\}$$

hold true for the mapping $\mathbf{h} := (f, g) \in C^{0,1}(B \setminus \{0\}, \mathbb{C})$. Here we have chosen again $\varepsilon < \varepsilon_0(\alpha)$ sufficiently small. Theorem 1 in [1, Section 8.1] (see also the original work [5]) now provides the asymptotic relation

$$\mathbf{h}(w) = \mathbf{a}w^{m-1} + o(|w|^{m-1}) \quad \text{as } w \to 0$$

with some $m \in \mathbb{N}$ and a vector $\mathbf{a} = (a^1, a^2) \in \mathbb{C}^2 \setminus \{\mathbf{0}\}$. For the functions $f^m = w^{-m}F$, $g^m = w^{-m}G$ we find

$$f^m(w) = a^1 + o(1), \quad g^m(w) = a^2 + o(1) \quad \text{as } w \to 0,$$

and from (3.2) we derive

$$\begin{aligned} |f^m_{\overline{w}}(w)| &\leq c\Big\{|w|^{m-\beta}|f^m(w)|^2 + |w|^{m+1-3\alpha}|g^m(w)|^2\Big\} \leq c|w|^{m-\beta}, \\ |g^m_{\overline{w}}(w)| &\leq c\Big\{|w|^{m+1-3\beta}|f^m(w)|^2 + |w|^{m-\alpha}|g^m(w)|^2\Big\} \leq c|w|^{m+1-3\beta} \end{aligned} \tag{3.19}$$

using $\alpha \leq \frac{1}{2}$. Therefore, Proposition 3.4 yields $f^m \in C^\mu(B, \mathbb{C})$ for all $\mu \in (0, 1)$. Consequently, we again have proved case (i) in Lemma 3.1 provided $f^m(0) = a^1 \neq 0$ holds true. Otherwise, we find $|f^m(w)| \leq c|w|^\mu$ for any $\mu \in (0, 1)$. Inserting this into (3.19), we finally obtain

$$\begin{aligned} |f^m_{\overline{w}}(w)| &\leq c\big\{|w|^{m-\beta+2\mu} + |w|^{m+1-3\alpha}\big\} \leq c|w|^{m+1-3\alpha}, \\ |g^m_{\overline{w}}(w)| &\leq c\big\{|w|^{m+1-3\beta+2\mu} + |w|^{m-\alpha}\big\} \leq c|w|^{m-\alpha} \quad \text{in } B \setminus \{0\}, \end{aligned}$$

whenever we select $\mu = 1 - 2\alpha + \frac{3}{2}\varepsilon \in (0, 1)$ with $\varepsilon < \frac{4}{3}\alpha$. By virtue of Proposition 3.4, we infer $g^m \in C^\mu(B, \mathbb{C})$ for all $\mu \in (0, 1)$. Because $g^m(0) = a^2 \neq 0$ must hold true, the case (ii) in Lemma 3.1 is fulfilled. This completes the proof of Lemma 3.1. $\square$

4. The generalized reflection

Let us suppose that the boundary configuration $\{\Gamma, \mathcal{S}\}$ satisfies assumption (A) with sufficiently small $r > 0$. Reflecting γ_k, $k = 1, 2$, by virtue of

$$\hat{\gamma}_k(t) := \begin{cases} \gamma_k(t), & t \in [0, r] \\ \gamma_k(-t), & t \in [-r, 0] \end{cases}, \quad k = 1, 2,$$

we obtain functions $\hat{\gamma}_1, \hat{\gamma}_2 \in C^2([-r, r], \mathbb{R})$, which describe the extended Jordan arc $\hat{\Gamma}$. We set

$$\mathbf{h}(x^3) := \frac{1}{1 + \hat{\gamma}_1'(x^3)^2 + \hat{\gamma}_2'(x^3)^2} \begin{pmatrix} \hat{\gamma}_1'(x^3) \\ \hat{\gamma}_2'(x^3) \\ 1 \end{pmatrix}, \quad x^3 \in [-r, r],$$

and define the reflection across $\hat{\Gamma}$ by

$$R_\Gamma(x^3)\mathbf{p} := 2\big(\mathbf{h}(x^3) \cdot \mathbf{p}\big)\mathbf{h}(x^3) - \mathbf{p} \quad \text{for } \mathbf{p} \in \mathbb{R}^3,\ x^3 \in [-r, r]. \tag{4.1}$$

Now let $\mathbf{x} = \mathbf{x}(w) : B^+ \to \mathbb{R}^3$ be a stationary, partially free H-surface, which fulfils assumption (B). From the boundary condition $\mathbf{x}(I^-) \subset \Gamma$ and the conformality relations we then infer (see [3, Lemma 1]):

$$R_\Gamma(x^3(w))\mathbf{x}_w(w) = \mathbf{x}_{\overline{w}}(w) \quad \text{for all } w \in I^-. \tag{4.2}$$

As in [3] and [1, Section 8.4], we apply a straightening procedure. However, we have to restrict ourselfes to $\hat{\Gamma}$ in contrast to the minimal surface case, since the eigenvalues of the refelction $R_{\mathcal{S}} = R_{\mathcal{S}}(\mathbf{x})$ depend on the particular point $\mathbf{x} \in \mathcal{B}_r(\mathbf{0})$; compare the considerations below.

Let us define the cone

$$C_\lambda := \big\{\mathbf{x} = (x^1, x^2, x^3) \in \mathbb{R}^3 \ :\ (x^1)^2 + (x^2)^2 \le \lambda (x^3)^2\big\}.$$

For sufficiently small $r > 0$ and $\lambda > 0$ we surely have $\hat{\Gamma} \subset C_\lambda$ and $(\mathcal{S} \setminus \{\mathbf{0}\}) \subset \mathbb{R}^3 \setminus C_{2\lambda}$. Let $\eta = \eta(\mathbf{x}) \in C^1(\mathbb{R}^3 \setminus \{\mathbf{0}\}, [0, 1])$ be a function with the properties

$$\eta(\mathbf{x}) = \begin{cases} 1, & \text{on } C_\lambda \\ 0, & \text{on } \mathbb{R}^3 \setminus C_{2\lambda} \end{cases}$$

and

$$|\nabla \eta(\mathbf{x})| \le \frac{c}{|\mathbf{x}|} \quad \text{on } \mathbb{R}^3 \setminus \{\mathbf{0}\}. \tag{4.3}$$

Setting additionally $\eta(\mathbf{0}) = 0$, we consider the matrix-valued mapping

$$T(\mathbf{x}) = \eta(\mathbf{x})(T_\Gamma(x^3) - \mathrm{Id}) + \mathrm{Id}, \quad \mathbf{x} \in \mathcal{B}_r(\mathbf{0}). \tag{4.4}$$

Here Id denotes the 3×3-identity matrix and $T_\Gamma = T_\Gamma(x^3)$ is defined as follows: We identify $R_\Gamma = R_\Gamma(x^3)$ with its matrix representation and write $O_\Gamma \in C^1([-r, r], \mathbb{R}^{3\times 3})$ for the orthogonal matrix with the property

$$R_\Gamma(x^3) = O_\Gamma(x^3) \circ \mathrm{Diag}[-1, -1, 1] \circ O_\Gamma(x^3)^t.$$

(M^t denotes the transposed of a matrix M.) Then T_Γ is defined as

$$T_\Gamma(x^3) := O_\Gamma(0) \circ O_\Gamma(x^3)^t, \quad x^3 \in [-r, r]. \tag{4.5}$$

Writing $|M|$ for the l_2-Norm of Matrix M, we find

$$|T_\Gamma(x^3) - \mathrm{Id}| \le c|x^3| \quad \text{for } x^3 \in [-r, r]. \tag{4.6}$$

Hence we conclude $T = T(\mathbf{x}) \in C^{0,1}(\mathcal{B}_r(\mathbf{0}), \mathbb{R}^{3\times 3})$ according to (4.3). In particular, we note

$$\lim_{\mathbf{x}\to\mathbf{0}} T(\mathbf{x}) = \mathrm{Id}. \tag{4.7}$$

Therefore, the function of inverse matrices $T^{-1} \in C^{0,1}(\mathcal{B}_r(\mathbf{0}), \mathbb{R}^{3\times 3})$ exists for sufficiently small radius $r > 0$, and we obtain

$$|T(\mathbf{x})| + |T(\mathbf{x})^{-1}| + |\nabla T(\mathbf{x})| \le c \quad \text{for all } \mathbf{x} \in \mathcal{B}_r(\mathbf{0}) \setminus \{\mathbf{0}\}. \tag{4.8}$$

Because our stationary H-surface will not meet $\mathcal{S}$ perpendicularly, in general, reflection across $\mathcal{S}$ as described in [3, Lemma 1] or [1, Section 8.4] will not be appropriate. We rather introduce the generalized reflection matrix $R_\mathcal{S}(\mathbf{x})$ by

$$R_\mathcal{S}(\mathbf{x}) := \frac{1}{p}\begin{pmatrix} 1 + q^2 - \psi_1^2 + \psi_2^2 & -2(\psi_1\psi_2 - iq) & 2(\psi_1 + iq\psi_2) \\ -2(\psi_1\psi_2 + iq) & 1 + q^2 + \psi_1^2 - \psi_2^2 & 2(\psi_2 - iq\psi_1) \\ 2(\psi_1 - iq\psi_2) & 2(\psi_2 + iq\psi_1) & -1 + q^2 + \psi_1^2 + \psi_2^2 \end{pmatrix} \tag{4.9}$$

for $|\mathbf{x}| < r$. In (4.9) we have abbreviated

$$\begin{aligned} \psi_1 &= \psi_1(\mathbf{x}) := \psi_{x^1}(x^1, x^2), \\ \psi_2 &= \psi_2(\mathbf{x}) := \psi_{x^2}(x^1, x^2), \\ q &= q(\mathbf{x}) := Q^3 - \psi_1 Q^1 - \psi_2 Q^2, \\ p &= p(\mathbf{x}) := 1 - q^2 + \psi_1^2 + \psi_2^2. \end{aligned} \tag{4.10}$$

Observe that $R_\mathcal{S}(\mathbf{x})$ agrees with the usual reflection matrix for $q(\mathbf{x}) \equiv 0$, that is $\mathbf{Q}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) = 0$ on $\mathcal{S}$. The identities

$$R_\mathcal{S}(\mathbf{x})^t = \overline{R_\mathcal{S}(\mathbf{x})} = R_\mathcal{S}(\mathbf{x})^{-1} \quad \text{for } \mathbf{x} \in \mathcal{B}_r(\mathbf{0}) \tag{4.11}$$

are easily verified, and we note

$$|R_\mathcal{S}(\mathbf{x})| + |\nabla R_\mathcal{S}(\mathbf{x})| \le c \quad \text{for all } \mathbf{x} \in \mathcal{B}_r(\mathbf{0}). \tag{4.12}$$

We recall the relation $\mathbf{x}(\overline{B^+}) \subset \mathcal{B}_r(0)$ for the considered H-surface $\mathbf{x} = \mathbf{x}(w)$.

Proposition 4.1. *Let us define the mapping* $\mathbf{y} := T(\mathbf{x})\mathbf{x}_w : \overline{B^+} \setminus \{0\} \to \mathbb{C}^3$. *Then* $\mathbf{y} \in C^{0,1}(\overline{B^+} \setminus \{0\}, \mathbb{C}^3)$ *holds true, and the boundary conditions*

$$\begin{aligned} R_\Gamma(0)\mathbf{y}(w) &= \overline{\mathbf{y}(w)} \quad \text{for all } w \in I^- = (-1,0), \\ R_{\mathcal{S}}(\mathbf{x}(w))\mathbf{y}(w) &= \overline{\mathbf{y}(w)} \quad \text{for all } w \in I^+ = (0,+1) \end{aligned} \tag{4.13}$$

are satisfied.

Proof. The Lipschitz continuity of $\mathbf{y}$ follows from $\mathbf{x} \in C^2(\overline{B^+} \setminus \{0\}, \mathbb{R}^3)$ and $T \in C^{0,1}(\mathcal{B}_r(\mathbf{0}), \mathbb{R}^{3\times 3})$. The first boundary condition in (4.13) is an immediate consequence of (4.2) and the definition of T. Indeed, we calculate

$$\begin{aligned} \overline{\mathbf{y}} &= T(\mathbf{x})\mathbf{x}_{\overline{w}} = T_\Gamma(x^3)\mathbf{x}_{\overline{w}} = T_\Gamma(x^3) \circ R_\Gamma(x^3)\mathbf{x}_w \\ &= O_\Gamma(0) \circ \mathrm{Diag}[-1,-1,1] \circ O_\Gamma(x^3)^t \mathbf{x}_w \\ &= R_\Gamma(0) \circ O_\Gamma(0) \circ O_\Gamma(x^3)^t \mathbf{x}_w \\ &= R_\Gamma(0) \circ T(\mathbf{x})\mathbf{x}_w = R_\Gamma(0)\mathbf{y} \quad \text{on } I^-. \end{aligned}$$

In order to verify the second boundary condition, we define the auxiliary function $\boldsymbol{\zeta} = (\zeta^1, \zeta^2, \zeta^3) : \overline{B^+} \setminus \{0\} \to \mathbb{C}^3$ by

$$\boldsymbol{\zeta}(w) := \begin{pmatrix} 1 & iq(\mathbf{x}) & \psi_{x^1}(x^1,x^2) \\ -iq(\mathbf{x}) & 1 & \psi_{x^2}(x^1,x^2) \\ -i\psi_{x^1}(x^1,x^2) & -i\psi_{x^2}(x^1,x^2) & i \end{pmatrix} \mathbf{x}_w(w).$$

We now claim $\operatorname{Im}\boldsymbol{\zeta} = \mathbf{0}$ on I^+ and therefore

$$\boldsymbol{\zeta}(w) = \overline{\boldsymbol{\zeta}(w)} \quad \text{for all } w \in I^+. \tag{4.14}$$

Indeed, $\operatorname{Im}\zeta^3$ vanishes on I^+ due to $\mathbf{x}(I^+) \in \mathcal{S}$. And evaluating the relation (2.2) on I^+ for the tangential vectors $(1, 0, \psi_{x^1}(x^1,x^2))$, $(0, 1, \psi_{x^2}(x^1,x^2))$ yields $\operatorname{Im}\zeta^1 = \operatorname{Im}\zeta^2 = 0$ on I^+; see [9, Lemma 2] for the details.

By virtue of

$$\begin{pmatrix} 1 & -iq & \psi_{x^1} \\ iq & 1 & \psi_{x^2} \\ i\psi_{x^1} & i\psi_{x^2} & -i \end{pmatrix}^{-1} = \frac{1}{p}\begin{pmatrix} 1+\psi_{x^2}^2 & -\psi_{x^1}\psi_{x^2}+iq & q\psi_{x^2}-i\psi_{x^1} \\ -\psi_{x^1}\psi_{x^2}-iq & 1+\psi_{x^1}^2 & -q\psi_{x^1}-i\psi_{x^2} \\ \psi_{x^1}-iq\psi_{x^2} & \psi_{x^2}+iq\psi_{x^1} & i(1-q^2) \end{pmatrix},$$

we obtain

$$\begin{pmatrix} 1 & -iq & \psi_{x^1} \\ iq & 1 & \psi_{x^2} \\ i\psi_{x^1} & i\psi_{x^2} & -i \end{pmatrix}^{-1} \circ \begin{pmatrix} 1 & iq & \psi_{x^1} \\ -iq & 1 & \psi_{x^2} \\ -i\psi_{x^1} & -i\psi_{x^2} & i \end{pmatrix} = R_{\mathcal{S}}.$$

Because $T(\mathbf{x}(w)) = \mathrm{Id}$ holds true for $w \in I^+$, the second boundary condition in (4.13) follows from (4.14). $\square$

Next we reflect $\mathbf{y}$ across I via

$$\mathbf{z}(w) := \begin{cases} \mathbf{y}(w), & w \in \overline{B^+} \setminus \{0\} \\ \overline{R_{\mathcal{S}}(\mathbf{x}(\overline{w}))\mathbf{y}(\overline{w})}, & w \in \overline{B^-} \setminus \overline{I} \end{cases} . \tag{4.15}$$

In accordance with Proposition 4.1 and formula (4.11), we infer $\mathbf{z} = \mathbf{z}(w) \in C^{0,1}(\overline{B} \setminus \overline{I^-}, \mathbb{C}^3)$. Moreover, we note the relation

$$\lim_{v\to 0+} \mathbf{z}(u, v) = R_\Gamma(0) \circ R_{\mathcal{S}}(\mathbf{x}(u, 0)) \lim_{v\to 0-} \mathbf{z}(u, v) \quad \text{for all } u \in I^-. \tag{4.16}$$

Formulas (4.8), (4.12) in conjunction with the H-surface system in (1.2), which can be written as $\mathbf{x}_{w\overline{w}} = iH(\mathbf{x})\mathbf{x}_{\overline{w}} \times \mathbf{x}_w$, imply the estimate

$$|\mathbf{z}_{\overline{w}}(w)| \le c|\mathbf{z}(w)|^2 \quad \text{on } \overline{B} \setminus I. \tag{4.17}$$

Finally, [12, Theorem 1] yields the growth property

$$|\mathbf{z}(w)| \le c|w|^{\nu-1}, \quad w \in \overline{B} \setminus \{0\}, \tag{4.18}$$

with arbitrary $\nu \in (0, \mu)$ and a constant $c > 0$ (depending on ν, the data, and the modulus of continuity of $\mathbf{x} = \mathbf{x}(w)$, but clearly not on the particular point $w \in \overline{B} \setminus \{0\}$).

We now consider the matrix $R = R(\mathbf{x}) := R_\Gamma(0) \circ R_{\mathcal{S}}(\mathbf{x})$, which is given by

$$R(\mathbf{x}) = \frac{1}{p} \begin{pmatrix} -(1+q^2-\psi_1^2+\psi_2^2) & 2(\psi_1\psi_2 - iq) & -2(\psi_1 + iq\psi_2) \\ 2(\psi_1\psi_2 + iq) & -(1+q^2+\psi_1^2-\psi_2^2) & -2(\psi_2 - iq\psi_1) \\ 2(\psi_1 - iq\psi_2) & 2(\psi_2 + iq\psi_1) & -1+q^2+\psi_1^2+\psi_2^2 \end{pmatrix} \tag{4.19}$$

for $|\mathbf{x}| < r$, compare (4.1) and (4.9). In order to smooth the jump of $\mathbf{z}$ on I^-, we multiply $\mathbf{z}$ by a singular matrix, which is built up by the eigenvalues and eigenvectors of the jump matrix R. This procedure follows Dziuk's arguments [2, 3] and goes back to E. Heinz's paper [7] on the Marx Shiffman problem. For the problem at hand, we have to distinguish between two cases and this will be done in the following sections.

5. The regular case

In the present section we suppose the assumptions of Theorem 2.2 to be satisfied. Choosing $r > 0$ sufficiently small, the relation (2.5) implies

$$|\mathbf{Q}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x})| < \frac{|\nabla\psi(x^1, x^2)|}{\sqrt{1 + |\nabla\psi(x^1, x^2)|^2}} \quad \text{for } |\mathbf{x}| < r$$

and consequently

$$q(\mathbf{x})^2 < |\nabla\psi(x^1,x^2)|^2, \quad |\mathbf{x}| < r;$$

compare formula (4.10). An elementary calculation shows that $R(\mathbf{x})$ possesses the eigenvalues

$$\begin{aligned} \lambda_1(\mathbf{x}) &= \frac{-1-q^2+|\nabla\psi|^2+2i\sqrt{|\nabla\psi|^2-q^2}}{1-q^2+|\nabla\psi|^2}, \\ \lambda_2(\mathbf{x}) &= -1, \\ \lambda_3(\mathbf{x}) &= \frac{-1-q^2+|\nabla\psi|^2-2i\sqrt{|\nabla\psi|^2-q^2}}{1-q^2+|\nabla\psi|^2}, \quad |\mathbf{x}| < r, \end{aligned} \tag{5.1}$$

and the associated eigenvectors

$$\begin{aligned} \mathbf{v}_1(\mathbf{x}) &= \Big(q\psi_2-\psi_1\sqrt{|\nabla\psi|^2-q^2}, -q\psi_1-\psi_2\sqrt{|\nabla\psi|^2-q^2}, i|\nabla\psi|^2\Big), \\ \mathbf{v}_2(\mathbf{x}) &= (\psi_2, -\psi_1, iq), \\ \mathbf{v}_3(\mathbf{x}) &= \Big(q\psi_2+\psi_1\sqrt{|\nabla\psi|^2-q^2}, -q\psi_1+\psi_2\sqrt{|\nabla\psi|^2-q^2}, i|\nabla\psi|^2\Big) \end{aligned} \tag{5.2}$$

for $|\mathbf{x}| < r$. Writing $U := \big(\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\big) \in C^1(\mathcal{B}_r(\mathbf{0}),\mathbb{C}^{3\times 3})$, we easily calculate $\det U(\mathbf{x}) = 2i|\nabla\psi(x^1,x^2)|^2(|\nabla\psi(x^1,x^2)|^2-q(\mathbf{x})^2)^{\frac{3}{2}} \neq 0$, $|\mathbf{x}| < r$. Consequently, the inverse matrix function $U^{-1} \in C^1(\mathcal{B}_r(\mathbf{0}),\mathbb{C}^{3\times 3})$ exists. If we set

$$\gamma(\mathbf{x}) := \frac{1}{2\pi}\arccos\left(\frac{-1-q^2+|\nabla\psi|^2}{1-q^2+|\nabla\psi|^2}\right) = \frac{1}{\pi}\operatorname{arccot}\left(\sqrt{|\nabla\psi|^2-q^2}\right) \in \left(0,\frac{1}{2}\right)$$

for $|\mathbf{x}| < r$, the relation

$$R(\mathbf{x}) = U(\mathbf{x}) \circ \operatorname{Diag}\big[e^{2i\pi(\gamma(\mathbf{x})-1)}, e^{-i\pi}, e^{-2i\pi\gamma(\mathbf{x})}\big] \circ U(\mathbf{x})^{-1}, \quad |\mathbf{x}| < r, \tag{5.3}$$

follows from (5.1) and the definition of U. Now we reflect $\mathbf{x}$ trivially across I by virtue of

$$\hat{\mathbf{x}}(w) := \begin{cases} \mathbf{x}(w), & w \in \overline{B^+} \\ \mathbf{x}(\overline{w}), & w \in \overline{B^-} \end{cases}$$

and consider the vector-valued function

$$\mathbf{F}(w) := \operatorname{Diag}\big[w^{1-\gamma(\hat{\mathbf{x}}(w))}, w^{\frac{1}{2}}, w^{\gamma(\hat{\mathbf{x}}(w))}\big] \circ U(\hat{\mathbf{x}}(w))^{-1}\mathbf{z}(w), \quad w \in \overline{B}\setminus\{0\}. \tag{5.4}$$

Observe that $\mathbf{F} \in C^{0,1}(\overline{B}\setminus\overline{I^-},\mathbb{C}^3)$ holds true. This follows from $\hat{\mathbf{x}} \in C^{0,1}(\overline{B}\setminus\{0\})$, $\mathbf{z} \in C^{0,1}(\overline{B}\setminus\overline{I^-})$, $U^{-1}, \gamma \in C^1$ on $|\mathbf{x}| < r$, and from the analyticity of a function

w^β, $\beta \in (0,1)$, in the set $\overline{B} \setminus \overline{I^-}$. Furthermore, we note the relation

$$\begin{aligned}
&\lim_{v\to 0+} \mathrm{Diag}\big[w^{1-\gamma(\hat{\mathbf{x}}(w))}, w^{\frac12}, w^{\gamma(\hat{\mathbf{x}}(w))}\big]\\
&\quad = \mathrm{Diag}\big[|u|^{1-\gamma(\hat{\mathbf{x}}(u))}e^{i\pi[1-\gamma(\hat{\mathbf{x}}(u))]}, |u|^{\frac12}e^{i\frac{\pi}{2}}, |u|^{\gamma(\hat{\mathbf{x}}(u))}e^{i\pi\gamma(\hat{\mathbf{x}}(u))}\big]\\
&\quad = \mathrm{Diag}\big[|u|^{1-\gamma(\hat{\mathbf{x}}(u))}e^{-i\pi[1-\gamma(\hat{\mathbf{x}}(u))]}, |u|^{\frac12}e^{-i\frac{\pi}{2}}, |u|^{\gamma(\hat{\mathbf{x}}(u))}e^{-i\pi\gamma(\hat{\mathbf{x}}(u))}\big]\\
&\qquad \circ \mathrm{Diag}\big[e^{2i\pi[1-\gamma(\hat{\mathbf{x}}(u))]}, e^{i\pi}, e^{2i\pi\gamma(\hat{\mathbf{x}}(u))}\big]\\
&\quad = \lim_{v\to 0-} \mathrm{Diag}\big[w^{1-\gamma(\hat{\mathbf{x}}(w))}, w^{\frac12}, w^{\gamma(\hat{\mathbf{x}}(w))}\big]\circ U(\hat{\mathbf{x}}(u))^{-1}\circ R(\hat{\mathbf{x}}(u))^{-1}\circ U(\hat{\mathbf{x}}(u))
\end{aligned}$$

for $u \in I^-$. Therefore, the definition of $\mathbf{F}$ and the jump-property (4.16) yield

$$\begin{aligned}
&\lim_{v\to 0+} \mathbf{F}(u,v)\\
&\quad = \lim_{v\to 0+} \mathrm{Diag}\big[w^{1-\gamma(\hat{\mathbf{x}}(w))}, w^{\frac12}, w^{\gamma(\hat{\mathbf{x}}(w))}\big]\circ U(\hat{\mathbf{x}}(u))^{-1}\lim_{v\to 0+} \mathbf{z}(u,v)\\
&\quad = \lim_{v\to 0-} \mathrm{Diag}\big[w^{1-\gamma(\hat{\mathbf{x}}(w))}, w^{\frac12}, w^{\gamma(\hat{\mathbf{x}}(w))}\big]\circ U(\hat{\mathbf{x}}(u))^{-1}\lim_{v\to 0-} \mathbf{z}(u,v)\\
&\quad = \lim_{v\to 0-} \mathbf{F}(u,v) \quad \text{for all } u\in I^-.
\end{aligned}$$

Hence we conclude $\mathbf{F} \in C^0(\overline{B}\setminus\{0\}, \mathbb{C}^3)$. And we even see $\mathbf{F} \in C^{0,1}(\overline{B}\setminus\{0\}, \mathbb{C}^3)$, because $|\nabla \mathbf{F}(w)|$ remains bounded for $w \to u \in I^-$. Moreover, we learn from the definition of $\mathbf{z}$ and the properties of $R_{\mathcal{S}}$ and T (see (4.8) and (4.12)) that

$$\begin{aligned}
&|F^1_{\overline{w}}(w)| \le c|w|^{1-\hat{\gamma}(w)-\tau}|\mathbf{z}(w)|^2,\\
&|F^2_{\overline{w}}(w)| \le c|w|^{\frac12}|\mathbf{z}(w)|^2,\\
&|F^3_{\overline{w}}(w)| \le c|w|^{\hat{\gamma}(w)-\tau}|\mathbf{z}(w)|^2 \quad \text{a.e. in } B\setminus\{0\}
\end{aligned} \tag{5.5}$$

holds true for $\mathbf{F} = (F^1, F^2, F^3)$ with an arbitrarily small $\tau > 0$ and a constant $c = c(\tau) > 0$. Here we have written $\hat{\gamma} = \hat{\gamma}(w) := \gamma(\hat{\mathbf{x}}(w))$. From the definition (5.4) we infer

$$|\mathbf{z}(w)|^2 \le c\Big\{|w|^{-2+2\hat{\gamma}}|F^1(w)|^2 + |w|^{-1}|F^2(w)|^2 + |w|^{-2\hat{\gamma}}|F^3(w)|^2\Big\}$$

for $w \in B\setminus\{0\}$ and this, together with (5.5), gives

$$\begin{aligned}
&|F^1_{\overline{w}}| \le c\Big\{|w|^{-1+\hat{\gamma}-\tau}|F^1(w)|^2 + |w|^{-\hat{\gamma}-\tau}|F^2(w)|^2 + |w|^{1-3\hat{\gamma}-\tau}|F^3(w)|^2\Big\},\\
&|F^2_{\overline{w}}| \le c\Big\{|w|^{-\frac32+2\hat{\gamma}}|F^1(w)|^2 + |w|^{-\frac12}|F^2(w)|^2 + |w|^{\frac12-2\hat{\gamma}}|F^3(w)|^2\Big\},\\
&|F^3_{\overline{w}}| \le c\Big\{|w|^{-2+3\hat{\gamma}-\tau}|F^1(w)|^2 + |w|^{-1+\hat{\gamma}-\tau}|F^2(w)|^2 + |w|^{-\hat{\gamma}-\tau}|F^3(w)|^2\Big\}
\end{aligned} \tag{5.6}$$

almost everywhere in $B\setminus\{0\}$.

We now intend to apply Lemma 3.1 to the functions $F^1(w)$, $F^3(w)$. To this end, we need the following

Proposition 5.1. *There is* $\delta \in (0,1)$ *and a constant* $c > 0$ *such that the components of* $\mathbf{F} = (F^1, F^2, F^3)$*, defined in* (5.4)*, satisfy the estimate*

$$|F^2(w)|^2 \le c\Big\{|w|^{-1+2\hat{\gamma}(w)}|F^1(w)|^2 + |w|^{1-2\hat{\gamma}(w)}|F^3(w)|^2\Big\}, \quad w \in \overline{B_\delta(0)} \setminus \{0\}. \tag{5.7}$$

Proof. Let us introduce the vector-valued function

$$\mathbf{G}(w) = \begin{pmatrix} G^1(w) \\ G^2(w) \\ G^3(w) \end{pmatrix} := \begin{pmatrix} w^{-1+\hat{\gamma}(w)} F^1(w) \\ w^{-\frac{1}{2}} F^2(w) \\ w^{-\hat{\gamma}(w)} F^3(w) \end{pmatrix}, \quad w \in \overline{B} \setminus \overline{I}.$$

At first, we note the relation

$$\hat{\mathbf{x}}_w(w) = \begin{cases} T(\hat{\mathbf{x}})^{-1} \circ U(\hat{\mathbf{x}})\, \mathbf{G}(w), & w \in \overline{B^+} \setminus \overline{I} \\ T(\hat{\mathbf{x}})^{-1} \circ R_{\mathcal{S}}(\hat{\mathbf{x}}) \circ U(\hat{\mathbf{x}})\, \mathbf{G}(w), & w \in \overline{B^-} \setminus \overline{I} \end{cases}. \tag{5.8}$$

This follows immediately from $\mathbf{z} = U(\hat{\mathbf{x}})\,\mathbf{G}$ on $B \setminus \{0\}$, the definitions of $\mathbf{z}$ and $\mathbf{y}$, and from relation (4.11). Formulas (5.8) and (4.11) now imply

$$\hat{\mathbf{x}}_w^t T(\hat{\mathbf{x}})^t \circ T(\hat{\mathbf{x}})\, \hat{\mathbf{x}}_w = \mathbf{G}^t U(\hat{\mathbf{x}})^t \circ U(\hat{\mathbf{x}})\, \mathbf{G} \quad \text{on } \overline{B} \setminus \overline{I}. \tag{5.9}$$

According to the relation

$$T(\mathbf{x})^t \circ T(\mathbf{x}) = \eta(\mathbf{x})\big(1-\eta(\mathbf{x})\big)\big[T_\Gamma(x^3)^t + T_\Gamma(x^3) - 2\mathrm{Id}\big] + \mathrm{Id}, \quad |\mathbf{x}| < r,$$

(compare formulas (4.4), (4.5)), we can also write

$$\hat{\mathbf{x}}_w^t T(\hat{\mathbf{x}})^t \circ T(\hat{\mathbf{x}})\, \hat{\mathbf{x}}_w = \eta(\hat{\mathbf{x}})\big(1-\eta(\hat{\mathbf{x}})\big)\hat{\mathbf{x}}_w^t\big[T_\Gamma(\hat{x}^3)^t + T_\Gamma(\hat{x}^3) - 2\mathrm{Id}\big]\hat{\mathbf{x}}_w$$

on $\overline{B} \setminus \overline{I}$, where we have used the conformality relations $\hat{\mathbf{x}}_w \cdot \hat{\mathbf{x}}_w = 0$ on $\overline{B} \setminus \overline{I}$. Inserting this into (5.9), we obtain the identity

$$\mathbf{G}^t\big[U(\hat{\mathbf{x}})^t \circ U(\hat{\mathbf{x}}) - M\big]\mathbf{G} = 0 \quad \text{on } \overline{B} \setminus \overline{I} \tag{5.10}$$

from (5.8). Here $M = (m_{ij}(w))_{ij}$ denotes a (discontinuous) matrix, which satisfies the estimate

$$|M(w)| \le c|\hat{\mathbf{x}}(w)| \quad \text{for } w \in \overline{B} \setminus \overline{I}, \tag{5.11}$$

due to the boundedness of T^{-1}, $R_{\mathcal{S}}$, U, and the inequality (4.6). From (5.10) and (5.11) we deduce

$$|G^2|^2|\mathbf{v}_2(\hat{\mathbf{x}}) \cdot \mathbf{v}_2(\hat{\mathbf{x}}) - m_{22}| \le c\left\{\left(1 + \frac{1}{\varepsilon}\right)\left(|G^1|^2 + |G^3|^2\right) + \varepsilon|G^2|^2\right\}$$

on $\overline{B} \setminus \overline{I}$ with arbitrary $\varepsilon > 0$. By virtue of $\mathbf{v}_2 \cdot \mathbf{v}_2 = |\nabla\psi|^2 - q^2 > 0$, the formula (5.11), $\hat{\mathbf{x}}(0) = \mathbf{0}$, and the continuity of $\hat{\mathbf{x}}$, we may choose $\delta \in (0, 1)$ small enough to ensure

$$\big|\mathbf{v}_2(\hat{\mathbf{x}}) \cdot \mathbf{v}_2(\hat{\mathbf{x}}) - m_{22}\big| \geq \frac{1}{2}\big[|\nabla\psi(\hat{x}^1, \hat{x}^2)|^2 - q(\hat{\mathbf{x}})^2\big] \quad \text{on } \overline{B_\delta(0)} \setminus I.$$

Taking $\varepsilon > 0$ sufficiently small, we arrive at

$$|G^2(w)|^2 \leq c\big\{|G^1(w)|^2 + |G^3(w)|^2\big\}, \quad w \in \overline{B_\delta(0)} \setminus I.$$

Inequality (5.7) now follows immediately from the definition of $\mathbf{G}$ and the continuity of $\mathbf{F}$ on $\overline{B} \setminus \{0\}$. $\square$

Let us define

$$\gamma_0 := \gamma(\mathbf{0}) = \frac{1}{\pi}\operatorname{arccot}\left(\cot\alpha\sqrt{1 - \frac{[\mathbf{Q}(\mathbf{0}) \cdot \mathbf{n}(\mathbf{0})]^2}{\cos^2\alpha}}\right) \in \left(0, \frac{1}{2}\right). \tag{5.12}$$

Observe that $\gamma_0 = \frac{\alpha}{\pi}$ holds true for $\mathbf{Q}(\mathbf{0}) \cdot \mathbf{n}(\mathbf{0}) = 0$.

Lemma 5.2. *Define the mapping* $\mathbf{F} = (F^1, F^2, F^3) \in C^{0,1}(\overline{B} \setminus 0, \mathbb{C}^3)$ *by (5.4). Then we may choose* $m \in \mathbb{N} \cup \{0\}$ *such that the following holds true with* $f^{j,m}(w) := w^{-m} F^j(w)$, $j = 1, 2, 3$*: Either the mapping*

$$\mathbf{\Psi}_I^m = \begin{pmatrix} \Psi_I^{1,m}(w) \\ \Psi_I^{2,m}(w) \\ \Psi_I^{3,m}(w) \end{pmatrix} := \begin{pmatrix} w^{\gamma(\mathbf{x}(w)) - \gamma_0} f^{1,m}(w) \\ w^{\frac{1}{2} - \gamma_0} f^{2,m}(w) \\ w^{1 - \gamma(\mathbf{x}(w)) - \gamma_0} f^{3,m}(w) \end{pmatrix}, \quad w \in \overline{B^+} \setminus \{0\}, \tag{5.13}$$

can be extended continuously to $w = 0$ *with the properties* $\Psi_I^{1,m}(0) \neq 0$ *and* $\Psi_I^{2,m}(0) = \Psi_I^{3,m}(0) = 0$. *Or the mapping*

$$\mathbf{\Psi}_{II}^m = \begin{pmatrix} \Psi_{II}^{1,m}(w) \\ \Psi_{II}^{2,m}(w) \\ \Psi_{II}^{3,m}(w) \end{pmatrix} := \begin{pmatrix} w^{-1 + \gamma(\mathbf{x}(w)) + \gamma_0} f^{1,m}(w) \\ w^{-\frac{1}{2} + \gamma_0} f^{2,m}(w) \\ w^{-\gamma(\mathbf{x}(w)) + \gamma_0} f^{3,m}(w) \end{pmatrix}, \quad w \in \overline{B^+} \setminus \{0\}, \tag{5.14}$$

can be extended continuously to $w = 0$ *with* $\Psi_{II}^{1,m}(0) = \Psi_{II}^{2,m}(0) = 0$ *and* $\Psi_{II}^{3,m}(0) \neq 0$.

Proof.

1. At first, we note the growth estimates

$$\begin{aligned} |F^1(w)| &\leq |w|^{\nu - \gamma(\hat{\mathbf{x}}(w))}, \\ |F^2(w)| &\leq |w|^{\nu - \frac{1}{2}}, \\ |F^3(w)| &\leq |w|^{\nu - 1 + \gamma(\hat{\mathbf{x}}(w))}, \quad w \in \overline{B} \setminus \{0\}, \end{aligned} \tag{5.15}$$

with arbitrary $\nu \in (0, \mu)$. These follow immediately from the formulas (4.18) and (5.4) as well as the boundedness of U^{-1}. Choose $\varepsilon > 0$ and then $\delta = \delta(\varepsilon) > 0$ as well as $\tau = \tau(\varepsilon) > 0$ sufficiently small to ensure

$$0 < \gamma_0 - \frac{\varepsilon}{2} \le \gamma(\hat{\mathbf{x}}(w)) - \tau \le \gamma(\hat{\mathbf{x}}(w)) + \tau \le \gamma_0 + \frac{\varepsilon}{2} < \frac{1}{2}, \quad w \in B_\delta(0).$$

From (5.15) we then infer

$$|F^1(w)| \le |w|^{\nu-\gamma_0-\frac{\varepsilon}{2}}, \quad |F^3(w)| \le |w|^{\nu-1+\gamma_0-\frac{\varepsilon}{2}}, \quad w \in B_\delta(0) \setminus \{0\}. \tag{5.16}$$

Furthermore, Proposition 5.1 and formula (5.6) yield

$$\begin{aligned} |F^1_{\overline{w}}(w)| &\le c\Big\{|w|^{-1+\gamma_0-\frac{\varepsilon}{2}}|F^1(w)|^2 + |w|^{1-3\gamma_0-\frac{3}{2}\varepsilon}|F^3(w)|^2\Big\}, \\ |F^3_{\overline{w}}(w)| &\le c\Big\{|w|^{-2+3\gamma_0-\frac{3}{2}\varepsilon}|F^1(w)|^2 + |w|^{-\gamma_0-\frac{\varepsilon}{2}}|F^3(w)|^2\Big\} \end{aligned} \tag{5.17}$$

a.e. on $B_\delta(0) \setminus \{0\}$ with possibly diminished $\delta(\varepsilon) > 0$. (In the sequel we may have to choose $\varepsilon > 0$ and consequently $\delta(\varepsilon), \tau(\varepsilon) > 0$ even smaller; we will indicate this at the respective points in the proof.)
Setting $\alpha := \gamma_0 + \frac{\varepsilon}{2} \in (0, \frac{1}{2})$, $\beta := 1 - \alpha + \varepsilon$ as well as $F := F^1$, $G := F^2$, formulas (5.16) and (5.17) have the form (3.1) and (3.2), respectively. If we assume $\varepsilon < \varepsilon_0$ additionally, we may therefore apply Lemma 3.1 with $\varepsilon_0 = \varepsilon_0(\alpha, \nu) > 0$ chosen as in the cited Lemma. We deduce the existence of a number $m \in \mathbb{N} \cup \{0\}$ such that the functions $f^{j,m}(w) = w^{-m}F^j(w)$, $j = 1, 3$, satisfy $f^{1,m} \in C^\mu(B_\delta(0), \mathbb{C})$ for all $0 < \mu < \min\{1, m + \gamma_0 - \frac{\varepsilon}{2}\}$ as well as one of the following conditions:

(i) $f^{1,m}(0) \ne 0$ and

$$\begin{aligned} |f^{1,m}_{\overline{w}}(w)| &\le c|w|^{m-1+\gamma_0-\frac{\varepsilon}{2}}, \\ |f^{3,m}_{\overline{w}}(w)| &\le c|w|^{m-2+3\gamma_0-\frac{3}{2}\varepsilon} \quad \text{a.e. on } B_\delta(0) \setminus \{0\}. \end{aligned} \tag{5.18}$$

If $m > 0$ holds true, then $f^{3,m}$ is also continuous in $B_\delta(0)$.

(ii) $f^{1,m}(0) = 0$, $f^{3,m} \in C^\mu(B_\delta(0), \mathbb{C})$ for all $\mu < \min\{1, m + 1 - \gamma_0 - \frac{\varepsilon}{2}\}$, $f^{3,m}(0) \ne 0$, and

$$\begin{aligned} |f^{1,m}_{\overline{w}}(w)| &\le c|w|^{m+1-3\gamma_0-\frac{3}{2}\varepsilon}, \\ |f^{3,m}_{\overline{w}}(w)| &\le c|w|^{m-\gamma_0-\frac{\varepsilon}{2}} \quad \text{a.e. on } B_\delta(0) \setminus \{0\}. \end{aligned} \tag{5.19}$$

2. Suppose that case (i) in the above alternative is fulfilled. Then we consider $\mathbf{\Psi}^m_I$. Note that $\zeta(w) := [\gamma(\mathbf{x}(w)) - \gamma_0] \log w$ is continuous in $\overline{B^+}$ with $\zeta(w) \to 0$ for $w \to 0$, because γ is of class $C^1(\mathcal{B}_r(\mathbf{0}))$ and $\mathbf{x}$ is of class $C^\mu(\overline{B^+})$. Consequently, we infer

$$\Psi^{1,m}_I = e^\zeta f^{1,m} \in C^0(\overline{B^+})$$

with $\Psi^{1,m}_I(0) = f^{1,m}(0) \ne 0$.

Next we show $\Psi_I^{3,m} \in C^0(\overline{B^+})$ and $\Psi_I^{3,m}(0) = 0$. Observe that $wf^{3,m}(w)$ is continuous on $B_\delta(0)$ and satisfies $wf^{3,m}(w) \to 0$ for $w \to 0$ (see (5.16) if $m = 0$). Because (5.18) implies

$$\left|[wf^{3,m}(w)]_{\overline{w}}\right| \le c|w|^{m-1+3\gamma_0-\frac{3}{2}\varepsilon} \quad \text{a.e. on } B_\delta(0) \setminus \{0\},$$

Proposition 3.4 yields

$$|wf^{3,m}(w)| \le c|w|^\mu \quad \text{for any } \mu < \min\left\{1, m + 3\gamma_0 - \frac{3}{2}\varepsilon\right\}. \tag{5.20}$$

Thus we find $|\Psi_I^{3,m}(w)| \le c|w|^{\mu-2\gamma_0-\frac{\varepsilon}{2}} \to 0$ for $w \to 0$, whenever $\varepsilon > 0$ is sufficiently small.

Concerning $\Psi_I^{2,m}$ we note the estimates

$$|f^{2,m}(w)| \le c|w|^{-\frac{1}{2}+\gamma_0-\frac{\varepsilon}{2}}, \quad |f^{2,m}_{\overline{w}}(w)| \le c|w|^{m-\frac{3}{2}+2\gamma_0-\varepsilon}$$

a.e. on $B_\delta(0) \setminus \{0\}$; compare (5.6), (5.7), and (5.20). By applying Proposition 3.4 to $wf^{2,m}(w)$, we deduce

$$|wf^{2,m}(w)| \le c|w|^\mu \quad \text{for any } \mu < \min\left\{1, m + \frac{1}{2} + 2\gamma_0 - \varepsilon\right\}.$$

Therefore, we arrive at $\Psi_I^{2,m} \in C^0(\overline{B^+})$ and $\Psi_I^{2,m}(0) = 0$.

3. Now assume case (ii) to be valid and consider $\boldsymbol{\Psi}_{II}^m$. As above we have $w^{-\gamma(\mathbf{x}(w))+\gamma_0} \to 1$ for $w \to 0$ and consequently $\Psi_{II}^{3,m} \in C^0(\overline{B^+})$, $\Psi_{II}^{3,m}(0) \ne 0$. Furthermore, the estimate (5.19) and the fact $f^{1,m}(0) = 0$ render

$$|f^{1,m}(w)| \le c|w|^\mu \quad \text{for all } \mu < \min\left\{1, m + 2 - 3\gamma_0 - \frac{3}{2}\varepsilon\right\}, \tag{5.21}$$

according to Proposition 3.4. This yields

$$|\Psi_{II}^{1,m}(w)| \le c|w|^{-1+2\gamma_0-\frac{\varepsilon}{2}+\mu} \to 0 \quad \text{for } w \to 0$$

as asserted. Finally, the formulas (5.6), (5.7), (5.21) imply

$$|f^{2,m}(w)| \le c|w|^{\frac{1}{2}-\gamma_0-\frac{\varepsilon}{2}}, \quad |f^{2,m}_{\overline{w}}(w)| \le c|w|^{m+\frac{1}{2}-2\gamma_0-\varepsilon}$$

a.e. on $B_\delta(0) \setminus \{0\}$, and we infer

$$|f^{2,m}(w)| \le c|w|^\mu \quad \text{for any } \mu < \left\{1, m + \frac{3}{2} - 2\gamma_0 - \varepsilon\right\},$$

again by virtue of Proposition 3.4. The definition (5.14) therefore yields $\Psi_{II}^{2,m} \in C^0(\overline{B^+})$ and $\Psi_{II}^{2,m}(0) = 0$, which completes the proof of Lemma 5.2. $\square$

Now we are able to give the

Proof of Theorem 2.2. The definitions of $\mathbf{y}$, $\mathbf{z}$, and $\mathbf{F}$ imply the relation

$$\mathbf{x}_w(w) = T(\mathbf{x}(w))^{-1} \circ U(\mathbf{x}(w)) \begin{pmatrix} w^{-1+\gamma(\mathbf{x}(w))} F^1(w) \\ w^{-\frac{1}{2}} F^2(w) \\ w^{-\gamma(\mathbf{x}(w))} F^3(w) \end{pmatrix}, \quad w \in \overline{B^+} \setminus \{0\}. \tag{5.22}$$

Observe that the matrix-valued function $T(\mathbf{x})^{-1} \circ U(\mathbf{x})$ is continuous on $\overline{B^+}$ and we have $T(\mathbf{0})^{-1} = \mathrm{Id}$; confer (4.7). Define $\mathbf{\Psi}_I^m$ and $\mathbf{\Psi}_{II}^m$ by (5.13) and (5.14), respectively.

At first, let us assume $\mathbf{\Psi}_I^m \in C^0(\overline{B^+}, \mathbb{C}^3)$ as well as $\Psi_I^{1,m}(0) \neq 0$, $\Psi_I^{2,m}(0) = \Psi_I^{3,m}(0) = 0$ with some $m \in \mathbb{N} \cup \{0\}$; see Lemma 5.2. If we set

$$\mathbf{\Phi}_I := T(\mathbf{x})^{-1} \circ U(\mathbf{x}) \mathbf{\Psi}_I^m \in C^0(\overline{B^+}, \mathbb{C}^3),$$

the representation

$$\mathbf{x}_w(w) = w^{m-1+\gamma_0} \mathbf{\Phi}_I(w), \quad w \in \overline{B^+} \setminus \{0\},$$

follows, *i.e.* (2.6) holds true with $\kappa = 1 - \gamma_0$ and $\mathbf{\Phi} := \mathbf{\Phi}_I$. Recalling $U = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ and (5.2), we find

$$\mathbf{\Phi}_I(0) = \Psi_I^{1,m}(0) \mathbf{v}_1(0) = \Psi_I^{1,m}(0) \begin{pmatrix} -a\sqrt{a^2 - q(\mathbf{0})^2} \\ -q(\mathbf{0})a \\ ia^2 \end{pmatrix}.$$

If we utilize $q(\mathbf{0}) = \sqrt{1+a^2}\, \mathbf{Q}(\mathbf{0}) \cdot \mathbf{n}(\mathbf{0})$ and $a = \cot\alpha$ with $\alpha \in (0, \frac{\pi}{2})$, we arrive at

$$\mathbf{\Phi}(0) = \mathbf{\Phi}_I(0) = \Psi_I^{1,m}(0) \frac{\cos\alpha}{\sin^2\alpha} \begin{pmatrix} -\sqrt{\cos^2\alpha - [\mathbf{Q}(\mathbf{0}) \cdot \mathbf{n}(\mathbf{0})]^2} \\ -\mathbf{Q}(\mathbf{0}) \cdot \mathbf{n}(\mathbf{0}) \\ i\cos\alpha \end{pmatrix}. \tag{5.23}$$

Consequently, we have deduced the relations (2.7).

On the other hand, if the mapping $\mathbf{\Psi}_{II}^m$ is continuous on $\overline{B^+}$ and if $\Psi_{II}^{1,m}(0) = \Psi_{II}^{2,m}(0) = 0$, $\Psi_{II}^{3,m}(0) \neq 0$ hold true with some $m \in \mathbb{N} \cup \{0\}$, then we define

$$\mathbf{\Phi}_{II} := T(\mathbf{x})^{-1} \circ U(\mathbf{x}) \mathbf{\Psi}_{II}^m \in C^0(\overline{B^+}, \mathbb{C}^3).$$

Relation (5.22) yields

$$\mathbf{x}_w(w) = w^{m-\gamma_0} \mathbf{\Phi}_{II}(w), \quad w \in \overline{B^+} \setminus \{0\},$$

that means we have found (2.6) with $\kappa = \gamma_0$ and $\mathbf{\Phi} := \mathbf{\Phi}_{II}$. Furthermore, similar arguments as above imply

$$\mathbf{\Phi}(0) = \mathbf{\Phi}_{II}(0) = \Psi_{II}^{3,m}(0)\frac{\cos\alpha}{\sin^2\alpha}\begin{pmatrix}\sqrt{\cos^2\alpha - [\mathbf{Q}(\mathbf{0})\cdot\mathbf{n}(\mathbf{0})]^2}\\ -\mathbf{Q}(\mathbf{0})\cdot\mathbf{n}(\mathbf{0})\\ i\cos\alpha\end{pmatrix} \tag{5.24}$$

and (2.7) follows again.

In both cases the relation $e^{-i\pi\kappa}\Phi^3(0) \in \mathbb{R}\setminus\{0\}$ is a consequence of the boundary condition (4.2). Indeed, if we employ (2.6) and let $w \in I^-$ tend to 0, then we infer the identity

$$e^{-i\pi\kappa}R_\Gamma(0)\mathbf{\Phi}(0) = e^{i\pi\kappa}\overline{\mathbf{\Phi}(0)}.$$

Because we have $R_\Gamma(0) = \mathrm{Diag}[-1, -1, 1]$, it follows $e^{-i\pi\kappa}\Phi^3(0) \in \mathbb{R}$ and (5.23) or (5.24) implies $\Phi^3(0) \neq 0$. This completes the proof of Theorem 2.2. $\square$

Remark 5.3. Observe that $\Phi^2(0) = 0$ holds true if and only if $\mathbf{Q}(\mathbf{0})\cdot\mathbf{n}(\mathbf{0}) = 0$ is satisfied. Then we also have $\Phi^1(0) = \pm i\Phi^3(0)$ as well as $\gamma_0 = \frac{\alpha}{\pi}$. This is exactly the situation described in [3] and [1, Section 8.4] for the minimal surface case. There the authors also considered the case $\alpha = \frac{\pi}{2}$, which is excluded in Theorem 2.2 according to assumption (2.5). Furthermore, it was shown in [3, Section 7] that the mapping $\mathbf{\Phi}$ is Hölder-continuous on $\overline{B^+}$. By similar estimates this can also be proved in the situation considered here. We left the details to the reader.

Let us conclude this section with some geometric consequences of Theorem 2.2, collected in the following

Theorem 5.4. *Let the assumptions of Theorem 2.2 be satisfied and define*

$$\vartheta := \frac{\mathbf{Q}(\mathbf{0})\cdot\mathbf{n}(\mathbf{0})}{\cos\alpha} \in [0, 1).$$

Then there exists a number $s > 0$ and an integer $k \in \mathbb{N}\cup\{0\}$ such that the components x^1, x^2, x^3 of the given H-surface $\mathbf{x}$ satisfy one of the four expansions

$$\big(x^1+i\sqrt{1-\vartheta^2}x^3\big)(w)=is\begin{cases} w^{2k+\gamma_0}\big[\sqrt{1-\vartheta^2}e^{-i\pi\gamma_0}+o(1)\big], & w\to 0\\ w^{2k+1+\gamma_0}\big[-\sqrt{1-\vartheta^2}e^{-i\pi\gamma_0}+o(1)\big], & w\to 0\\ \overline{w}^{2k+1-\gamma_0}\big[-\sqrt{1-\vartheta^2}e^{-i\pi\gamma_0}+o(1)\big], & w\to 0\\ \overline{w}^{2k+2-\gamma_0}\big[\sqrt{1-\vartheta^2}e^{-i\pi\gamma_0}+o(1)\big], & w\to 0\end{cases} \tag{5.25}$$

and

$$\big(x^2+i\vartheta x^3\big)(w) = is\begin{cases} w^{2k+\gamma_0}\big[\vartheta e^{-i\pi\gamma_0}+o(1)\big], & w\to 0\\ w^{2k+1+\gamma_0}\big[-\vartheta e^{-i\pi\gamma_0}+o(1)\big], & w\to 0\\ w^{2k+1-\gamma_0}\big[-\vartheta e^{i\pi\gamma_0}+o(1)\big], & w\to 0\\ w^{2k+2-\gamma_0}\big[\vartheta e^{i\pi\gamma_0}+o(1)\big], & w\to 0.\end{cases} \tag{5.26}$$

The l-th expansion in (5.26) *occurs if and only if we have the l-th expansion in* (5.25).

Furthermore, the surface normal $\mathbf{N} = |\mathbf{x}_u \times \mathbf{x}_v|^{-1}\mathbf{x}_u \times \mathbf{x}_v$ can be extended to a mapping of class $C^0(\overline{S_\delta(0)}, \mathbb{R}^3)$ for small $\delta > 0$, and $\mathbf{N}(w)$ tends to the limit

$$\lim_{w\to 0} \mathbf{N}(w) = \frac{1}{\cos\alpha}\begin{pmatrix} \mathbf{Q}(\mathbf{0})\cdot\mathbf{n}(\mathbf{0}) \\ \pm\sqrt{\cos^2\alpha - [\mathbf{Q}(\mathbf{0})\cdot\mathbf{n}(\mathbf{0})]^2} \\ 0 \end{pmatrix} = \begin{pmatrix} \vartheta \\ \pm\sqrt{1-\vartheta^2} \\ 0 \end{pmatrix}. \tag{5.27}$$

Finally, the oriented tangential vector $\mathbf{T}(u) = |\mathbf{x}_u(u)|^{-1}\mathbf{x}_u(u)$, $u \in I^+ \cap B_\delta(0)$, can be extended to a $C^0(\overline{I^+ \cap B_\delta(0)}, \mathbb{R}^3)$-mapping, and we have

$$\lim_{u\to 0+} \mathbf{T}(u) = \pm\begin{pmatrix} \sqrt{1-\vartheta^2}\,\sin(\pi\gamma_0) \\ \pm\vartheta\,\sin(\pi\gamma_0) \\ \cos(\pi\gamma_0) \end{pmatrix}. \tag{5.28}$$

Proof.

1. According to $\mathbf{x}(0) = \mathbf{0}$ and $\mathbf{\Phi} \in C^0(\overline{B^+}, \mathbb{C}^3)$, we obtain

$$\begin{aligned}\mathbf{x}(w) &= 2\int_0^1 \mathrm{Re}\big\{w\mathbf{x}_w(tw)\big\}\,dt \\ &= w^{m+1-\kappa}\big[\mathbf{\Phi}(0) + \boldsymbol{o}(1)\big] + \overline{w}^{m+1-\kappa}\big[\overline{\mathbf{\Phi}(0)} + \boldsymbol{o}(1)\big], \quad w \to 0,\end{aligned}$$

with $\kappa = \gamma_0$ or $\kappa = 1-\gamma_0$ and $m \in \mathbb{N}\cup\{0\}$ from (2.6). Formulas (2.7) therefore imply

$$\big(x^1 + i\sqrt{1-\vartheta^2}x^3\big)(w) = \begin{cases} w^{m+1-\kappa}\big[2i\sqrt{1-\vartheta^2}\,\Phi^3(0) + o(1)\big] \\ \overline{w}^{m+1-\kappa}\big[2i\sqrt{1-\vartheta^2}\,\overline{\Phi^3(0)} + o(1)\big] \end{cases} \tag{5.29}$$

as well as

$$\big(x^2 + i\vartheta x^3\big)(w) = w^{m+1-\kappa}\big[2i\vartheta\,\Phi^3(0) + o(1)\big] \tag{5.30}$$

for $w \to 0$. The proof of Theorem 2.2 shows that the first expansion in (5.29) occurs for $\kappa = 1-\gamma_0$, whereas $\kappa = \gamma_0$ holds true in the second expansion of (5.29). Distinguishing between the cases $m = 2k$, $m = 2k+1$ and recalling $\mathbf{x}(u) \in \Gamma$ for $u \in I^-$ as well as the property $e^{-i\pi\kappa}\Phi^3(0) \in \mathbb{R}\setminus\{0\}$, we arrive at the alternatives (5.25) and (5.26).

2. Due to $|\mathbf{\Phi}(0)| = \sqrt{2}|\Phi^3(0)| > 0$, there exists a number $\delta > 0$ such that $\mathbf{\Phi}(w) \neq \mathbf{0}$ is valid for $w \in \overline{S_\delta(0)}$. Consequently, the relation

$$\mathbf{N}(w) = -i\frac{\mathbf{x}_w \times \mathbf{x}_{\overline{w}}(w)}{|\mathbf{x}_w \times \mathbf{x}_{\overline{w}}(w)|} = -i\frac{\mathbf{\Phi}(w)\times\overline{\mathbf{\Phi}(w)}}{|\mathbf{\Phi}(w)\times\overline{\mathbf{\Phi}(w)}|}, \quad w \in \overline{S_\delta(0)}\setminus\{0\},$$

implies the announced regularity of $\mathbf{N}$. Furthermore, a direct computation yields (5.27), if one employs the relations (2.7) or, alternatively, one of the identities (5.23) and (5.24).

3. Similarly, we deduce the regularity of $\mathbf{T} = \mathbf{T}(u)$ and the asymptotic behaviour (5.28). Indeed, one just has to use the relation

$$\mathbf{T}(u) = \frac{\operatorname{Re}(\mathbf{\Phi}(u))}{|\operatorname{Re}(\mathbf{\Phi}(u))|}, \quad u \in I^+ \cap B_\delta(0),$$

the property $e^{-i\pi\kappa}\Phi^3(0) \in \mathbb{R} \setminus \{0\}$, and one of the formulas (5.23) or (5.24). $\square$

Remark 5.5. In Figure 1 below, we have plotted the four possible asymptotic shapes of a stationary H-surface prescribed by (5.25) and (5.26) with $k = 0$. The wire netting, which forms a disc, represents the tangential plane $T_{\mathbf{x}}\mathcal{S}$ at $\mathbf{x} = \mathbf{0}$, and the positive x^3-axis is tangential to Γ there; compare assumption (A). In the case $k > 0$, the surface simply wraps k-times around $\mathbf{x} = \mathbf{0}$, such that there appears a multiply covered disc, asymptotically. Figure 1 was produced with MAPLE 10 for the values $\alpha = \frac{\pi}{3}$ and $\mathbf{Q}(\mathbf{0}) \cdot \mathbf{n}(\mathbf{0}) = -\frac{2}{5}$.

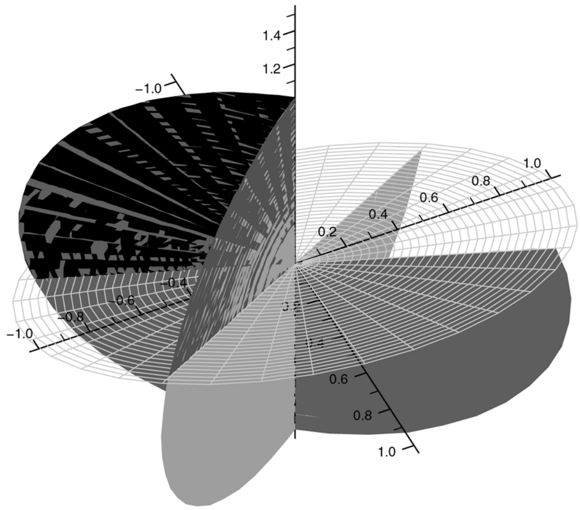

Figure 1. Asymptotic shape of an H-surface for $k = 0$.

Remark 5.6. Let us conclude this subsection with a note on the borderline case

$$|\mathbf{Q}(\mathbf{0}) \cdot \mathbf{n}(\mathbf{0})| = \cos\alpha.$$

For $\alpha = \frac{\pi}{2}$ one may apply the method above, whenever we have

$$|\mathbf{Q}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x})| = \frac{|\nabla\psi(x^1, x^2)|}{\sqrt{1 + |\nabla\psi(x^1, x^2)|^2}} \quad \text{for } |\mathbf{x}| < r \tag{5.31}$$

with small $r > 0$ and *if both sides vanish there*. But this situation was already studied by G. Dziuk [3], at least for minimal surfaces. For $\alpha < \frac{\pi}{2}$ the above technique seems not to be applicable even if the more restrictive relation (5.31) is fulfilled. Indeed, the matrix $R(\mathbf{x})$, $|\mathbf{x}| < r$, possesses the triple eigenvalue $\lambda = -1$ in that case, but the corresponding eigenspace is 1-dimensional. Consequently, we cannot decompose $R(\mathbf{x})$ as in (5.3) for $\alpha < \frac{\pi}{2}$ and a Jordan decomposition seems not to be useful for our purposes.

6. The irregular case

In contrast to the preceding section, we now suppose the assumptions of Theorem 2.3 to be fulfilled. From (2.8) we then infer

$$|\mathbf{Q}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x})| > \frac{|\nabla\psi(x^1, x^2)|}{\sqrt{1 + |\nabla\psi(x^1, x^2)|^2}} > 0 \quad \text{for } |\mathbf{x}| < r \tag{6.1}$$

for sufficiently small $r > 0$. Note that the surface normal $\mathbf{N}$ of our stationary, partially free H-surface cannot be expected to be continuous up to $w = 0$ in that case. Indeed, this would imply the relations $\mathbf{N}(0) \cdot (0, 0, 1) = 0$ and $\mathbf{N}(0) \cdot \mathbf{n}(\mathbf{0}) = -\mathbf{Q}(\mathbf{0}) \cdot \mathbf{n}(\mathbf{0})$, according to the boundary conditions in assumption (B). Because we have $\mathbf{n}(\mathbf{0}) = (-\cos\alpha, 0, \sin\alpha)$, these relations would imply

$$|N^1(0)| = \frac{|\mathbf{Q}(\mathbf{0}) \cdot \mathbf{n}(\mathbf{0})|}{\cos\alpha} > 1$$

contrary to $|\mathbf{N}| \equiv 1$.

Let the assumption (6.1) be satisfied. Define the matrix-valued function R as in (4.19) with the abbreviations (4.10). Then $R(\mathbf{x})$, $|\mathbf{x}| < r$, possesses the eigenvalues

$$\begin{aligned}
\lambda_1(\mathbf{x}) &= -\frac{1 - \sqrt{q^2 - |\nabla\psi|^2}}{1 + \sqrt{q^2 - |\nabla\psi|^2}}, \\
\lambda_2(\mathbf{x}) &= -1, \\
\lambda_3(\mathbf{x}) &= -\frac{1 + \sqrt{q^2 - |\nabla\psi|^2}}{1 - \sqrt{q^2 - |\nabla\psi|^2}}
\end{aligned}$$

and the eigenvectors

$$\begin{aligned}
\mathbf{v}_1(\mathbf{x}) &= \Big(q\psi_2 - i\psi_1\sqrt{q^2 - |\nabla\psi|^2}, -q\psi_1 - i\psi_2\sqrt{q^2 - |\nabla\psi|^2}, i|\nabla\psi|^2\Big), \\
\mathbf{v}_2(\mathbf{x}) &= \Big(\psi_2, -\psi_1, iq\Big), \\
\mathbf{v}_3(\mathbf{x}) &= \Big(q\psi_2 + i\psi_1\sqrt{q^2 - |\nabla\psi|^2}, -q\psi_1 + i\psi_2\sqrt{q^2 - |\nabla\psi|^2}, i|\nabla\psi|^2\Big),
\end{aligned}$$

which we collect to the matrix $U := (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) \in C^1(\mathcal{B}_r(\mathbf{0}), \mathbb{C}^{3\times 3})$. Due to $\det U = 2|\nabla\psi|^2(q^2 - |\nabla\psi|^2)^{\frac{3}{2}} \neq 0$ on $\mathcal{B}_r(\mathbf{0})$, the inverse matrix function $U^{-1} \in$

$C^1(\mathcal{B}_r(\mathbf{0}), \mathbb{C}^{3\times 3})$ exists. Defining

$$\sigma(\mathbf{x}) := 1 + \frac{i}{\pi} \ln\left(\frac{1 - \sqrt{q^2 - |\nabla\psi|^2}}{1 + \sqrt{q^2 - |\nabla\psi|^2}}\right), \quad |\mathbf{x}| < r, \tag{6.2}$$

we therefore find

$$R(\mathbf{x}) = U(\mathbf{x}) \circ \mathrm{Diag}\big[e^{-i\pi\sigma(\mathbf{x})}, e^{-i\pi}, e^{-i\pi\overline{\sigma(\mathbf{x})}}\big] \circ U(\mathbf{x})^{-1}, \quad |\mathbf{x}| < r. \tag{6.3}$$

Now we consider the function

$$\mathbf{F}(w) := \mathrm{Diag}\big[w^{\frac{1}{2}\sigma(\hat{\mathbf{x}}(w))}, w^{\frac{1}{2}}, w^{\frac{1}{2}\overline{\sigma(\hat{\mathbf{x}}(w))}}\big] \circ U(\hat{\mathbf{x}}(w))^{-1}\mathbf{z}(w), \quad w \in B\backslash\{0\}, \tag{6.4}$$

and claim $\mathbf{F} \in C^{0,1}(\overline{B} \setminus \{0\}, \mathbb{C}^3)$. Because we know $\hat{\mathbf{x}} \in C^{0,1}(\overline{B} \setminus \{0\})$, $\mathbf{z} \in C^{0,1}(\overline{B} \setminus \overline{I^-})$ as well as $U^{-1}, \sigma \in C^1$ on $|\mathbf{x}| < r$, the behaviour of $\mathbf{F}$ near I^- remains to be studied. To this aim, we note the relation

$$\lim_{v\to 0+} w^{\frac{1}{2}\sigma(\hat{\mathbf{x}}(w))} = \exp\left[\frac{\sigma(\hat{\mathbf{x}}(u))}{2}\big(\ln|u| + i\pi\big)\right] = e^{i\pi\sigma(\hat{\mathbf{x}}(u))} \lim_{v\to 0-} w^{\frac{1}{2}\sigma(\hat{\mathbf{x}}(w))}$$

for any $u \in I^-$. Taking (6.3) and the jump-property (4.16) into consideration, we consequently find

$$\begin{aligned}\lim_{v\to 0+} \mathbf{F}(w) &= \lim_{v\to 0-} \mathrm{Diag}\big[w^{\frac{1}{2}\sigma(\hat{\mathbf{x}}(w))}, w^{\frac{1}{2}}, w^{\frac{1}{2}\overline{\sigma(\hat{\mathbf{x}}(w))}}\big] \\ &\qquad \circ U(\hat{\mathbf{x}}(u))^{-1} \circ R(\hat{\mathbf{x}}(u))^{-1} \lim_{v\to 0+} \mathbf{z}(w) \\ &= \lim_{v\to 0-} \mathbf{F}(w),\end{aligned}$$

when we recall the definition $R_\Gamma(0) \circ R_{\mathcal{S}}(\mathbf{x}) =: R(\mathbf{x})$. This reveals $\mathbf{F} \in C^0(\overline{B} \setminus \{0\}, \mathbb{C}^3)$, and since $|\nabla\mathbf{F}(w)|$ remains bounded for $w \to u \in I^-$, the function $\mathbf{F}$ is even Lipschitz continuous in $\overline{B} \setminus \{0\}$. Furthermore, we estimate

$$|\mathbf{F}_{\overline{w}}(w)| \le c|w|^{\frac{1}{2}-\tau}|\mathbf{z}(w)|^2 \quad \text{a.e. on } B \tag{6.5}$$

with an arbitrarily small $\tau > 0$ and a constant $c = c(\tau) > 0$. This follows directly from the definition (6.4) and from the relation

$$\sqrt{\lambda(\hat{\mathbf{x}}(w))}|w|^{\frac{1}{2}} \le \big|w^{\frac{1}{2}\sigma(\hat{\mathbf{x}}(w))}\big|, \big|w^{\frac{1}{2}\overline{\sigma(\hat{\mathbf{x}}(w))}}\big| \le \frac{1}{\sqrt{\lambda(\hat{\mathbf{x}}(w))}}|w|^{\frac{1}{2}} \quad \text{on } B \setminus \{0\}. \tag{6.6}$$

Here we have abbreviated

$$\lambda(\mathbf{x}) := \frac{1 - \sqrt{q(\mathbf{x})^2 - |\nabla\psi(x^1, x^2)|^2}}{1 + \sqrt{q(\mathbf{x})^2 - |\nabla\psi(x^1, x^2)|^2}}, \quad |\mathbf{x}| < r. \tag{6.7}$$

Furthermore, formula (6.4) yields

$$|\mathbf{z}(w)| \le c|w|^{-\frac{1}{2}}|\mathbf{F}(w)| \quad \text{on } B \setminus \{0\},$$

again with the aid of (6.6). Together with (4.18) and (6.5) we deduce

$$|\mathbf{F}_{\overline{w}}(w)| \le c|w|^{\nu-1}|\mathbf{F}(w)| \quad \text{a.e. on } B \setminus \{0\} \tag{6.8}$$

with arbitrary $\nu \in (0, \mu)$. In addition, we infer the inequality

$$|\mathbf{F}(w)| \le c|w|^{\nu-\frac{1}{2}} \quad \text{on } B \setminus \{0\} \tag{6.9}$$

from (6.4), (6.6), and (4.18). According to Theorem 1 in [1, Section 8.1], the formulas (6.8) and (6.9) imply the existence of an integer $m \in \mathbb{N} \cup \{0\}$ and a complex vector $\mathbf{a} \in \mathbb{C}^3 \setminus \{\mathbf{0}\}$ such that the asymptotic relation

$$\mathbf{F}(w) = \mathbf{a}\, w^m + o(|w|^m) \quad \text{as } w \to 0 \tag{6.10}$$

is satisfied.

Proof of Theorem 2.3. The relation (2.9) with

$$\mathbf{\Phi}(w) := T(\mathbf{x}(w))^{-1} \circ U(\mathbf{x}(w)) \begin{pmatrix} w^{-\frac{i}{2\pi} \ln \lambda(\mathbf{x}(w))} f^{1,m}(w) \\ f^{2,m}(w) \\ w^{\frac{i}{2\pi} \ln \lambda(\mathbf{x}(w))} f^{3,m}(w) \end{pmatrix} \tag{6.11}$$

is immediate from (6.4), (6.2), and the identity $\mathbf{z}(w) = T(\mathbf{x}(w))\mathbf{x}_w(w)$ on $\overline{B^+} \setminus \{0\}$. In (6.11) we have defined $\lambda(\mathbf{x})$ as in (6.7), and we have abbreviated

$$\mathbf{f}^m = \big(f^{1,m}(w), f^{2,m}(w), f^{3,m}(w)\big) := w^{-m}\mathbf{F}(w) \in C^0(\overline{B}, \mathbb{C}^3)$$

with $m \in \mathbb{N} \cup \{0\}$ chosen as in (6.10). As in Proposition 5.1, we prove the inequality

$$\big|w^{-\frac{1}{2}} F^2(w)\big|^2 \le c\Big\{\big|w^{-\frac{1}{2}\sigma(\hat{\mathbf{x}}(w))} F^1(w)\big|^2 + \big|w^{-\frac{1}{2}\overline{\sigma(\hat{\mathbf{x}}(w))}} F^3(w)\big|^2\Big\}$$

on $\overline{B_\delta(0) \setminus \{0\}}$ with sufficiently small $\delta > 0$. According to (6.6) and the continuity of $\mathbf{f}^m$ on $\overline{B_\delta(0)}$, this relation implies

$$|f^{2,m}(w)|^2 \le c\big\{|f^{1,m}(w)|^2 + |f^{3,m}(w)|^2\big\}, \quad w \in \overline{B_\delta(0)},$$

with some constant $c > 0$. By virtue of $\mathbf{f}^m(0) = \mathbf{a} \ne \mathbf{0}$, we conclude that $f^{1,m}(0)$, $f^{3,m}(0)$ cannot vanish simultaneously. On the other hand, the functions $w^{\pm\frac{i}{2\pi} \ln \lambda(\mathbf{x}(w))}$ behave discontinuously for $w \to 0$, and we note that

$$0 < \sqrt{\lambda(\mathbf{x}(w))} \le \big|w^{\pm\frac{i}{2\pi} \ln \lambda(\mathbf{x}(w))}\big| \le \frac{1}{\sqrt{\lambda(\mathbf{x}(w))}}, \quad w \in \overline{B^+} \setminus \{0\}.$$

Because $T(\mathbf{x})^{-1} \circ U(\mathbf{x})$ is invertible, the assertions concerning $\mathbf{\Phi}$ follow immediately. □

7. Finiteness of branch points and a Gauß-Bonnet formula

We call a point $w_0 \in \overline{B^+}$ *branch point* of an H-surface $\mathbf{x} : \overline{B^+} \to \mathbb{R}^3$, if the limit $\lim\limits_{w \to w_0} |\nabla \mathbf{x}(w)|$ exists and vanishes. Observe that, under the assumptions of Theorem 2.2 or 2.3, $w_0 = 0$ is a branch point, if and only if $m \in \mathbb{N} \setminus \{0\}$ holds true in the asymptotic expansion (2.6) or (2.9), respectively. Then $m = m(w_0)$ is called the *order of the branch point* $w_0 = 0$.

We will now apply the local results of the Theorems 2.2 and 2.3 to stationary solutions $\mathbf{x}$ of (1.1), (1.2). Here $\{\Gamma, \mathcal{S}\}$ with $\Gamma, \mathcal{S} \in C^2$ denotes a partially free boundary configuration as described in the introduction, and the prescribed mean curvature function $H \in C^0(\mathbb{R}^3, \mathbb{R})$ is generated by some vector-field $\mathbf{Q} \in C^1(\mathbb{R}^3, \mathbb{R}^3)$ with the properties (2.1).

At first, we find the following

Corollary 7.1. *Let $\mathbf{x} = \mathbf{x}(w)$ be a stationary, partially free H-surface as described just above. Suppose that $|\mathbf{Q}(\mathbf{p}_k) \cdot \mathbf{n}(\mathbf{p}_k)| \neq \cos \alpha_k$ holds true for $k = 1, 2$, where $\alpha_k \in (0, \frac{\pi}{2})$ denote the angles between Γ and $\mathcal{S}$ at the meeting points $\mathbf{p}_k$, $k = 1, 2$. Then $\mathbf{x}$ possesses at most finitely many branch points in $\overline{B^+}$.*

Proof. The asymptotic expansions in B^+, on $C \setminus \{-1, +1\}$, and on I imply that branch points have to lie isolated in $\overline{B^+} \setminus \{-1, +1\}$; see [6, 8, 11]. Consequently, there are at most finitely many branch points in the compact set $\Theta_\varepsilon := \overline{B^+} \setminus \big(B_\varepsilon(-1) \cup B_\varepsilon(+1)\big)$ for arbitrary $\varepsilon > 0$.

Writing $H^+ := \{\zeta = \xi + i\eta \in \mathbb{C} : \eta > 0\}$, we next apply the conformal mappings $w = f_k(\zeta) : H^+ \to B^+$, $k = 1, 2$, which are given by their inverses $f_1^{-1}(w) = (\frac{w+1}{w-1})^2 : B^+ \to H^+$ and $f_2^{-1}(w) = -(\frac{w-1}{w+1})^2 : B^+ \to H^+$. After a suitable translation and rotation - which do not affect the size of $\mathbf{Q}(\mathbf{p}_k) \cdot \mathbf{n}(\mathbf{p}_k)$ and α_k -, the mappings $\mathbf{y}_k(\zeta) := \mathbf{x}(f_k(\sigma\zeta))$, $\zeta \in \overline{B^+}$, can be supposed to satisfy assumption (B) by choosing $\sigma > 0$ sufficiently small. According to Theorem 2.2 or Theorem 2.3, we infer the asymptotic representations

$$\mathbf{x}_w(w) = (w - w_k)^{2(m_k - \kappa_k)} \mathbf{\Theta}_k(w), \quad w \in \overline{B^+} \setminus \{0\}, \quad k = 1, 2, \tag{7.1}$$

with $w_1 = -1$, $w_2 = +1$. Here the functions $\mathbf{\Theta}_k = \mathbf{\Theta}_k(w)$ are continuous in $\overline{S_\varepsilon(w_k)}$ with $\mathbf{\Theta}_k(w_k) \neq \mathbf{0}$ (see Theorem 2.2), or we have at least $c^{-1} \le |\mathbf{\Theta}_k(w)| \le c$ on $\overline{S_\varepsilon(w_k)} \setminus \{1\}$ with a constant $c \ge 1$ and some $\varepsilon > 0$ (compare Theorem 2.3). Furthermore, $m_k \in \mathbb{N} \cup \{0\}$ are nonnegative integers and we have defined $\kappa_k := \frac{1}{2} \pm \tau_k \in (0, 1)$ with

$$\tau_k = \begin{cases} \dfrac{1}{2} - \dfrac{1}{\pi} \operatorname{arccot}\Big(\cot \alpha_k \sqrt{1 - \vartheta_k^2}\Big), & \text{if } |\vartheta_k| < 1 \\ 0, & \text{if } |\vartheta_k| > 1 \end{cases}$$

and

$$\vartheta_k := \frac{\mathbf{Q}(\mathbf{p}_k)\cdot\mathbf{n}(\mathbf{p}_k)}{\cos\alpha_k}, \quad k = 1, 2.$$

Due to (7.1), branch points cannot accumulate at $w = w_k = \pm 1$, and this completes the proof. □

Next we present a formula of Gauß-Bonnet type in the *regular case*, that is $|\mathbf{Q}(\mathbf{p}_k)\cdot\mathbf{n}(\mathbf{p}_k)| < \cos\alpha_k$ for $k = 1, 2$. By virtue of Corollary 7.1, any stationary, partially free H-surface $\mathbf{x}$ then possesses at most finitely many branch points in $\overline{B^+}$. Using the asymptotic expansions at branch points, we prove the following

Theorem 7.2. *Let* $\mathbf{x}$ *be a stationary* H*-surface solving* (1.1), (1.2) *as described at the beginning of this section. We assume*

$$|\mathbf{Q}(\mathbf{p}_k)\cdot\mathbf{n}(\mathbf{p}_k)| < \cos\alpha_k, \quad k = 1, 2,$$

for the angles $\alpha_k \in (0, \frac{\pi}{2})$ *between* Γ *and* $\mathcal{S}$ *at the meeting points* $\mathbf{p}_k$, $k = 1, 2$. *Let us denote the finitely many branch points in* $\overline{B^+} \setminus \{-1, +1\}$ *by* $w_3, \dots, w_{M+2} \in \partial B^+ \setminus \{-1, +1\}$ *and* $w_{M+3}, \dots, w_{M+N+2} \in B^+$, *where* $M, N \in \mathbb{N} \cup \{0\}$ *are integers. Finally, we write* $w_1 = -1$, $w_2 = +1$ *and define the sets* $C^\circ := C \setminus \{w_1, \dots, w_{M+2}\}$, $I^\circ := I \setminus \{w_3, \dots, w_{M+2}\}$, $B^\circ := B^+ \setminus \{w_{M+3}, \dots, w_{M+N+2}\}$. *Then the identity*

$$\begin{aligned}\iint\limits_{B^\circ} K\,E\,du\,dv = \pi + \pi\sum_{j=1}^{2}(m_j - \kappa_j) + 2\pi\sum_{j=3}^{M+2} m_j + \pi\sum_{j=M+3}^{M+N+2} m_j \\ - \int\limits_{C^\circ} k_g\sqrt{E}\,ds - \int\limits_{I^\circ} k_g\sqrt{E}\,du,\end{aligned} \tag{7.2}$$

holds true, whenever the integral $\iint_{B^\circ} K\,E\,du\,dv$ *exists as a Cauchy principle value. Here* $K = K(w)$ *denotes the Gaussian curvature of* $\mathbf{x}|_{B^\circ}$, k_g *is the geodesic curvature of the family of surface curves* $\mathbf{x}|_{I^\circ \cup C^\circ}$, *the numbers* $m_j = m(w_j) \in \mathbb{N}$, $j = 3, \dots, M + N + 2$, *denote the order of the branch points* w_j, *and* m_j, κ_j, $j = 1, 2$, *are taken from (7.1).*

Proof. We refer to [1, Section 7.11]. The additional exception points $w_1 = -1$, $w_2 = +1$ can be handled as the other branch points by using Lemma 2 there. Here one has to employ the representations (7.1). Formula (7.2) then follows by the Gaussian integral theorem applied to the domains

$$B^l := \left\{ w \in B^+ : |w - w_j| > \varepsilon_j^{(l)} \text{ for all } j = 1, \dots, M + N + 2 \right\}$$

with suitable $\varepsilon_j^{(l)} \downarrow 0$ $(l \to \infty)$ and an obvious limit procedure, whenever the integral on the left-hand side exists. □

Remark 7.3. If one succeeds in estimating the integrals in the Gauß-Bonnet formula (7.2) adequately, this formula may serve to exclude (or estimate the number of) branch points in $\overline{B^+}$. But there seems to arise some complications from the second integral on the right-hand side: It is not clear how to estimate $|k_g|$ with the aid of the given data, since $\mathbf{x}(w)$ does not meet $\mathcal{S}$ perpendicularly along I. This is also the reason for the additional assumption that the curvatura integra $\iint_{B^\circ} K\,E\,du\,dv$ exists as a Cauchy principle value.

References

[1] U. DIERKES, S. HILDEBRANDT, A. KÜSTER and O. WOHLRAB, "Minimal Surfaces I, II", Grundlehren Math. Wiss. Vol. 295, 296. Springer, Berlin Heidelberg New York, 1992.

[2] G. DZIUK, *Über quasilineare elliptische Systeme mit isothermen Parametern an Ecken der Randkurve*, Analysis **1** (1981), 63–81.

[3] G. DZIUK, *On the boundary behaviour of partially free minimal surfaces*, Manuscripta Math. **35** (1981), 105–123.

[4] K. GOLDHORN and S. HILDEBRANDT, *Zum Randverhalten der Lösungen gewisser zweidimensionaler Variationsprobleme mit freien Randbedingungen*, Math. Z. **118** (1970), 241–253.

[5] P. HARTMAN and A. WINTNER, *On the local behavior of solutions of nonparabolic partial differential equations*, Amer. J. Math. **75** (1953), 449–476.

[6] E. HEINZ, *Über das Randverhalten quasilinearer elliptischer Systeme mit isothermen Parametern*, Math. Z. **113** (1970), 99–105.

[7] E. HEINZ, *Über die analytische Abhängigkeit der Lösungen eines linearen elliptischen Randwertproblems von Parametern*, Nachr. Akad. Wiss. Göttingen **II**, Math.-Phys. Kl. (1979), 1–20.

[8] S. HILDEBRANDT, *Einige Bemerkungen über Flächen beschränkter mittlerer Krümmung*,- Math. Z. **115** (1970), 169-178.

[9] F. MÜLLER, *A priori bounds for H-surfaces in a partially free configuration*, Analysis **26** (2006), 471–489.

[10] F. MÜLLER, *Investigations on the regularity of surfaces with prescribed mean curvature and partially free boundaries*, Habilitationsschrift, BTU Cottbus (2006).

[11] F. MÜLLER, *On the regularity of H-surfaces with free boundaries on a smooth support manifold*, BTU Cottbus (2007), preprint.

[12] F. MÜLLER, *Growth estimates for the gradient of an H-surface near singular points of the boundary configuration*, Z. Anal. Anwendungen, to appear.

[13] J. C. C. NITSCHE, *Minimal surfaces with partially free boundary. Least area property and Hölder continuity for boundaries satisfying a chord-arc condition*, Arch. Ration. Mech. Anal. **39** (1970), 131–145.

[14] F. SAUVIGNY, "Partial Differential Equations 1: Foundations and Integral Representations", Springer, Berlin Heidelberg, 2006.

[15] F. SAUVIGNY, "Partial Differential Equations 2: Functional Analytic Methods", Springer, Berlin Heidelberg, 2006.

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