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Addendum to: On volumes of arithmetic quotients of SO $(1, n)$

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Abstract. There are errors in the proof of uniqueness of arithmetic subgroups of the smallest covolume. In this note we correct the proof, obtain certain results which were stated as a conjecture, and we give several remarks on further developments.

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1.1. Let us recall some notation and basic notions. Following [1] we will assume that n *is even* and $n \geq 4$. The group of orientation preserving isometries of hyperbolic n-space is isomorphic to $\mathrm{SO}(1, n)^o$, the connected component of the identity of the special orthogonal group of signature $(1, n)$, which can be identified with $\mathrm{SO}_0(1, n)$, the subgroup of $\mathrm{SO}(1, n)$ preserving the upper half space. This group is not Zariski closed in SL_{n+1} thus in order to construct arithmetically defined subgroups of $\mathrm{SO}(1, n)^o$ we consider arithmetic subgroups of the orthogonal group $\mathrm{SO}(1, n)$ or, more precisely, of groups $\mathrm{G} = \mathrm{SO}(f)$ where f is an admissible quadratic form defined over a totally real number field k (see [1, Section 2.1]).

We have an exact sequence of k-isogenies:

$$1 \to \mathrm{C} \to \widetilde{\mathrm{G}} \stackrel{\phi}{\to} \mathrm{G} \to 1, \tag{1.1}$$

where $\widetilde{\mathrm{G}}(k) \simeq \mathrm{Spin}(f)$ is the simply connected cover of G and $\mathrm{C} \simeq \mu_2$ is the center of $\widetilde{\mathrm{G}}$. This induces an exact sequence in Galois cohomology (see [5, Section 2.2.3])

$$\widetilde{\mathrm{G}}(k) \stackrel{\phi}{\to} \mathrm{G}(k) \stackrel{\delta}{\to} \mathrm{H}^1(k, \mathrm{C}) \to \mathrm{H}^1(k, \widetilde{\mathrm{G}}). \tag{1.2}$$

The main idea of this note is that by using (1.2) certain questions about arithmetic subgroups of G can be reduced to questions about the Galois cohomology group $\mathrm{H}^1(k, \mathrm{C})$.

A coherent collection of parahoric subgroups $P = (P_v)_{v \in V_f}$ of $\widetilde{\mathrm{G}}$ ($V_f = V_f(k)$ denotes the set of finite places of the field k) defines a principal arithmetic subgroup

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$\Lambda = \widetilde{\mathrm{G}}(k) \cap \prod_{v \in V_f} P_v \subset \widetilde{\mathrm{G}}(k)$ (see [2]). We fix an infinite place v of k for which $\mathrm{G}(k_v) \simeq \mathrm{SO}(1, n)$ and denote it by Id. The image of Λ under the central k-isogeny ϕ is an arithmetic subgroup of G and every maximal arithmetic subgroup of $\mathrm{G}(k_{Id})$ can be obtained as a normalizer of some $\phi(\Lambda)$ [2, Proposition 1.4]. We will also consider the local stabilizers of P in the adjoint group $\mathrm{G}(= \overline{\mathrm{G}})$, defining $\overline{P}_v$ to be the stabilizer of P_v in $\mathrm{G}(k_v)$ and $\overline{P} = (\overline{P}_v)_{v \in V_f}$. Clearly, $\overline{P}_v \supset \phi(P_v)$. In the notation of [1] the subgroups $\phi(P_v)$ are called parahoric subgroups of G, however this terminology is non-standard and we will avoid using it here.

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1.2. Given a totally real number field k with the group of units U, let

$$k^*_\infty = \{a \in k^* \mid a_v > 0 \text{ for } v \in V_\infty \setminus Id\}, \quad U_\infty = U \cap k^*_\infty.$$

Lemma 1.1. $\mathrm{Im}(\delta) \simeq k^*_\infty/(k^*)^2$.

Proof. From (1.2) we have $\mathrm{Im}(\delta : \mathrm{G}(k) \to \mathrm{H}^1(k, \mu_2)) = \mathrm{Ker}(\mathrm{H}^1(k, \mu_2) \to \mathrm{H}^1(k, \widetilde{\mathrm{G}}))$. The Hasse principle for simply connected k-groups implies that $\mathrm{H}^1(k, \widetilde{\mathrm{G}})$ is isomorphic to $\prod_{v \in V_\infty} \mathrm{H}^1(k_v, \widetilde{\mathrm{G}})$ [5, Theorem 6.6], and hence

$$\mathrm{Im}(\delta) = \mathrm{Ker}(\mathrm{H}^1(k, \mu_2) \to \textstyle\prod_{v \in V_\infty} \mathrm{H}^1(k_v, \widetilde{\mathrm{G}})).$$

The group $\mathrm{H}^1(k, \mu_2)$ is canonically isomorphic to $k^*/(k^*)^2$ [5, Lemma 2.6]. It is well known that for all $v \in V_\infty$ such that the group $\mathrm{G}(k_v)$ is anisotropic, the map ϕ in (1.2) is surjective and hence for all such v, $\mathrm{Im}(\delta_v) = \mathrm{Ker}(\mathrm{H}^1(k_v, \mu_2) \to \mathrm{H}^1(k_v, \widetilde{\mathrm{G}}))$ is trivial. For the remaining one infinite place $v(= Id) \in V_\infty$, $\phi(\widetilde{\mathrm{G}}(k_v))$ is a subgroup of index 2 in $\mathrm{G}(k_v)$ which consists of the orthogonal transformations with the trivial spinor norm. Collecting this information together we obtain the required isomorphism. □

1.3 The proof of the uniqueness part in [1, Theorem 4.1] contains errors but the result is correct. We will now give another argument for it. In order to do so we first establish a more general fact and then apply it to the cases considered in [1].

Let $P = (P_v)_{v \in V_f}$ and $P' = (P'_v)_{v \in V_f}$ be two coherent collections of parahoric subgroups of $\widetilde{\mathrm{G}}$ such that for all $v \in V_f$, P'_v is conjugate to P_v under an element of $\mathrm{G}(k_v)$. For all but finitely many v, $P_v = P'_v$ hence there is an element $g \in \mathrm{G}(\mathbb{A}_f)$ ($\mathbb{A}_f$ denotes the ring of finite adèles of k) such that P' is the conjugate of P under g. We have $\overline{P} = \prod_{v \in V_f} \overline{P}_v$ is the stabilizer of P in $\mathrm{G}(\mathbb{A}_f)$. The number of distinct

$\mathrm{G}(k)$-conjugacy classes of coherent collections P' as above is the cardinality $c(\overline{P})$ of $\mathcal{C}(\overline{P}) = \mathrm{G}(k)\backslash\mathrm{G}(\mathbb{A}_f)/\overline{P}$, which is called the class group of G relative to $\overline{P}$. The class number $c(\overline{P})$ is known to be finite (see *e.g.* [2, Proposition 3.9]). The following result can be used for obtaining further information about its value.

Proposition 1.2. *Let* $\mathrm{G} = \mathrm{SO}(f)$, $\widetilde{\mathrm{G}} = \mathrm{Spin}(f)$ *for an admissible quadratic form* f *defined over* k *and let* $P = (P_v)_{v\in V_f}$ *a coherent collection of parahoric subgroups of* $\widetilde{\mathrm{G}}$*. The class number* $c(P)$ *divides the order* $h_{\infty,2}$ *of a restricted* 2*-class group of* k *given by*

$$h_{\infty,2} = \frac{2^{[k:\mathbb{Q}]-1}h_2}{[U : U_\infty]},$$

where h_2 *is the order of the* 2*-class group of* k.

Proof. Recall two isomorphisms (see [5, Proposition 8.8], a minor modification is needed in order to adjust the statement to our setting but the argument remains the same):

$$\mathrm{G}(k)\backslash\mathrm{G}(\mathbb{A}_f)/\overline{P} \simeq \mathrm{G}(\mathbb{A}_f)/\overline{P}\mathrm{G}(k);$$

$$\mathrm{G}(\mathbb{A}_f)/\overline{P}\mathrm{G}(k) \simeq \delta_{\mathbb{A}_f}(\mathrm{G}(\mathbb{A}_f))/\delta_{\mathbb{A}_f}(\overline{P}\mathrm{G}(k)),$$

where $\delta_{\mathbb{A}_f}$ is the restriction of the product map $\prod_v \mathrm{G}(k_v) \to \prod_v \mathrm{H}^1(k_v, \mathrm{C})$ to $\mathrm{G}(\mathbb{A}_f)$.

For every finite place v, $\mathrm{H}^1(k_v, \widetilde{\mathrm{G}})$ is trivial (see [5, Theorem 6.4]) which implies $\delta_v : \mathrm{G}(k_v) \to \mathrm{H}^1(k_v, \mathrm{C})$ is surjective. Thus the image of $\delta_{\mathbb{A}_f}(\mathrm{G}(\mathbb{A}_f))$ can be identified with the restricted direct product $\prod' \mathrm{H}^1(k_v, \mathrm{C})$ with respect to the subgroups $\delta_v(\overline{P}_v)$. Also $\delta_{\mathbb{A}_f}(\mathrm{G}(k))$ naturally identifies with the image of $\delta(\mathrm{G}(k))$ in $\mathrm{H}^1(k, \mathrm{C})$ under the embedding $\psi : \mathrm{H}^1(k, \mathrm{C}) \to \prod' \mathrm{H}^1(k_v, \mathrm{C})$. Hence we have an isomorphism

$$\delta_{\mathbb{A}_f}(\mathrm{G}(\mathbb{A}_f))/\delta_{\mathbb{A}_f}(\overline{P}\mathrm{G}(k)) \simeq \textstyle\prod' \mathrm{H}^1(k_v, \mathrm{C})/\left(\prod_v \delta_v(\overline{P}_v) \cdot \psi(\mathrm{Im}\,\delta(\mathrm{G}(k)))\right).$$

The group $\mathrm{H}^1(k_v, \mu_2)$ is canonically isomorphic to $k_v^* / (k_v^*)^2$, by Lemma 1.1 $\mathrm{Im}\,\delta(\mathrm{G}(k)) \simeq k_\infty^*/(k^*)^2$, so we obtain

$$\frac{\prod' \mathrm{H}^1(k_v, \mathrm{C})}{\prod_v \delta_v(\overline{P}_v) \cdot \psi(\mathrm{Im}\,\delta(\mathrm{G}(k)))} \simeq \frac{\prod' k_v^*/(k_v^*)^2}{\delta_P \cdot k_\infty^*/(k^*)^2} \simeq \frac{J_f}{\delta_P \cdot J_f^2 k^*} \cdot \frac{k^*}{k_\infty^*},$$

where J_f is the ring of finite idèles of k and δ_P denotes $\prod_v \delta_v(\overline{P}_v)$.

Now, $\#(J_f/J_f^2k^*) = h_2$, the group k^*/k_∞^* splits as a product of local factors and $\#(k^*/k_\infty^*) = 2^{[k:\mathbb{Q}]-1}/[U : U_\infty]$ (see [4, Chapter 6]). This implies the proposition. □

In order to give a precise formula for the class number $c(P)$ one has to analyze the image of $\prod_v \delta_v(\overline{P}_v)$ in $\prod' \mathrm{H}^1(k_v, \mathrm{C})$. Still in many practical cases this appears to be unnecessary. Thus in order to prove the uniqueness of the minimal hyperbolic orbifolds we need to consider $k = \mathbb{Q}[\sqrt{5}]$ (in the compact case) and $k = \mathbb{Q}$ (for the non-compact orbifolds). In both cases $h_2 = h = 1$. For $k = \mathbb{Q}[\sqrt{5}]$, $U_\infty = \{1, \frac{1-\sqrt{5}}{2}\}$ and thus $[U : U_\infty] = 2$ which implies $h_{\infty,2} = 1$. For $k = \mathbb{Q}$, clearly, $h_{\infty,2} = 1$ as well. So in all the cases $c(P) = 1$ which implies that the corresponding arithmetic subgroups are defined uniquely up to a conjugation by $g \in \mathrm{SO}(1,n)$. It is clear that we can always chose $g \in \mathrm{SO}_0(1,n)$ and therefore the smallest orbifolds constructed in [1] are unique up to an (orientation preserving) isometry.

1.4. We now turn to Conjecture 4.1 and its analogue for the non-cocompact orbifolds in [1, Section 4.4]. Recall that in [1] the numbers $N(r)$, $N'(r)$ were defined for every $r \geq 2$ and estimated from above. These numbers are related to the index of the principal arithmetic subgroups in their normalizers. We now prove

Proposition 1.3. *For every $r \geq 2$, $N(r) = N'(r) = 1$.*

Proof. Let Λ be a principal arithmetic subgroup of $\widetilde{\mathrm{G}}$ which corresponds to a compact or non-compact hyperbolic n-orbifold of the minimal volume, $\Lambda' = \phi(\Lambda)$ and $\Gamma = N_{\mathrm{G}}(\Lambda')$.

From [2, Proposition 2.9], which in turn follows from the work of J. Rohlfs, using the fact that the center of our group G is trivial, we obtain:

$$[\Gamma : \Lambda'] = \#(\mathrm{H}^1(k, \mu_2)_\Theta \cap \delta(\mathrm{G}(k))) = \#\mathrm{Im}(\delta : \mathrm{G}(k) \to \mathrm{H}^1(k, \mu_2)).$$

By Lemma 1.1 we can identify the image of δ. The cases we are interested in are

$$k = \mathbb{Q} : \ \mathrm{Im}(\delta) = \left\{k^{*2}, (-1)k^{*2}\right\};$$
$$k = \mathbb{Q}[\sqrt{5}] : \ \mathrm{Im}(\delta) = \left\{k^{*2}, \frac{1-\sqrt{5}}{2}k^{*2}\right\}.$$

In both cases $[\Gamma : \Lambda'] = \#\mathrm{Im}(\delta) = 2$. Now it is easy to see that $\Lambda' = \phi(\Lambda) \subset \mathrm{SO}_0(1,n)$. From the other side there always exists $g \in \mathrm{SO}(1,n) \setminus \mathrm{SO}_0(1,n)$ which normalizes $\phi(\Lambda)$. For example take $g = \mathrm{diag}(-1,-1,1,\ldots,1)$. As in all the cases under consideration the quadratic form associated to Λ is diagonal [1, Sections 4.3, 4.4], g stabilizes Λ and clearly $g \in \mathrm{SO}(1,n) \setminus \mathrm{SO}_0(1,n)$. From these facts it follows that Λ' is a maximal arithmetic subgroup in $\mathrm{SO}_0(1,n)$ and thus $N(r)$ (or $N'(r)$)$=1$. □

This proposition makes *precise* the statements of Theorem 4.1 and 4.4 of [1]. It also implies that Table 2 of *loc. cit.* gives the precise values of the covolumes of the smallest n-dimensional hyperbolic orbifolds in even dimensions up to 18.

One other corollary is that cocompact and non-cocompact arithmetic subgroups of $\mathrm{SO}(1, 2r)^o$ of the smallest covolumes can be obtained as the stabilizers of certain lattices described in [1, Section 4.3]. We remark that since the fields of definition of the groups have class number 1, the lattices in both cases are free as $\mathcal{O}_k$-modules.

1.5. Correction: on p. 765, l. 9 one should read "grow super-exponentially" instead of "grow exponentially". (It follows from [1] that the Euler characteristic is bounded from below by const $\cdot\, (\prod_{i=1}^{r} \frac{(2i-1)!}{(2\pi)^{2i}})^{[k:\mathbb{Q}]}$ which for large enough r is $\geq$ const $\cdot\, (2r-1)!$)

We conclude this addendum with a few remarks on related results which appeared after the paper was published.

1.6. In [1, Section 4.5] we observed that for $r > 2$ the minimal covolume among the arithmetic lattices in $\mathrm{SO}(1, 2r)$ is attained on a non-uniform lattice. This interesting phenomenon was first discovered by A. Lubotzky for SL_2 over local fields of positive characteristic. Recently, in [6] A. Salehi Golsefidy proved that lattices of minimal covolume in classical Chevalley groups over local fields of characteristic $p > 7$ are all non-uniform. This result gives further support to a **conjecture** that *generically* (*i.e. for groups of high enough rank or fields of large enough positive characteristic*) *the minimal covolume is always attained on a non-uniform lattice*.

1.7. In [3] M. Conder and C. Maclachlan constructed a compact orientable hyperbolic 4-manifold which has Euler characteristic 16. The previously known smallest example which was used in order to formulate the main result in [1, Section 5] had $\chi = 26$. The construction of [3] agrees with our Theorem 5.5 and it also allows us to give a more precise formulation of the theorem:

Theorem 5.5′. *If there exists a compact orientable arithmetic hyperbolic* 4*-manifold* M *with* $\chi(M) \leq 16$, *then* M *is defined over* $\mathbb{Q}[\sqrt{5}]$ *and has the form* $\Gamma_M \backslash \mathcal{H}^4$ *with* Γ_M *being a torsion-free subgroup of index* $7200\chi(M)$ *of the group* Γ_1 *of the smallest arithmetic hyperbolic* 4*-orbifold.*

References

[1] M. BELOLIPETSKY, *On volumes of arithmetic quotients of* SO(1, n), Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5) **3** (2004), 749–770.

[2] A. BOREL and G. PRASAD, *Finiteness theorems for discrete subgroups of bounded covolume in semi-simple groups*, Inst. Hautes Études Sci. Publ. Math. **69** (1989), 119–171; *Addendum*, ibid. **71** (1990), 173–177.

[3] M. CONDER and C. MACLACHLAN, *Compact hyperbolic 4-manifolds of small volume*, Proc. Amer. Math. Soc. **133** (2005), 2469–2476.

[4] S. LANG, "Algebraic Number Theory", Graduate Texts in Mathematics, Vol. 110. Springer-Verlag, New York, 1994.

[5] V. P. PLATONOV and A. S. RAPINCHUK, “Algebraic Groups and Number Theory”, Pure and Applied Mathematics, Vol. 139. Academic Press, Inc., Boston, MA, 1994.
[6] A. SALEHI GOLSEFIDY, *Lattices of minimum covolume in Chevalley groups over positive characteristic local fields*, preprint.

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