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Erratum and addendum to: The BV-energy of maps into a manifold: relaxation and density results

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In Step 3 of the proof of Theorem 2.14, in Section 4 of [1], there is a mistake. The same mistake also appears in [2, Section 7.5]. Using arguments from Step 3 and Step 4 of [1], here we correct such a mistake.

Letting $B_j = \overline{B}_r(x_0) \in \mathcal{F}'_m$, we first deform the sliced current $\langle T, d_{x_0}, r\rangle$ to a Cartesian current $\widetilde{T}_j$ with support in $\partial B_\delta(x_0) \times B_{\mathcal{Y}}(y_j, \varepsilon_m)$, where $\delta < r$ and $B_{\mathcal{Y}}(y_j, \varepsilon_m)$ is a small geodesic ball in $\mathcal{Y}$ centered at some point $y_j \in \mathcal{Y}$. We then regularize the boundary data $\widetilde{T}_j$ and apply a standard convolution and projection argument. For the reader's convenience, we present a quite complete outline of the proof.

Keeping Steps 1 and 2 as in Section 4 of [1], we proceed as follows.

Step 3: *Projecting the boundary data*. For any $\rho > 0$, we set $Q^n_\rho := [-\rho, \rho]^n \subset \mathbb{R}^n$ and denote by Σ^i_ρ the i-dimensional skeleton of Q^n_ρ, so that $\bigcup \Sigma^{n-1}_\rho = \partial Q^n_\rho$. Also, let $\|x\| := \max\{|x_1|, \dots, |x_n|\}$. In the sequel, we say that an i-dimensional current S belongs to $\mathrm{cart}^{1,1}(\Sigma^i_r \times \mathcal{Y})$ if for any i-face F of Σ^i_r its restriction $S \llcorner (F \times \mathcal{Y})$ belongs to $\mathrm{cart}^{1,1}(F \times \mathcal{Y})$ and, for any i-faces F_1 and F_2 of Σ^i_r that intersect on a common $(i-1)$-face I, we have

$$\partial(S \llcorner (F_1 \times \mathcal{Y})) \llcorner I \times \mathcal{Y} = -\partial(S \llcorner (F_2 \times \mathcal{Y})) \llcorner I \times \mathcal{Y}\,. \tag{1.1}$$

In this case, moreover, we will denote by $\mathcal{E}_{1,1}(S, \Sigma^i_r)$ the sum of the BV-energies of the restrictions $S \llcorner (F \times \mathcal{Y})$ of S to all the i-faces F of Σ^i_r. We also recall that $\mathcal{Y} \subset \mathbb{R}^N$, and denote by

$$B_{\mathcal{Y}}(y, \varepsilon) := \overline{B}^N(y, \varepsilon) \cap \mathcal{Y}$$

the intersection of $\mathcal{Y}$ with the closed N-ball of radius ε centered at y. If $y \in \mathcal{Y}$ and $0 < \varepsilon < \varepsilon_0$, we let $\Psi_{(y,\varepsilon)} : \mathbb{R}^N \to \overline{B}_{\mathcal{Y}}(y, \varepsilon)$ be the retraction map given by $\Psi_{(y,\varepsilon)}(z) := \Pi_\varepsilon \circ \xi_{(y,\varepsilon)}$, where

$$\xi_{(y,\varepsilon)}(z) := \begin{cases} z & \text{if} \quad z \in \overline{B}^N(y, \varepsilon) \\ \varepsilon \dfrac{z-y}{|z-y|} & \text{if} \quad z \in \mathbb{R}^N \setminus \overline{B}^N(y, \varepsilon) \end{cases}$$

and $\Pi_\varepsilon : \mathcal{Y}_\varepsilon \to \mathcal{Y}$ is the projection map given by Remark 1.9 in [1], so that $\Psi_{(y,\varepsilon)}$ is a Lipschitz continuous function with $\mathrm{Lip}\,\Psi_{(y,\varepsilon)} = \mathrm{Lip}\,\Pi_\varepsilon \to 1^+$ as $\varepsilon \to 0^+$.

Let $B_j = \overline{B}_r(x_0) \in \mathcal{F}'_m$. By means of a deformation and slicing argument, we may and do define a bilipschitz homeomorphism $\psi_j : \overline{B}_r(x_0) \to Q^n_r$ such that $\|D\psi_j\|_\infty \leq K$, $\|D\psi_j^{-1}\|_\infty \leq K$ for some absolute constant $K > 0$, only depending on n, and

$$\psi_j(\overline{B}_\rho(x_0)) = Q^n_\rho \qquad \forall\, \rho \in (r/2, r)\,. \tag{1.2}$$

Setting

$$T_j := (\psi_j \bowtie Id_{\mathbb{R}^N})_\#(T \llcorner \overline{B}_r(x_0) \times \mathcal{Y})$$

we clearly have

$$\partial T_j = (\psi_j \bowtie Id_{\mathbb{R}^N})_\# \langle T, d_{x_0}, r\rangle \in \mathrm{cart}^{1,1}(\partial Q^n_r \times \mathcal{Y})\,.$$

Moreover, denoting by $T_j \llcorner (\Sigma^i_r \times \mathcal{Y})$ the i-dimensional slice of T_j on $\Sigma^i_r \times \mathcal{Y}$, we also may and do define ψ_j in such a way that $T_j \llcorner (\Sigma^i_r \times \mathcal{Y})$ is an i-dimensional current in $\mathrm{cart}^{1,1}(\Sigma^i_r \times \mathcal{Y})$ satisfying the energy estimate

$$\mathcal{E}_{1,1}(T_j, \Sigma^i_r) \leq C \cdot \frac{1}{r} \cdot \mathcal{E}_{1,1}(T_j, \Sigma^{i+1}_r) \qquad \forall\, i = 1, \ldots, n-2\,,$$

where $C > 0$ is an absolute constant, not depending on T_j, and $\mathcal{E}_{1,1}(T_j, \Sigma^i_r)$ denotes the BV-energy of the i-current $T_j \llcorner (\Sigma^i_r \times \mathcal{Y})$. With this notation, we have $T_j \llcorner (\Sigma^{n-1}_r \times \mathcal{Y}) = \partial T_j$. Since by the construction

$$\mathcal{E}_{1,1}(\langle T, d_{x_0}, r\rangle, \partial B_r(x_0) \times \mathcal{Y}) \leq \frac{2}{r}\, \mathcal{E}_{1,1}(T, \overline{B}_{2r}(x_0) \times \mathcal{Y}) \tag{1.3}$$

and

$$\frac{1}{(2r)^{n-1}}\, \mathcal{E}_{1,1}(T, \overline{B}_{2r}(x_0) \times \mathcal{Y}) \leq \frac{1}{m}\,, \tag{1.4}$$

we infer that on one hand

$$\mathcal{E}_{1,1}(T_j, \Sigma^i_r) \leq \widetilde{C}\, r^{i-n}\, \mathcal{E}_{1,1}(T, \overline{B}_{2r}(x_0) \times \mathcal{Y}) \qquad \forall\, i = 1, \ldots, n-1 \tag{1.5}$$

and on the other hand

$$\frac{1}{r^{i-1}} \mathcal{E}_{1,1}(T_j, \Sigma^i_r) \leq \widetilde{C}\, \frac{1}{m} \qquad \forall\, i = 1, \ldots, n\,, \tag{1.6}$$

where $\widetilde{C} > 0$ is an absolute constant.

Remark 1.1. Let $u_j := u_T \circ \psi_j^{-1}$ denote the BV-function corresponding to T_j, and $u_{j|\Sigma_r^i}$ the restriction of u_j to Σ_r^i. The inequality (1.6), with $i = 1$, yields on one hand that the concentration part of the 1-current $T_j \llcorner \Sigma_r^1 \times \mathcal{Y}$ is zero, and on the other hand that

$$|D(u_{j|\Sigma_r^1})|(\Sigma_r^1) \leq \widetilde{C}\,\frac{1}{m}\,.$$

Therefore, setting $\varepsilon_m := 1/\sqrt{m}$, for $m \in \mathbb{N}$ sufficiently large the image $u_j(\Sigma_r^1)$ is contained in a small geodesic ball $B_{\mathcal{Y}}(y_j, \varepsilon_m/2)$ centered at some given point $y_j \in \mathcal{Y}$. As a consequence, we obtain that

$$\operatorname{spt}(T_j \llcorner (\Sigma_r^1 \times \mathcal{Y})) \subset \Sigma_r^1 \times \overline{B}_{\mathcal{Y}}(y_j, \varepsilon_m/2)\,.$$

Let $q \in \mathbb{N}^+$. In the case of dimension $n \geq 3$, following an argument by Bethuel, if S_h is one of the $(n-1)$-faces of Σ_r^{n-1}, where $h = 1, \dots, 2n$, we may and do define a partition of S_h into $(q+1)^{n-1}$ small $(n-1)$-dimensional “cubes” $C_{l,h}$ in such a way that the following facts hold:

i) If $[C_{l,h}]_i$ denotes the i-dimensional skeleton of the boundary of $C_{l,h}$, the restriction of T_j to $[C_{l,h}]_i \times \mathcal{Y}$ is an i-dimensional current in $\operatorname{cart}^{1,1}([C_{l,h}]_i \times \mathcal{Y})$ for every $i = 1, \dots, n-2$.

ii) If $n = 3$, we have

$$\sum_{l=1}^{(q+1)^2} \mathcal{E}_{1,1}(T_j, \partial C_{l,h}) \leq K\left(\mathcal{E}_{1,1}(T_j, \partial S_h) + \frac{q}{r}\,\mathcal{E}_{1,1}(T_j, S_h)\right), \tag{1.7}$$

where $K > 0$ is an absolute constant.

iii) If $n \geq 4$, and $[S_h]_i$ denotes the i-dimensional skeleton of S_h, for every $i = 1, \dots, n-2$ we have

$$\sum_{l=1}^{(q+1)^{n-1}} \mathcal{E}_{1,1}(T_j, [C_{l,h}]_i) \leq K \cdot \sum_{t=i}^{n-1}\left(\frac{q}{r}\right)^{t-i} \cdot \mathcal{E}_{1,1}(T_j, [S_h]_t)\,, \tag{1.8}$$

where $K > 0$ is an absolute constant.

iv) All the $C_{l,h}$'s are bilipschitz homeomorphic to the $(n-1)$-cube $[-r/q, r/q]^{n-1}$ by linear maps $f_{l,h}$ such that $\|Df_{l,h}\|_\infty \leq K$, $\|Df_{l,h}^{-1}\|_\infty \leq K$.

Remark 1.2. By (1.6) and (1.7), or (1.8), we infer that

$$\sum_{l=1}^{(q+1)^{n-1}} \mathcal{E}_{1,1}(T_j, [C_{l,h}]_1) \leq \widehat{C}\,\frac{q^{n-2}}{m}\,,$$

where $\widehat{C} > 0$ is an absolute constant. Moreover, the image $u_j(\Sigma_r^1)$ is contained in $B_{\mathcal{Y}}(y_j, \varepsilon_m/2)$. Therefore, in the sequel we will take

$$q := \text{integer part of } ((2\widehat{C})^{-1} \cdot \varepsilon_m \cdot m)^{1/(n-2)}\,. \tag{1.9}$$

Arguing as in Remark 1.1, we then infer that the image of $[C_{l,h}]_1$ by u_j is contained in the geodesic ball $B_{\mathcal{Y}}(y_j, \varepsilon_m)$ and, moreover, that

$$\mathrm{spt}(T_j \llcorner ([C_{l,h}]_1 \times \mathcal{Y})) \subset [C_{l,h}]_1 \times \overline{B}_{\mathcal{Y}}(y_j, \varepsilon_m) \tag{1.10}$$

for every l and h.

Let $\delta := r(1 - q^{-1})$ and define $\Phi_q : Q^n_r \to Q^n_\delta$ by $\Phi_q(x) := (1 - q^{-1})\, x$ and $\pi_{(r,\delta)} : Q^n_r \setminus Q^n_\delta \to \partial Q^n_r$ by $\pi_{(r,\delta)}(x) := r\, x/\|x\|$. Setting

$$\mathcal{M}_{(r,\delta)} := \pi^{-1}_{(r,\delta)}\Big(\bigcup_{h=1}^{2n} \bigcup_{l=1}^{(q+1)^{n-1}} \partial C_{l,h}\Big)$$

it turns out that the $(n-1)$-skeleton

$$\mathcal{N}_{(r,\delta)} := \mathcal{M}_{(r,\delta)} \cup \partial Q^n_r \cup \partial Q^n_\delta$$

is the union of the boundaries of n-dimensional "cubes" $Q_{l,h}$, satisfying $C_{l,h} \subset \partial Q_{l,h}$ for every l and h, that partition $Q^n_r \setminus Q^n_\delta$. Moreover, each $Q_{l,h}$ is bilipschitz homeomorphic to the n-cube $[-r/q, r/q]^n$ by linear maps $\widetilde{f}_{l,h}$ such that $\|D\widetilde{f}_{l,h}\|_\infty \le K$, $\|D\widetilde{f}^{-1}_{l,h}\|_\infty \le K$, where $K > 0$ is an absolute constant. Finally, set

$$\widetilde{\Sigma}^i_r := \Big(\bigcup_{h=1}^{2n} \bigcup_{l=1}^{(q+1)^{n-1}} [C_{l,h}]_i\Big) \tag{1.11}$$

and denote by $\mathcal{N}^i_{(r,\delta)}$ the i-dimensional skeleton of $\mathcal{N}_{(r,\delta)}$, so that

$$\mathcal{N}^i_{(r,\delta)} = \widetilde{\Sigma}^i_r \cup \Phi_q(\widetilde{\Sigma}^i_r) \cup \pi^{-1}_{(r,\delta)}(\widetilde{\Sigma}^{i-1}_r) \qquad \forall\, i = 2, \dots, n-1\,.$$

We now define an n-current $\widehat{T}_j$ in $\mathrm{cart}^{1,1}(\mathrm{int}(Q^n_r \setminus Q^n_\delta) \times \mathcal{Y})$ and an $(n-1)$-current $\widetilde{T}_j \in \mathrm{cart}^{1,1}(\partial Q^n_\delta \times \mathcal{Y})$ such that the following properties hold:

(a) $\widehat{T}_j$ has small BV-energy;
(b) $\widetilde{T}_j$ is supported in $\partial Q^n_\delta \times \overline{B}_{\mathcal{Y}}(y_j, \varepsilon_m)$ and its BV-energy is comparable to the BV-energy of ∂T_j;
(c) the boundary of $\widehat{T}_j$ agrees with ∂T_j on $\partial Q^n_r \times \mathcal{Y}$;
(d) the boundary of $\widehat{T}_j$ agrees with $-\widetilde{T}_j$ on $\partial Q^n_\delta \times \mathcal{Y}$.

To this purpose we first define a 2-current $S^{(2)}_j$ on $\mathcal{N}^2_{(r,\delta)} \times \mathcal{Y}$ by setting

$$S^{(2)}_j := \begin{cases} T_j \llcorner (\widetilde{\Sigma}^2_r \times \mathcal{Y}) & \text{on } \ \widetilde{\Sigma}^2_r \times \mathcal{Y} \\ (\Phi_q \bowtie \Psi_{(y_j,\varepsilon_m)})_\# T_j \llcorner (\widetilde{\Sigma}^2_r \times \mathcal{Y}) & \text{on } \ \Phi_q(\widetilde{\Sigma}^2_r) \times \mathcal{Y} \end{cases}$$

whereas on $\pi^{-1}_{(r,\delta)}(\widetilde{\Sigma}^1_r)\times\mathcal{Y}$ we set

$$S^{(2)}_j := \sum_{h=1}^{2n}\sum_{l=1}^{(q+1)^{n-1}} H_\#([\![0,1]\!]\times(T_j \llcorner ([C_{l,h}]_1\times\mathcal{Y})))$$

where H is the affine homotopy map

$$H(t,x,y) := \left(t\delta\frac{x}{\|x\|}+(1-t)x,\, y\right), \qquad t\in[0,1]\,,\ x\in[C_{l,h}]_1\,,\ y\in\mathcal{Y}\,.$$

By (1.10) and (1.11) we infer that $S^{(2)}_j$ is well-defined in $\mathcal{N}^2_{(r,\delta)}\times\mathcal{Y}$ in such a way (1.1) holds, with $S=S^{(2)}_j$, for every 2-faces F_1 and F_2 of $\mathcal{N}^2_{(r,\delta)}$ that intersect on a common 1-face I.

The case $n=3$. We then define $\widehat{T}_j$ on each $Q_{l,h}$ by

$$\widehat{T}_j \llcorner (Q_{l,h}\times\mathcal{Y}) := \widehat{H}_\#([\![0,1]\!]\times(S^{(2)}_j \llcorner (\partial Q_{l,h}\times\mathcal{Y})))\,, \tag{1.12}$$

where $\widehat{H}(t,x,y) := (tc_{l,h}+(1-t)x,\, y)$ and $c_{l,h}$ is the barycenter of $Q_{l,h}$. On account of (1.1), with $S=S^{(2)}_j$, we infer that the current $\widehat{T}_j \llcorner \mathrm{int}(Q_{l,h})\times\mathcal{Y}$ actually belongs to $\mathrm{cart}^{1,1}$ for every h and l. In fact, the boundary of $\widehat{T}_j$ is computed on 2-forms in $\mathcal{Z}^{2,1}$, whence it cannot see the "singular" set $\{c_{l,h}\}\times\mathcal{Y}$ of $\widehat{T}_j$. Moreover, it is readily checked that $\widehat{T}_j$ satisfies the energy estimate

$$\mathcal{E}_{1,1}(\widehat{T}_j, \mathrm{int}(Q_{l,h})\times\mathcal{Y}) \le C\,\frac{r}{q}\,\mathcal{E}_{1,1}(S^{(2)}_j, \partial Q_{l,h})\,,$$

whereas by the definition of $S^{(2)}_j$ we obtain

$$\mathcal{E}_{1,1}(S^{(2)}_j, \partial Q_{l,h}) \le C\left(\mathcal{E}_{1,1}(T_j, C_{l,h}) + \frac{r}{q}\,\mathcal{E}_{1,1}(T_j, \partial C_{l,h})\right).$$

Therefore, by (1.7), and by summing on l and h, we estimate

$$\mathcal{E}_{1,1}(\widehat{T}_j, (Q^3_r\setminus Q^3_\delta)\times\mathcal{Y}) \le C\left(\frac{r}{q}\,\mathcal{E}_{1,1}(T_j,\Sigma^2_r) + \left(\frac{r}{q}\right)^2\mathcal{E}_{1,1}(T_j,\Sigma^1_r)\right).$$

In conclusion, for m large, and $n=3$, by (1.9) and (1.5) we obtain the energy estimate

$$\mathcal{E}_{1,1}(\widehat{T}_j, (Q^n_r\setminus Q^n_\delta)\times\mathcal{Y}) \le C\,(\varepsilon_m\cdot m)^{1/(2-n)}\,\mathcal{E}_{1,1}(T, \overline{B}_{2r}(x_0)\times\mathcal{Y}) \tag{1.13}$$

where, we recall, $(\varepsilon_m\cdot m)^{1/(2-n)}\to 0$ as $m\to+\infty$, since $\varepsilon_m\cdot m=\sqrt{m}$.

The case $n \geq 4$. We define an i-current $S_j^{(i)}$ on $\mathcal{N}^i_{(r,\delta)} \times \mathcal{Y}$ arguing by iteration on the dimension $i = 3, \dots, n-1$. More precisely, if F is any i-face of $[Q_{l,h}]_i$, we distinguish two cases. If F is contained in ∂Q_r^n we set

$$S_j^{(i)} \llcorner (F \times \mathcal{Y}) := T_j \llcorner (F \times \mathcal{Y}) .$$

Otherwise, we define $S_j^{(i)}$ on $F \times \mathcal{Y}$ by means of a "radial" extension of the boundary datum $S_j^{(i-1)} \llcorner (\partial F \times \mathcal{Y})$ similar to the one in (1.12), so that

$$\mathcal{E}_{1,1}(S_j^{(i)}, F) \leq C \, \frac{r}{q} \, \mathcal{E}_{1,1}(S_j^{(i-1)}, \partial F) .$$

Notice that for every $(i-1)$-faces F_1 and F_2 of $\mathcal{N}^{i-1}_{(r,\delta)}$ that intersect on a common $(i-2)$-face I, we again have that (1.1) holds true, with $S = S_j^{(i-1)}$. We finally define $\widehat{T}_j$ on each $Q_{l,h}$ by (1.12), with $S_j^{(n-1)}$ instead of $S_j^{(2)}$. By the construction, and for (1.8), we readily infer that

$$\mathcal{E}_{1,1}(\widehat{T}_j, (Q_r^n \setminus Q_\delta^n) \times \mathcal{Y}) \leq C \sum_{i=1}^{n-1} \left(\frac{r}{q} \right)^{n-i} \mathcal{E}_{1,1}(T_j, \Sigma_r^i) ,$$

so that by (1.9) and (1.5) we obtain again (1.13), for m large. Now, for any $n \geq 3$ the current $\widehat{T}_j$ this way constructed belongs to $\text{cart}^{1,1}(\text{int}(Q_r^n \setminus Q_\delta^n), \mathcal{Y})$. In fact, the boundary of $\widehat{T}_j$ is computed on $(n-1)$-forms in $\mathcal{Z}^{n-1,1}$, hence it cannot see a "singular" set that lives on $\Sigma \times \mathcal{Y}$ for some $(n-3)$-dimensional skeleton Σ. Moreover, the above properties (a)–(d) follow from the construction, as required.

In conclusion, setting

$$S_j := (\psi_j^{-1} \bowtie Id_{\mathbb{R}^N})_\# (\widehat{T}_j \llcorner \text{int}(Q_r^n \setminus Q_\delta^n) \times \mathcal{Y}) ,$$

on account of (1.2) we infer that S_j belongs to $\text{cart}^{1,1}((B_r(x_0) \setminus \overline{B}_\delta(x_0)) \times \mathcal{Y})$, and by (1.13) it satisfies the energy estimate

$$\mathcal{E}_{1,1}(S_j, (B_r(x_0) \setminus \overline{B}_\delta(x_0)) \times \mathcal{Y}) \leq C \, (\varepsilon_m \cdot m)^{1/(2-n)} \, \mathcal{E}_{1,1}(T, \overline{B}_{2r}(x_0) \times \mathcal{Y}) . \quad (1.14)$$

Finally, by the properties (c) (d) we infer that S_j satisfies the boundary conditions

$$\partial S_j \llcorner (\partial B_r(x_0) \times \mathcal{Y}) = \langle T, d_{x_0}, r \rangle$$

and

$$\partial S_j \llcorner (\partial B_\delta(x_0) \times \mathcal{Y}) = -\widetilde{T}_j ,$$

where $\widetilde{T}_j \in \text{cart}^{1,1}(\partial B_\delta(x_0) \times \mathcal{Y})$ has support

$$\text{spt}\, \widetilde{T}_j \subset \partial B_\delta(x_0) \times B_{\mathcal{Y}}(y_j, \varepsilon_m) \quad (1.15)$$

and BV-energy

$$\mathcal{E}_{1,1}(\widetilde{T}_j, \partial B_\delta(x_0) \times \mathcal{Y}) \le C \cdot \mathcal{E}_{1,1}(\langle T, d_{x_0}, r\rangle, \partial B_r(x_0) \times \mathcal{Y}) .$$

In the case of dimension $n = 2$ we simply take $\delta = r$ and $\widetilde{T}_j := \langle T, d_{x_0}, r\rangle$. The energy bounds (1.3) and (1.4) yield that (1.15) holds true, see Remark 1.1.

Step 4: *Approximation on the balls of* $\mathcal{F}'_m$. Set $\widehat{B}_j := B_\delta(x_0)$ and let $v_j \in BV(\partial \widehat{B}_j, \mathcal{Y})$ denote the BV-function corresponding to $\widetilde{T}_j$. Using the argument in Step 3 of [1, Section 4], due to the inductive hypothesis, we find a sequence of smooth maps $\{v_h^{(j)}\} \subset W^{1,1}(\partial \widehat{B}_j, \mathcal{Y})$ such that $\| v_h^{(j)} - v_j \|_{L^1(\partial \widehat{B}_j)} \to 0$,

$$G_{v_h^{(j)}} \rightharpoonup \widetilde{T}_j \qquad \text{weakly in} \quad \mathcal{Z}_{n-1,1}(\partial \widehat{B}_j \times \mathcal{Y})$$

as $h \to \infty$ and

$$\int_{\partial \widehat{B}_j} |D_\tau v_h^{(j)}| \, d\mathcal{H}^{n-1} \le \mathcal{E}_{1,1}(\widetilde{T}_j, \partial \widehat{B}_j \times \mathcal{Y}) \cdot (1 + 2^{-h})$$

for every h. By property (1.15), we may and do assume that the approximating sequence satisfies

$$v_h^{(j)}(\partial \widehat{B}_j) \subset B_{\mathcal{Y}}(y_j, \varepsilon_m) \tag{1.16}$$

for every h. Taking k sufficiently large, and using the argument by Gagliardo, we then define a map $W_k^{(j)} \in W^{1,1}(A^\delta_{\rho_k}, \mathbb{R}^N)$, where $0 < \rho_k < \delta$ and $A^R_\rho = A^R_\rho(x_0)$ denotes the annulus

$$A^R_\rho := \overline{B}_R(x_0) \setminus B_\rho(x_0) , \qquad 0 < \rho < R ,$$

in such a way that $W^{(j)}_{k|\partial B_\delta(x_0)} = v_{j|\partial B_\delta(x_0)}$ in the sense of traces,

$$W_k^{(j)}\left(x_0 + \rho_k \, \frac{x - x_0}{|x - x_0|} \right) = v_k^{(j)}\left(x_0 + \delta \, \frac{x - x_0}{|x - x_0|} \right)$$

and the energy $\int_{A^\delta_{\rho_k}} |DW_k^{(j)}| \, dx$ is arbitrarily small, if $\rho_k \nearrow \delta$ sufficiently rapidly. Condition (1.16) yields

$$\operatorname{dist}(W_k^{(j)}(x), \mathcal{Y}) < \varepsilon_0 \qquad \text{for } \mathcal{L}^n\text{-a.e. } x \in A^\delta_{\rho_k} \tag{1.17}$$

for m large enough, hence we may and do define $w_k^{(j)} := \Pi_{\varepsilon_0} \circ W_k^{(j)}$ on $A^\delta_{\rho_k}$, where Π_{ε_0} is the Lipschitz projection onto $\mathcal{Y}$ given by Remark 1.9 in [1], so that $w_k^{(j)}(A^\delta_{\rho_k}) \subset B_{\mathcal{Y}}(y_j, \varepsilon_m)$.

We now extend $w_k^{(j)}$ to the whole ball $\widehat{B}_j$ by the map $\widetilde{w}_k^{(j)} : \overline{B}_{\rho_k}(x_0) \to B_{\mathcal{Y}}(y_j, \varepsilon_m)$ given by

$$\widetilde{w}_k^{(j)}(x) := \begin{cases} w_k^{(j)} \circ \psi_{(\delta,\sigma)}(x) & \text{if} \quad x \in A_{\delta-2\sigma}^{\delta-\sigma} \\ \Psi_{(y_j,\varepsilon_m)} \circ u \circ \phi_{(\delta,\sigma,r)}(x) & \text{if} \quad x \in B_{\delta-2\sigma}(x_0)\,, \end{cases} \tag{1.18}$$

where $u = u_T$, $\sigma := \delta - \rho_k$, $\psi_{(\delta,\sigma)} : A_{\delta-2\sigma}^{\delta-\sigma} \to A_{\delta-\sigma}^{\delta}$ is the reflection map

$$\psi_{(\delta,\sigma)}(x) := \big(-|x-x_0| + 2\,(\delta-\sigma)\big)\,\frac{x-x_0}{|x-x_0|}$$

and $\phi_{(\delta,\sigma,r)} : B_{\delta-2\sigma}(x_0) \to B_r(x_0)$ is the homothetic map

$$\phi_{(\delta,\sigma)}(x) := x_0 + \frac{r}{\delta-2\sigma}\,(x-x_0)\,.$$

Set now $\rho := \rho_k = \delta - \sigma$. Since the image of $B_\rho(x_0)$ by $\widetilde{w}_k^{(j)}$ is contained in the geodesic ball $B_{\mathcal{Y}}(y_j, \varepsilon_m)$, by means of a convolution argument we can approximate $\widetilde{w}_k^{(j)}$ on $B_\rho(x_0)$ by a smooth sequence $v_\varepsilon^{(j)} : B_\rho(x_0) \to \overline{B}^N(y_j, \varepsilon_m)$ that converges in the L^1-sense to $\widetilde{w}_{k|B_\rho(x_0)}^{(j)}$ and with total variation converging to the total variation $|D\widetilde{w}_k^{(j)}|(B_\rho(x_0))$. We finally set $w_\varepsilon^{(j)} := \Pi_{\varepsilon_m} \circ v_\varepsilon^{(j)} : B_\rho(x_0) \to B_{\mathcal{Y}}(y_j, \varepsilon_m)$, so that clearly $w_\varepsilon^{(j)} \rightharpoonup \widetilde{w}_k^{(j)}$ weakly in $BV(B_\rho(x_0), \mathbb{R}^N)$, whereas

$$|Dw_\varepsilon^{(j)}|(B_\rho(x_0)) \le (\operatorname{Lip} \Pi_{\varepsilon_m}) \cdot |Dv_\varepsilon^{(j)}|(B_\rho(x_0))\,.$$

Therefore, the energy of $\widetilde{w}_k^{(j)}$ being small on $A_{\delta-2\sigma}^{\delta-\sigma}$, we may and do assume that

$$\limsup_{\varepsilon\to 0} |Dw_\varepsilon^{(j)}|(B_\rho(x_0)) \le (\operatorname{Lip} \Pi_{\varepsilon_m})^2 \cdot |Du|(B_r(x_0)) + \frac{2^{-j}}{k}\,. \tag{1.19}$$

Moreover, by suitably defining the convolution kernel, we may and do assume that the traces are equal, so that $w_{\varepsilon|\partial B_\rho(x_0)}^{(j)} = v_{\varepsilon|\partial B_\rho(x_0)}^{(j)} = \widetilde{w}_{k|\partial B_\rho(x_0)}^{(j)}$. Most importantly, by the construction we may and do assume that the boundaries of the graphs agree on $\partial B_\rho(x_0)$, so that

$$\partial G_{w_\varepsilon^{(j)}} \llcorner \partial B_\rho(x_0) \times \mathcal{Y} = \partial G_{v_\varepsilon^{(j)}} \llcorner \partial B_\rho(x_0) \times \mathcal{Y} = \partial G_{\widetilde{w}_k^{(j)}} \llcorner \partial B_\rho(x_0) \times \mathcal{Y}\,. \tag{1.20}$$

We then define $u_k^{(j)} : \overline{B}_\delta(x_0) \to \mathcal{Y}$ by

$$u_k^{(j)}(x) := \begin{cases} w_k^{(j)}(x) & \text{if} \quad x \in A_{\rho_k}^{\delta} \\ w_{\varepsilon_k}^{(j)}(x) & \text{if} \quad x \in B_{\rho_k}(x_0) \end{cases}$$

where $\rho = \rho_k$ and $\varepsilon_k \searrow 0$ along a sequence. Moreover, for every m and k we let $T_k^{(m)} \in \mathcal{D}_{n,1}(B^n \times \mathcal{Y})$ be given by

$$T_k^{(m)} := T \llcorner (B^n \setminus \Omega_m) \times \mathcal{Y} + \sum_{j=1}^{\infty} (S_j + G_{u_k^{(j)}} \llcorner \operatorname{int}(\widehat{B}_j) \times \mathcal{Y}), \tag{1.21}$$

where S_j is defined in Step 3, so that $T_k^{(m)} \in \operatorname{cart}^{1,1}(B^n \times \mathcal{Y})$.

Step 5: *Approximating maps on the whole domain.* As in Step 5 of [1, Section 4], a part from the fact that the Cantor part of Du_m is not zero on the annuli $B_r(x_0) \setminus B_\delta(x_0)$. However, due to the energy estimate (1.13), by summing on j, we may and do assume that for m large enough

$$|D^C u_m|(B^n) \le \frac{1}{2} |D^C u_T|(B^n) . \tag{1.22}$$

Step 6: *Approximating currents.* Set $T_m := T_{k_m}^{(m)}$, where the sequence $k_m \to \infty$ is defined as in Step 5. We show that the flat distance of T_m from T is small. Recall that the flat norm $T \mapsto \mathbf{F}(T)$ is given by

$$\mathbf{F}(T) := \sup\{T(\phi) \mid \phi \in \mathcal{Z}^{n-1,1}(B^n \times \mathcal{Y}) , \ \mathbf{F}(\phi) \le 1\} ,$$

where

$$\mathbf{F}(\phi) := \max\left\{ \sup_{z \in B^n \times \mathcal{Y}} \|\phi(z)\| , \ \sup_{z \in B^n \times \mathcal{Y}} \|d\phi(z)\| \right\} .$$

In fact, by (1.21) we infer that

$$\mathbf{F}(T_k^{(m)} - T) \le \sum_{j=1}^{\infty} \mathbf{F}((T_k^{(m)} - T) \llcorner \operatorname{int}(B_j) \times \mathcal{Y})) .$$

Moreover, condition

$$\mu_{Jc}(J_c(T) \setminus J_m) < \frac{1}{m}$$

yields that the Jump-concentration part of the energy of T and of $T_k^{(m)}$ is small on the union of the balls B_j. Therefore, using the L^1-convergence of u_m to u, for every $\varepsilon \in (0, 1)$, possibly passing to a subsequence, we have

$$\mathbf{F}(T_m - T) \le \varepsilon^m \qquad \forall m .$$

On account of (1.22), and using an iteration argument similar to the one used in [1, Section 5] to obtain Theorem 2.15 from Proposition 5.1, we find the approximating sequence $\{T_k\}$ such that $u_k := u_{T_k}$ in $BV(B^n, \mathcal{Y})$ has no Cantor part, $|D^C u_k| = 0$ for every k.

References

[1] M. GIAQUINTA and D. MUCCI, *The BV-energy of maps into a manifold: relaxation and density results*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5) **5** (2006), 483–548.
[2] M. GIAQUINTA and D. MUCCI, *Maps into manifolds and currents: area and $W^{1,2}$-, $W^{1/2}$-, BV-energies*, Edizioni della Normale, C.R.M. Series, Scuola Norm. Sup. Pisa, 2006.

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