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Sobolev versus Hölder local minimizers and existence of multiple solutions for a singular quasilinear equation

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Abstract. We investigate the following quasilinear and singular problem,

$$\begin{cases} -\Delta_p u = \dfrac{\lambda}{u^\delta} + u^q & \text{in } \Omega; \\ u|_{\partial\Omega} = 0, \quad u > 0 & \text{in } \Omega, \end{cases} \tag{P}$$

where Ω is an open bounded domain with smooth boundary, $1 < p < \infty$, $p-1 < q \leq p^* - 1$, $\lambda > 0$, and $0 < \delta < 1$. As usual, $p^* = \frac{Np}{N-p}$ if $1 < p < N$, $p^* \in (p, \infty)$ is arbitrarily large if $p = N$, and $p^* = \infty$ if $p > N$. We employ variational methods in order to show the existence of at least two distinct (positive) solutions of problem (P) in $W_0^{1,p}(\Omega)$. While following an approach due to Ambrosetti-Brezis-Cerami, we need to prove two new results of separate interest: a strong comparison principle and a regularity result for solutions to problem (P) in $C^{1,\beta}(\overline{\Omega})$ with some $\beta \in (0,1)$. Furthermore, we show that $\delta < 1$ is a reasonable sufficient (and likely optimal) condition to obtain solutions of problem (P) in $C^1(\overline{\Omega})$.

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1. Introduction

In this paper we are interested in the following quasilinear and singular problem:

$$\begin{cases} -\Delta_p u = \dfrac{\lambda}{u^\delta} + u^q & \text{in } \Omega; \\ u|_{\partial\Omega} = 0, \quad u > 0 & \text{in } \Omega. \end{cases} \tag{P}$$

Here, Ω is an open bounded domain with smooth boundary, $\Delta_p u = \nabla \cdot (|\nabla u|^{p-2} \nabla u)$ denotes the p-Laplace operator, $1 < p < \infty$, $p - 1 < q \leq p^* - 1$, $\lambda > 0$, and $0 < \delta < 1$. As usual, $p^* = \frac{Np}{N-p}$ if $1 < p < N$, $p^* \in (p, \infty)$ is arbitrarily large

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if $p = N$, and $p^* = \infty$ if $p > N$. Such problems arise, for instance, in models of pseudoplastic flows.

Our main concern is the question of existence and multiplicity of weak solutions to the Dirichlet boundary value problem (P) in $W_0^{1,p}(\Omega)$. To obtain multiple (at least two distinct, positive) solutions of problem (P), we combine some well-known variational methods (see *e.g.* Ambrosetti-Brezis-Cerami [2]) with a few new ideas of our own which employ two new results of separate interest: a regularity result for solutions to problem (P) in $C^{1,\beta}(\overline{\Omega})$ with some $\beta \in (0, 1)$, Theorem 2.2, and a strong comparison principle, Theorem 2.3. Our regularity result is obtained by adapting some ideas from Lieberman [33] for estimates in Campanato spaces. The strong comparison principle extends a result of Cuesta and Takáč [14]. More precisely, we look for solutions to problem (P) that are critical points of the energy functional $E_\lambda \colon W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\begin{aligned} E_\lambda(u) = {} & \frac{1}{p} \int_\Omega |\nabla u|^p \, \mathrm{d}x \\ & - \frac{\lambda}{1-\delta} \int_\Omega (u^+)^{1-\delta} \, \mathrm{d}x - \frac{1}{q+1} \int_\Omega (u^+)^{q+1} \, \mathrm{d}x \end{aligned} \tag{1.1}$$

in the Sobolev space $W_0^{1,p}(\Omega)$. As usual, $r^+ = \max\{r, 0\}$ and $r^- = \max\{-r, 0\}$ for $r \in \mathbb{R}$. Note that E_λ is not of class C^1 on $W_0^{1,p}(\Omega)$ because of the singular term $(u^+)^{1-\delta}$; consequently, one cannot directly apply classical variational methods, such as the Mountain Pass lemma of Ambrosetti-Rabinowitz [4].

First, we show that the number

$$\Lambda \stackrel{\text{def}}{=} \inf\{\lambda > 0 \colon \text{(P) has no weak solution}\} \tag{1.2}$$

satisfies $0 < \Lambda < \infty$. Then we prove the existence of multiple (at least two distinct, positive) solutions of problem (P) for every $\lambda \in (0, \Lambda)$: a local minimizer and a saddle point for the functional E_λ. Indeed, this existence and multiplicity result is a consequence of a competition between the positive and two negative terms in the energy functional E_λ. Notice that $E_\lambda(0) = 0$ and $0 < 1 - \delta < 1 < p < q + 1$. Let $0 < \lambda < \Lambda$. The first negative term,

$$-\frac{\lambda}{1-\delta} \int_\Omega (u^+)^{1-\delta} \, \mathrm{d}x,$$

dominates provided $u > 0$ is "small", the positive term,

$$\frac{1}{p} \int_\Omega |\nabla u|^p \, \mathrm{d}x,$$

becomes dominant for $u > 0$ "mid-sized", and the second negative term,

$$-\frac{1}{q+1} \int_\Omega (u^+)^{q+1} \, \mathrm{d}x,$$

becomes dominant for $u > 0$ "large". This intuitive picture clearly suggests two critical points for E_λ: a local minimizer between "small" and "mid-sized", and a saddle point between "mid-sized" and "large". As $\lambda \in (0, \Lambda)$ approaches Λ, the two critical points merge into a single one for $\lambda = \Lambda$ which disappears for $\lambda > \Lambda$; see definition (1.2).

The local minimizer is obtained first in the C^1 topology with the help of our $C^{1,\beta}$ regularity result (Theorem 2.2) combined with our strong comparison principle (Theorem 2.3). Then we take advantage of arguments due to Brezis and Nirenberg [12] and Ambrosetti, Brezis, and Cerami [2] in order to show that the local minimizer in the C^1 topology is also a local minimizer for E_λ in the $W_0^{1,p}$ topology. In contrast, the saddle point is obtained by a modification of the Mountain Pass lemma of Ambrosetti-Rabinowitz [4], *cf.* Ghoussoub and Preiss [25].

Before giving our main results, let us briefly recall the literature concerning related singular problems. When $p = 2$, the following problem has been investigated in quite a large number of papers:

$$\begin{cases} -\Delta u = \dfrac{\lambda\,\kappa(x)}{u^\delta} + \mu(x)\, u^q & \text{in } \Omega; \\ u|_{\partial\Omega} = 0, \quad u > 0 & \text{in } \Omega. \end{cases} \tag{1.3}$$

The weights $\kappa, \mu\colon \Omega \to \mathbb{R}$ are assumed to be nonnegative and (essentially) bounded. When $\mu = 0$ (the purely singular problem), Crandall, Rabinowitz, and Tartar [15] show that, for any $\delta > 0$, problem (1.3) admits a unique solution u_λ in $C^2(\Omega) \cap C(\overline{\Omega})$; furthermore, if $0 < \delta < 1$ then u_λ is in $C^2(\Omega) \cap C^1(\overline{\Omega})$. When $\mu > 0$ is small enough, Coclite and Palmieri [13] prove the existence of a solution to problem (1.3) for $0 < \lambda < \Lambda$, with Λ as in (1.2), $0 < \Lambda < \infty$. Assuming $0 < \delta < 1$, Yijing, Shaoping, and Yiming [44] apply variational arguments based on Nehari's method [35] to show the existence of at least two solutions for $q > 1$ subcritical: $q < \infty$ if $N = 1$ or 2, and $q < 2^* - 1 = \frac{N+2}{N-2}$ if $N \geq 3$. The critical case $q = 2^* - 1$ and $N \geq 3$ was settled almost simultaneously in Haitao [29] and Hirano, Saccon, and Shioji [31] by two different methods: Perron's method and Nehari's method, respectively. To get the existence of at least two solutions, Haitao [29] shows that for any $0 < \lambda < \Lambda$, the solution obtained by Perron's method is a local minimizer for the energy functional E_λ. His arguments depend on the strong maximum principle (see Brezis and Nirenberg [11, Theorem 3]). In Adimurthi and Giacomoni [1], the existence of at least two solutions in dimension $N = 2$ is extended to $0 < \delta < 3$ and to critical nonlinearities of Trudinger-Moser type (see Moser [34]). Note that $\delta < 3$ is the optimal condition on δ ($\delta > 0$) to obtain solutions in $W_0^{1,2}(\Omega)$.

In the case $p \neq 2$ the question of multiplicity of solutions has been investigated for problems with convex and concave nonlinearities of the following kind:

$$\begin{cases} -\Delta_p u = \lambda\, u^\delta + u^q & \text{in } \Omega; \\ u|_{\partial\Omega} = 0, \quad u > 0 & \text{in } \Omega, \end{cases} \tag{1.4}$$

where $0 < \delta < p - 1 < q \leq p^* - 1$. Ambrosetti, García Azorero, and Peral [3] establish the existence of at least two solutions to problem (1.4) for the subcritical

($q < p^* - 1$) and radially symmetric case ($\Omega = B_R(0)$ a ball). Their main tools are some uniform a priori estimates (that require radial symmetry) and global bifurcation theory. The critical case $q = p^* - 1$ is treated in García Azorero and Peral [20] with additional restrictions on p and $\lambda > 0$ small enough. These restrictions are used to prove that the levels of certain Palais-Smale sequences are strictly below the first critical level $S^{N/p}/N$ at which the Palais-Smale condition fails. Recall

$$S = \inf_{0 \neq u \in W_0^{1,p}(\Omega)} \frac{\int_\Omega |\nabla u|^p \, \mathrm{d}x}{\left(\int_\Omega |u|^{p^*} \, \mathrm{d}x\right)^{p/p^*}} .$$

Note that in this work, the existence of at least two solutions is not obtained for all $\lambda \in (0, \Lambda)$; only for $\lambda > 0$ small enough. The restriction that $\lambda > 0$ be small was removed in García Azorero, Peral, and Manfredi [21] using the approach of C^1 versus $W_0^{1,p}$ local minimizers ([2]). Essential elements in their approach are a $C^{1,\beta}$ regularity result of DiBenedetto [19] and a strong comparison principle of Guedda and Veron [27]. A similar result for problem (P) with radial symmetry is obtained in Giacomoni and Sreenadh [22] when $0 < \delta, \lambda > 0$ is small enough, and $q > p - 1 > 0$. The radially symmetric setting enables a shooting method to be employed; see also Atkinson and Peletier [8] and Prashanth and Sreenadh [37] for $1 < p < N$ and $\lambda \in (0, \Lambda)$.

The outline of this paper is as follows. Our main results are stated in Section 2. In Section 3 we prove the existence of a solution that is a local minimizer of E_λ in $W_0^{1,p}(\Omega)$ for $0 < \lambda < \Lambda$. In the proof we use Theorem 2.2 which follows from the regularity results contained in Appendices A and B. In Section 4, using Ekeland's principle and minimax arguments, we prove the existence of a second solution and thus finish the proof of Theorem 2.1.

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2. Main results

We look for *weak* solutions (solutions, for short) of problem (P), that is, for functions $u \in W_0^{1,p}(\Omega)$ satisfying $\operatorname{ess\,inf}_K u > 0$ over every compact set $K \subset \Omega$ and

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla \phi \, \mathrm{d}x = \lambda \int_\Omega u^{-\delta} \phi \, \mathrm{d}x + \int_\Omega u^q \phi \, \mathrm{d}x \tag{2.1}$$

for all $\phi \in C_c^\infty(\Omega)$. As usual, $C_c^\infty(\Omega)$ denotes the space of all C^∞ functions $\phi \colon \Omega \to \mathbb{R}$ with compact support. We denote by $p^* = Np/(N-p)$ the critical Sobolev exponent for $1 < p < N$; we take $p^* \in (p, \infty)$ arbitrarily large for $p = N$, and $p^* = \infty$ for $p > N$.

We introduce some notation which will be used throughout the paper. Given $1 \le p < \infty$, the norm in $L^p(\Omega)$ is denoted by

$$\|u\|_{L^p(\Omega)} \stackrel{\text{def}}{=} \left(\int_\Omega |u|^p \,\mathrm{d}x \right)^{1/p}$$

and the norm in $W_0^{1,p}(\Omega)$ by

$$\|u\|_{W_0^{1,p}(\Omega)} \stackrel{\text{def}}{=} \left(\int_\Omega |\nabla u|^p \,\mathrm{d}x \right)^{1/p} .$$

The normalized positive eigenfunction associated with the principal eigenvalue λ_1 of $-\Delta_p$ is denoted by ϕ_1:

$$-\Delta_p \phi_1 = \lambda_1 \, |\phi_1|^{p-2} \phi_1 \quad \text{in } \Omega; \qquad \phi_1 = 0 \quad \text{on } \partial\Omega, \tag{2.2}$$

$\phi_1 \in W_0^{1,p}(\Omega)$ is normalized by $\phi_1 > 0$ in Ω and $\int_\Omega \phi_1^p \,\mathrm{d}x = 1$.

The function $d(x)$ denotes the distance from a point $x \in \overline{\Omega}$ to the boundary $\partial\Omega$, where $\overline{\Omega} = \Omega \cup \partial\Omega$ is the closure of $\Omega \subset \mathbb{R}^N$. This means that

$$d(x) \stackrel{\text{def}}{=} \operatorname{dist}(x, \partial\Omega) \equiv \inf_{y \in \partial\Omega} |x - y|.$$

Note that the strong maximum and boundary point principles from Vázquez [43, Theorem 5, page 200] guarantee $\phi_1 > 0$ in Ω and $\frac{\partial \phi_1}{\partial \nu} < 0$ on $\partial\Omega$, respectively. Hence, since $\phi_1 \in C^1(\overline{\Omega})$, there are constants ℓ and L, $0 < \ell < L$, such that $\ell \, d(x) \le \phi_1(x) \le L \, d(x)$ for all $x \in \Omega$.

The open ball in $W_0^{1,p}(\Omega)$ of radius r centered at u is denoted by

$$\mathcal{B}_r(u) \stackrel{\text{def}}{=} \{ v \in W_0^{1,p}(\Omega) \colon \|u - v\|_{W_0^{1,p}(\Omega)} < r \},$$

for some $r \in (0, \infty)$ and $u \in W_0^{1,p}(\Omega)$. If $u = 0$, we abbreviate $\mathcal{B}_r \equiv \mathcal{B}_r(0)$. Finally, the open ball in $\mathbb{R}^N$ of radius r centered at x is denoted by $B_r(x)$.

Our main result is the following theorem.

Theorem 2.1. *Let the pair (p, q) satisfy either $p \in (1, \infty)$ and $q \in (p-1, p^*-1)$, or else $p \in \left(\frac{2N}{N+2}, 2 \right] \cup \left(\frac{3N}{N+3}, 3 \right)$ and $q = p^* - 1$. Then there exists $\Lambda \in (0, \infty)$ with the following properties*:

(i) *For every $0 < \lambda < \Lambda$ there exist at least two solutions of problem* (P)*, u_λ and v_λ, such that $u_\lambda, v_\lambda \in C^1(\overline{\Omega})$ and $u_\lambda \lneqq v_\lambda$.*

(ii) *For $\lambda = \Lambda$ there exists at least one solution of* (P) *in $C^1(\overline{\Omega})$.*

(iii) *For every $\lambda > \Lambda$ there is no solution of* (P).

To prove Theorem 2.1, we establish a $C^{1,\beta}(\overline{\Omega})$, Theorem B.1 in Appendix B. Theorem B.1 gives the following regularity result for weak solutions to problem (P).

Theorem 2.2. *Let* $0 < \delta < 1$, $1 < p < \infty$, *and* $p - 1 < q \le p^* - 1$. *Then any weak solution to problem* (P) *belongs to* $C^{1,\beta}(\overline{\Omega})$ *for some* $\beta \in (0, 1)$.

This regularity result motivates and complements the following new strong comparison principle.

Theorem 2.3. *Let* $u, v \in C^{1,\beta}(\overline{\Omega})$, *for some* $0 < \beta < 1$, *satisfy* $0 \lneqq u$, $0 \lneqq v$ *and*

$$-\Delta_p u - \lambda u^{-\delta} = f, \tag{2.3}$$

$$-\Delta_p v - \lambda v^{-\delta} = g, \tag{2.4}$$

with $u = v = 0$ *on* $\partial\Omega$, *where* $f, g \in C(\Omega)$ *are such that* $0 \le f < g$ *pointwise everywhere in* Ω. *Then, the following strong comparison principle holds*:

$$0 < u < v \ \text{ in } \ \Omega \quad \text{and} \quad \frac{\partial v}{\partial \nu} < \frac{\partial u}{\partial \nu} < 0 \ \text{ on } \ \partial\Omega. \tag{2.5}$$

Remark 2.4. Theorem 2.3 holds if we replace the p-Laplacian operator by a more general quasilinear operator; see, for instance, conditions (3)-(7) in Cuesta and Takáč [14].

Proof of Theorem 2.3. First, note that from the strong maximum of Vázquez (see Theorem 5 in [43]), we infer that $u > 0$ in Ω and $\frac{\partial u}{\partial \nu} < 0$ on $\partial\Omega$. Hence, since $u \in C^1(\overline{\Omega})$, there are constants ℓ and L, $0 < \ell < L$, such that $\ell\, d(x) \le u(x) \le L\, d(x)$ near the boundary $\partial\Omega$. Analogous results hold for v. Moreover, $f \le g$ in Ω guarantees $u \le v$ in Ω, by the weak comparison principle which can be proved by a standard variational argument. Consequently,

$$\ell\, d(x) \le u(x) \le v(x) \le L\, d(x) \tag{2.6}$$

near the boundary $\partial\Omega$. As in the proof of Proposition 2.4 in Cuesta and Takáč [14] (see page 729), we define an η-neighborhood $\Omega_\eta \subset \Omega$ of the boundary $\partial\Omega$,

$$\Omega_\eta \stackrel{\text{def}}{=} \{x \in \Omega \colon d(x) < \eta\}, \tag{2.7}$$

for $\eta > 0$, and set $w \stackrel{\text{def}}{=} v - u$, $0 \le w \in C^{1,\beta}(\overline{\Omega})$ with $w = 0$ on $\partial\Omega$. There exists $\eta > 0$ small enough, such that in the open set Ω_η we have

$$\begin{aligned} &- \operatorname{div}(A(x)\nabla w) - B(x)w \\ &= - \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\, \frac{\partial w}{\partial x_j}\right) - \lambda B(x) w = g - f > 0. \end{aligned} \tag{2.8}$$

The coefficients $a_{ij}(x)$ are given by

$$a_{ij}(x) = \int_0^1 |(1-t)\nabla u(x) + t\nabla v(x)|^{p-2} \times \left[\delta_{ij} + (p-2)\, \frac{\frac{\partial}{\partial x_i}((1-t)u + tv)\, \frac{\partial}{\partial x_j}((1-t)u + tv)}{|(1-t)\nabla u(x) + t\nabla v(x)|^2} \right] \mathrm{d}t \tag{2.9}$$

for $x \in \Omega_\eta$ and $i, j = 1, 2, \ldots, N$, where δ_{ij} denotes the Kronecker symbol: $\delta_{ij} = 1$ if $i = j$; $\delta_{ij} = 0$ if $i \neq j$. The differential operator above induced by the matrix $(a_{ij})_{i,j=1,2,\ldots,N}$ is uniformly elliptic in Ω_η with $a_{ij} \in C^{0,\beta}(\overline{\Omega_\eta})$ provided $\eta > 0$ is chosen small enough. The coefficient $B(x)$ satisfies

$$B(x) = -\delta \int_0^1 \frac{\mathrm{d}t}{((1-t)u(x) + tv(x))^{\delta+1}} < 0. \tag{2.10}$$

Inequalities in (2.6) guarantee that $B(x)$ satisfies the conditions of Lemma 2.7 in Hernández, Mancebo, and Vega [30]. We conclude that the (classical) strong maximum principle applies to inequality (2.8) in each connected component of the open set Ω_η, thus yielding inequalities (2.5) in Ω_η.

Finally, we will show that $u < v$ throughout Ω. Let $\eta' \in (0, \eta)$ and $\tilde{\Omega} \stackrel{\text{def}}{=} \Omega \setminus \overline{\Omega_{\eta'}}$. Employing $w > 0$ in Ω_η, we can find $c > 0$ such that $w \geq c$ on $\partial\tilde{\Omega} \subset \Omega_\eta$. Moreover, recalling $f, g \in C(\Omega)$ with $0 \leq f < g$ pointwise everywhere in Ω, we can choose $c > 0$ small enough, such that also

$$\frac{\lambda}{u^\delta} - \frac{\lambda}{(u+c)^\delta} \leq g - f \quad \text{holds in } \tilde{\Omega}.$$

It follows that $u + c \leq v$ on $\partial\tilde{\Omega}$ together with

$$-\Delta_p(u+c) - \frac{\lambda}{(u+c)^\delta} \leq f + (g - f) = g = -\Delta_p v - \frac{\lambda}{v^\delta} \quad \text{in } \tilde{\Omega}.$$

Consequently, we may apply the weak comparison principle (see Proposition 2.3 in [14]) in order to conclude that $u + c \leq v$ holds throughout $\tilde{\Omega}$. As $\Omega = \Omega_\eta \cup \tilde{\Omega}$, we have verified $u < v$ throughout Ω. □

3. Existence of weak solutions

3.1. Existence of a solution for $0 < \lambda \leq \Lambda$

First, let us consider the following purely singular Dirichlet problem:

$$\begin{cases} -\Delta_p u = \lambda u^{-\delta} & \text{in } \Omega; \\ u|_{\partial\Omega} = 0, \quad u > 0 & \text{in } \Omega. \end{cases} \tag{3.1}$$

Recall $0 < \delta < 1$. By requiring "$u > 0$ in Ω" we actually mean $\operatorname{ess\,inf}_K u > 0$ for any compact set $K \subset \Omega$. We look for a solution $u \in W_0^{1,p}(\Omega)$ that satisfies equation (3.1) in the sense of distributions. More precisely, if $u_0 \in W_0^{1,p}(\Omega)$ is a distributional solution of problem (3.1), with $\operatorname{ess\,inf}_K u_0 > 0$ for any compact set $K \subset \Omega$, then $u_0 \in C^1(\Omega)$ by interior regularity due (independently) to DiBenedetto [19, Theorem 2, page 829] and Tolksdorf [42, Theorem 1, page 127].

Lemma 3.1. *Assume* $0 < \delta < 1$ *and* $\lambda > 0$. *Then problem* (3.1) *has a unique weak solution in* $W_0^{1,p}(\Omega)$ *in the sense of distributions. This solution, denoted by* $\underline{u}_\lambda$, *satisfies* $\underline{u}_\lambda \geq \epsilon_\lambda \phi_1$ *a.e. in* Ω, *where* $\epsilon_\lambda > 0$ *is a constant.*

Proof. First, we observe that an energy functional on $W_0^{1,p}(\Omega)$ formally corresponding to problem (3.1) can be given by

$$\tilde{E}_\lambda(u) \stackrel{\text{def}}{=} \frac{1}{p}\int_\Omega |\nabla u|^p \,\mathrm{d}x - \frac{\lambda}{1-\delta}\int_\Omega (u^+)^{1-\delta}\,\mathrm{d}x\,, \quad u \in W_0^{1,p}(\Omega).$$

Owing to the Poincaré inequality and $0 < 1-\delta < 1 < p < \infty$, this functional is coercive and weakly lower semicontinuous on $W_0^{1,p}(\Omega)$. It follows that $\tilde{E}_\lambda$ possesses a global minimizer $u_0 \in W_0^{1,p}(\Omega)$. We have $u_0 \not\equiv 0$ in Ω, owing to $\tilde{E}_\lambda(0) = 0 > \tilde{E}_\lambda(\epsilon\phi_1)$ for every $\epsilon > 0$ small enough.

Second, the polar decomposition $u = u^+ - u^-$ of any function $u \in W_0^{1,p}(\Omega)$ gives $\nabla u = \nabla u^+ - \nabla u^-$. Thus, if u_0 is a global minimizer for $\tilde{E}_\lambda$, then so is its absolute value $|u_0|$, by $\tilde{E}_\lambda(|u_0|) \leq \tilde{E}_\lambda(u_0)$. The equality $\tilde{E}_\lambda(|u_0|) = \tilde{E}_\lambda(u_0)$ holds if and only if $u_0^- = 0$ a.e. in Ω, that is, if and only if $u_0 \geq 0$ a.e. in Ω. Thus, any global minimizer u_0 for $\tilde{E}_\lambda$ must satisfy $u_0 \geq 0$ a.e. in Ω. Equivalently, $u \in W_0^{1,p}(\Omega)_+$ where

$$W_0^{1,p}(\Omega)_+ \stackrel{\text{def}}{=} \left\{u \in W_0^{1,p}(\Omega) \colon u \geq 0 \text{ a.e. in } \Omega\right\}$$

stands for the positive cone in $W_0^{1,p}(\Omega)$.

Third, we will show that even $u_0 \geq \epsilon\phi_1$ holds almost everywhere in Ω with a constant $\epsilon > 0$ small enough. To this end, let us first remark that the Gâteaux derivative $\tilde{E}'_\lambda(\epsilon\phi_1)$ of $\tilde{E}_\lambda$ at $\epsilon\phi_1$ exists and satisfies

$$\begin{aligned}\tilde{E}'_\lambda(\epsilon\phi_1) &= -\Delta_p(\epsilon\phi_1) - \lambda(\epsilon\phi_1)^{-\delta} = \lambda_1(\epsilon\phi_1)^{p-1} - \lambda(\epsilon\phi_1)^{-\delta}\\ &= (\epsilon\phi_1)^{-\delta}\left(\lambda_1(\epsilon\phi_1)^{p-1+\delta} - \lambda\right)\\ &\leq -\frac{\lambda}{2}(\epsilon\phi_1)^{-\delta} < 0\end{aligned} \tag{3.2}$$

whenever $\epsilon > 0$ is small enough, say, $0 < \epsilon \leq \epsilon_\lambda$.

On the contrary to our claim above, suppose that the (nonnegative) function $v = (u_0 - \epsilon_\lambda \phi_1)^- = (\epsilon_\lambda \phi_1 - u_0)^+$ does not vanish identically in Ω. Denote

$$\Omega^+ = \{x \in \Omega \colon v(x) > 0\}.$$

Let us investigate the function $\xi(t) \stackrel{\text{def}}{=} \tilde{E}_\lambda(u_0 + tv)$ of $t \in \mathbb{R}_+ = [0, \infty)$. This function is convex thanks to the fact that the restriction of the functional $\tilde{E}_\lambda$ to the positive cone $W_0^{1,p}(\Omega)_+$ is convex. We have $\xi(t) \geq \xi(0)$ for all $t \geq 0$. Furthermore, owing to $u_0 + tv \geq \max\{u_0,\, t\epsilon_\lambda\phi_1\} \geq t\epsilon_\lambda\phi_1$ for $t > 0$, the Gâteaux derivative $\tilde{E}'_\lambda(u_0 + tv)$ of $\tilde{E}_\lambda$ at $u_0 + tv$ exists and yields $\xi'(t) = \langle \tilde{E}'_\lambda(u_0 + tv),\, v\rangle$ for $t > 0$. This derivative is nonnegative and nondecreasing. Consequently, for $0 < t < 1$ we have

$$\begin{aligned} 0 \leq \xi'(1) - \xi'(t) &= \langle \tilde{E}'_\lambda(u_0 + v) - \tilde{E}'_\lambda(u_0 + tv),\ v\rangle \\ &= \int_{\Omega^+} \tilde{E}'_\lambda(\epsilon_\lambda\phi_1)\, v \,\mathrm{d}x - \xi'(t) \\ &\leq -\frac{\lambda}{2}\int_{\Omega^+} (\epsilon_\lambda\phi_1)^{-\delta}\, v \,\mathrm{d}x < 0, \end{aligned} \tag{3.3}$$

by inequality (3.2) and $\xi'(t) \geq 0$, a contradiction. We have verified $v \equiv 0$ in Ω, that is, $u_0 \geq \epsilon_\lambda\phi_1$ a.e. in Ω.

Finally, we have proved that every global minimizer u_0 for $\tilde{E}_\lambda$ on $W_0^{1,p}(\Omega)$ must satisfy $u_0 \geq \epsilon_\lambda\phi_1$ a.e. in Ω. The functional $\tilde{E}_\lambda$ being strictly convex on $W_0^{1,p}(\Omega)_+$, we conclude that u_0 is the only critical point of $\tilde{E}_\lambda$ in $W_0^{1,p}(\Omega)_+$ with the property $\operatorname{ess\,inf}_K u_0 > 0$ for any compact set $K \subset \Omega$. Consequently, $\underline{u}_\lambda = u_0$ provides the unique weak solution to problem (3.1). □

Remark 3.2. In our proof of Lemma 3.1 above, $v_0 = 0$ is a critical point for the functional $v \mapsto \tilde{E}_\lambda(u_0 + v)$ defined for all $v \in C_c^\infty(\Omega)$ only. We have proved that the functional $\tilde{E}_\lambda$ restricted to $W_0^{1,p}(\Omega)_+$ has precisely one critical point that stays away from zero, uniformly on any compact set $K \subset \Omega$, namely, the global minimizer u_0.

We obtain the following result regarding Λ.

Lemma 3.3. *Let* $0 < \delta < 1$ *and* $p - 1 < q \leq p^* - 1$. *Then* $0 < \Lambda < \infty$.

Proof. We give the proof only in the critical case, *i.e.* $q = p^* - 1$. In the subcritical case, *i.e.*, $q < p^* - 1$, the proof is simpler since the energy functional $\underline{E}_\lambda$ defined below is weakly lower semicontinuous in $W_0^{1,p}(\Omega)$. Let $\underline{u}_\lambda$ be the unique solution to (3.1). Define

$$f_\lambda(x, s) \stackrel{\text{def}}{=} \begin{cases} \lambda s^{-\delta} + s^q & \text{if } s > \underline{u}_\lambda(x); \\ \lambda(\underline{u}_\lambda(x))^{-\delta} + (\underline{u}_\lambda(x))^q & \text{if } s \leq \underline{u}_\lambda(x). \end{cases} \tag{3.4}$$

Let $F_\lambda(x,s) = \int_0^s f_\lambda(x,t)\,\mathrm{d}t$. Define $\underline{E}_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ by

$$\underline{E}_\lambda(u) \stackrel{\text{def}}{=} \frac{1}{p}\int_\Omega |\nabla u|^p\,\mathrm{d}x - \int_\Omega F_\lambda(x,u)\,\mathrm{d}x. \tag{3.5}$$

From Lemma A.4, $\underline{E}_\lambda$ is $C^1(W_0^{1,p}(\Omega),\mathbb{R})$. We consider the following minimization problem:

$$I_\lambda = \min_{u\in\overline{\mathcal{B}}_r} \underline{E}_\lambda(u). \tag{3.6}$$

Clearly, we have $I_\lambda > -\infty$. Note that $\int_\Omega \left(\frac{1}{p}|\nabla u|^p - \frac{1}{q+1}|u|^{q+1}\right)\mathrm{d}x > 0$ for every $u \in \partial\mathcal{B}_r$ provided $r > 0$ is small enough. Fix such $r > 0$; the other negative term in $\underline{E}_\lambda(u)$ may be made arbitrarily small by taking $\lambda > 0$ small enough. Therefore we find r and λ such that

$$\min_{u\in\partial\mathcal{B}_r} \underline{E}_\lambda(u) > 0. \tag{3.7}$$

Moreover, since $\underline{E}_\lambda(tu) < 0$ for t small, we have

$$I_\lambda < 0. \tag{3.8}$$

Let $\{u_n\}_{n=1}^\infty$ be a minimizing sequence, *i.e.* $u_n \subset \mathcal{B}_r$ and $\underline{E}_\lambda(u_n) \to I_\lambda$ as $n \to \infty$. From (3.7) and (3.8), $\{u_n\}_{n=1}^\infty$ satisfies $\mathrm{dist}(u_n,\partial\mathcal{B}_r) \geq \eta_0$ for some $\eta_0 > 0$. Therefore, there exists $0 < r_0 < r$ such that

$$u_n \in \mathcal{B}_{r_0}. \tag{3.9}$$

Now, from Ekeland's variational principle, there exist $r_0 \leq r_1 < r$ and a sequence $\{v_n\}_{n=1}^\infty \subset \mathcal{B}_{r_1}$ satisfying

$$\begin{aligned} &\mathrm{dist}(u_n,v_n) \leq \frac{1}{n}, \quad \underline{E}_\lambda(u_n) \leq \underline{E}_\lambda(v_n) \ \text{ and} \\ &\underline{E}'_\lambda(v_n) \to 0 \ \text{ in } \ W^{-1,p'}(\Omega) \text{ as } n\to\infty. \end{aligned} \tag{3.10}$$

From the first statement of (3.10), $\{v_n\}_{n=1}^\infty$ is a minimizing sequence for I_λ and up to a subsequence satisfies $v_n \rightharpoonup \tilde{u}_\lambda$ as $n \to \infty$ with $\tilde{u}_\lambda \in \overline{\mathcal{B}_{r_1}}$. From the last statement of (3.10), we have

$$-\Delta_p(v_n) - f_\lambda(x,v_n) = o_n(1) \quad \text{in } \ W^{-1,p'}(\Omega). \tag{3.11}$$

From (3.11), Theorem 2.1 in Boccardo and Murat [9] with

$$\begin{aligned} f_n(x) &= (\max\{v_n(x),\underline{u}_\lambda(x)\})^{-\delta} + o_n(1), \\ g_n(x) &= (\max\{v_n(x),\underline{u}_\lambda(x)\})^q \end{aligned}$$

(note that from Hardy's inequality and since $q = p^* - 1$, $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ satisfy the conditions in Theorem 2.1 in [9]), Remark 2.1 in [9] and from Brezis and Lieb [10], it follows that

$$\begin{aligned} &\|v_n\|_{W_0^{1,p}(\Omega)} = \|v_n - \tilde{u}_\lambda\|_{W_0^{1,p}(\Omega)} + \|\tilde{u}_\lambda\|_{W_0^{1,p}(\Omega)} + o_n(1) \text{ and} \\ &\|v_n\|_{L^{q+1}(\Omega)} = \|v_n - \tilde{u}_\lambda\|_{L^{q+1}(\Omega)} + \|\tilde{u}_\lambda\|_{L^{q+1}(\Omega)} + o_n(1) \end{aligned} \tag{3.12}$$

as $n \to \infty$. From (3.9), (3.10) and (3.12), it follows that $\tilde{u}_\lambda, v_n - \tilde{u}_\lambda \in \mathcal{B}_r$. Thus,

$$\int_\Omega \left(\frac{1}{p} |\nabla v_n - \tilde{u}_\lambda|^p - \frac{1}{q+1} |v_n - \tilde{u}_\lambda|^{q+1} \right) \mathrm{d}x > 0\,. \tag{3.13}$$

From (3.12) and (3.13), we get

$$\begin{aligned} I_\lambda &= \underline{E}_\lambda(v_n) + o_n(1) \\ &= \underline{E}_\lambda(\tilde{u}_\lambda) + \frac{1}{p}\|v_n - \tilde{u}_\lambda\|^p_{W_0^{1,p}(\Omega)} - \frac{1}{q+1}\|v_n - \tilde{u}_\lambda\|^{q+1}_{L^{q+1}(\Omega)} + o_n(1) \\ &\geq \underline{E}_\lambda(\tilde{u}_\lambda) + o_n(1). \end{aligned}$$

Hence, $\underline{E}_\lambda(\tilde{u}_\lambda) = I_\lambda$ and

$$\begin{cases} -\Delta_p \tilde{u}_\lambda = f_\lambda(x, \tilde{u}_\lambda) \text{ in } \Omega; \\ \tilde{u}_\lambda|_{\partial\Omega} = 0. \end{cases}$$

Now, Theorem 2.3 imply that $\tilde{u}_\lambda > \underline{u}_\lambda$ in Ω, hence $\tilde{u}_\lambda$ is a weak solution to problem (P). Thus $\Lambda > 0$.

Now, let us show that $\Lambda < \infty$. We argue by contradiction: suppose there exists a sequence $\lambda_n \to \infty$ such that problem (P) admits a solution u_n. There exists $\lambda_* > 0$ such that

$$\frac{\lambda}{t^\delta} + t^q \geq (\lambda_1 + \epsilon) t^{p-1} \text{ for all } t > 0,\ \epsilon \in (0, 1) \text{ and } \lambda > \lambda_*.$$

Choose $\lambda_n > \lambda_*$. Clearly u_n is a supersolution of the problem

$$\begin{cases} -\Delta_p u = (\lambda_1 + \epsilon) u^{p-1} \text{ in } \Omega; \\ u > 0, \quad u|_{\partial\Omega} = 0. \end{cases} \tag{3.14}$$

for all $\epsilon \in (0, 1)$. We now use Lemma 3.1 to choose $\mu < \lambda_1 + \epsilon$ small enough so that $\mu\phi_1(x) < u_n(x)$ and $\mu\phi_1$ is a subsolution to problem (3.14). By a monotone interation procedure we obtain a solution to (3.14) for any $\epsilon \in (0, 1)$, contradicting the fact that λ_1 is an isolated point in the spectrum of $-\Delta_p$ in $W_0^{1,p}(\Omega)$ (see Anane [5]). □

We prove now the existence of positive weak solution to (P) for any $0 < \lambda < \Lambda$. Precisely, we have the following result:

Proposition 3.4. *For any $\lambda \in (0, \Lambda)$, there exists u_λ a positive weak solution to* (P). *Moreover, $E_\lambda(u_\lambda) < 0$.*

Proof. Fix $0 < \lambda < \lambda_2 < \Lambda$. λ_2 is such that there exist solutions to (P) for $\lambda = \lambda_2$. Let $\underline{u}_\lambda$ be the solution of (3.1) and u_{λ_2} is one solution of (P) (when $\lambda = \lambda_2$ in the equation of (P)). Clearly, from Theorem B.1 in Appendix B, $\underline{u}_\lambda, u_{\lambda_2}$ are in $C^{1,\beta}(\overline{\Omega})$ for some $0 < \beta < 1$ and $\underline{u}_\lambda \le u_{\lambda_2}$ in Ω. Indeed, setting $\underline{\Omega} \stackrel{\text{def}}{=} \{\underline{u}_\lambda - \bar{u} > 0\}$ and from the equations satisfied by $\underline{u}_\lambda$ and $\bar{u} = u_{\lambda_2}$ we have

$$\int_{\underline{\Omega}} (+\Delta_p \bar{u} - \Delta_p \underline{u}_\lambda)(\underline{u}_\lambda - \bar{u})\,\mathrm{d}x \le \lambda \int_{\underline{\Omega}} (\underline{u}_\lambda^{-\delta} - \bar{u}^{-\delta})(\underline{u}_\lambda - \bar{u})\,\mathrm{d}x \le 0 \quad (3.15)$$

and

$$\begin{aligned} &\int_{\underline{\Omega}} (+\Delta_p \bar{u} - \Delta_p \underline{u}_\lambda)(\underline{u}_\lambda - \bar{u})\,\mathrm{d}x \\ &\ge \int_{\underline{\Omega}} (|\nabla \underline{u}_\lambda|^{p-2}\nabla \underline{u}_\lambda - |\nabla \bar{u}|^{p-2}\nabla \bar{u})(\nabla \underline{u}_\lambda - \nabla \bar{u})\,\mathrm{d}x \\ &\ge \begin{cases} C_p \displaystyle\int_{\underline{u}_\lambda - \bar{u} > 0} \frac{|\nabla(\underline{u}_\lambda - \bar{u})|^2}{(|\nabla \underline{u}_\lambda| + |\nabla \bar{u}|)^{2-p}}\,\mathrm{d}x & \text{if } 1 < p < 2 \\ C_p \displaystyle\int_{\underline{u}_\lambda - \bar{u} > 0} |\nabla(\underline{u}_\lambda - \bar{u})|^p\,\mathrm{d}x & \text{if } p \ge 2 \end{cases} \\ &\ge 0 \end{aligned} \quad (3.16)$$

from Lemma 4.1 in Ghoussoub and Yuan [26]. Hence from (3.15) and (3.16), we get $\underline{u}_\lambda \le \bar{u}$.

By the strong comparison principle (Theorem 2.3), we obtain $\bar{u} > \underline{u}_\lambda$ in Ω, $\frac{\partial \bar{u}}{\partial \nu} < \frac{\partial \underline{u}_\lambda}{\partial \nu}$ on $\partial\Omega$. Define

$$\tilde{f}_\lambda(x, s) = \begin{cases} \lambda \bar{u}(x)^{-\delta} + \bar{u}(x)^q & \text{if } s > \bar{u}(x), \\ \lambda s^{-\delta} + s^q & \text{if } \underline{u}_\lambda(x) \le s \le \bar{u}(x), \\ \lambda \underline{u}_\lambda(x)^{-\delta} + \underline{u}_\lambda(x)^q & \text{if } s < \underline{u}_\lambda(x). \end{cases}$$

Let $\tilde{F}_\lambda(x, s) = \int_0^s \tilde{f}_\lambda(x, t)\,\mathrm{d}t$. Define the functional $\tilde{E}_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ by

$$\tilde{E}_\lambda(u) = \frac{1}{p} \int_\Omega |\nabla u|^p\,\mathrm{d}x - \int_\Omega \tilde{F}_\lambda(x, u)\,\mathrm{d}x.$$

$\tilde{E}_\lambda$ is bounded below in $W_0^{1,p}(\Omega)$ and is weakly lower semi-continuous. Hence, $\tilde{E}_\lambda$ achieves its global minimum at some $u_\lambda \in W_0^{1,p}(\Omega)$. Moreover, since $\tilde{E}_\lambda$ is

C^1 by Lemma A.4, u_λ solves the equation $-\Delta_p u_\lambda = \tilde{f}_\lambda(x, u_\lambda)$ in Ω. From the strong maximum principle of Vázquez (see Theorem 5 in [43]) we get $u_\lambda > 0$ in Ω. It follows by regularity results (see again Theorem B.1 in Appendix B) that $u_\lambda \in C^{1,\beta}(\overline{\Omega})$ for some $\beta \in (0, 1)$. Again, by Therorem 2.3, we conclude that $\underline{u}_\lambda < u_\lambda < \bar{u}$ in Ω and $\frac{\partial}{\partial \nu}(u_\lambda - \underline{u}_\lambda) < 0$, $\frac{\partial}{\partial \nu}(\bar{u} - u_\lambda) < 0$ on $\partial\Omega$. Hence, $\tilde{f}_\lambda(x, u_\lambda) = \lambda u_\lambda^{-\delta} + u_\lambda^q$ for $x \in \Omega$ and so u_λ is a weak solution to (P). Moreover, we have that

$$\tilde{E}_\lambda(u_\lambda) \le \tilde{E}_\lambda(\underline{u}_\lambda) = E_\lambda(\underline{u}_\lambda) < \frac{1}{p}\int_\Omega |\nabla \underline{u}_\lambda|^p \,\mathrm{d}x - \frac{\lambda}{1-\delta}\int_\Omega {\underline{u}_\lambda}^{1-\delta}\,\mathrm{d}x < 0.$$

This completes the proof of Proposition 3.4. □

Now, we show the following result.

Proposition 3.5. *There exists at least one positive weak solution for* $\lambda = \Lambda$ *to* (P).

Proof. Let $\{\lambda_k\}_{k=1}^\infty$ such that $\lambda_k \nearrow \Lambda$ as $k \to \infty$. Then, from Proposition 3.4, there exists $u_k = u_{\lambda_k} \ge \underline{u}_{\lambda_k}$ to a weak positive solution to (P) for $\lambda = \lambda_k$. Therefore, for any $\phi \in C_c^\infty(\Omega)$, we have

$$\int_\Omega |\nabla u_k|^{p-2} \nabla u_k \nabla \phi \,\mathrm{d}x = \lambda_k \int_\Omega (u_k)^{-\delta}\phi\,\mathrm{d}x + \int_\Omega u_k^q \phi\,\mathrm{d}x. \tag{3.17}$$

Since $u_k \in W_0^{1,p}(\Omega)$ and $u_k \ge \underline{u}_{\lambda_k}$ it is easy to see that (3.17) holds also for $\phi \in W_0^{1,p}(\Omega)$. Moreover, from Proposition 3.4

$$E_{\lambda_k}(u_k) < 0. \tag{3.18}$$

From (3.18), it follows that

$$\sup_k \|u_k\|_{W_0^{1,p}(\Omega)} < \infty. \tag{3.19}$$

Hence, there exists $u_\Lambda \ge \underline{u}_{\lambda_k}$ such that $u_k \rightharpoonup u_\Lambda$ in $W_0^{1,p}(\Omega)$ as $k \to \infty$ and then by Sobolev imbedding:

$$u_k \rightharpoonup u_\Lambda \text{ in } L^q(\Omega) \text{ and pointwise a.e. as } k \to \infty. \tag{3.20}$$

From (3.17), (3.19) and (3.20), we get for any $\phi \in W_0^{1,p}(\Omega)$:

$$\int_\Omega |\nabla u_\Lambda|^{p-2}\nabla u_\Lambda \nabla\phi\,\mathrm{d}x = \lambda \int_\Omega (u_\Lambda)^{-\delta}\phi\,\mathrm{d}x + \int_\Omega u_\Lambda^q \phi\,\mathrm{d}x \tag{3.21}$$

which completes the proof of Proposition 3.5. □

From the above propositions, we get the following corollary:

Corollary 3.6. *Let* $1 < p < \infty$, $p - 1 < q \le p^* - 1$, $0 < \delta < 1$, *and* $0 < \lambda \le \Lambda$. *Then there exists a minimal solution to* (P).

Proof. We use here the weak comparison principle (see Proposition 2.3 in Cuesta and Takáč [14] or Tolksdorf [41]) and the following monotone iterative scheme:

$$\begin{cases} -\Delta_p u_n - \dfrac{\lambda}{u_n^\delta} = u_{n-1}^q \ \text{ in } \Omega; \\ u_n|_{\partial\Omega} = 0, \end{cases} \tag{3.22}$$

where $u_0 = \underline{u}_\lambda$, the unique solution to (3.1). Note that u_0 is a weak subsolution to (P) and $u_0 \le u_\Lambda$ where u_Λ is the solution to (P) obtained in Proposition 3.5. Then, from the weak comparison principle, we get easily that $u_0 \le u_1$ and $\{u_n\}_{n=1}^\infty$ is nondecreasing. Furthermore, $u_n \le u_\Lambda$ and $\{u_n\}_{n=1}^\infty$ is uniformly bounded in $W_0^{1,p}(\Omega)$. Hence, it is easy to prove that $\{u_n\}_{n=1}^\infty$ converges weakly in $W_0^{1,p}(\Omega)$ and pointwise to $\hat{u}_\lambda$, a weak solution to (P). Let us show that $\hat{u}_\lambda$ is the minimal solution to (P) for any $\lambda \in (0, \Lambda]$. Let v_λ a weak solution to (P) for $\lambda \in (0, \Lambda]$. Then, $u_0 = \underline{u}_\lambda \le v_\lambda$. From the weak comparison principle, $u_n \le v_\lambda$ for any $n \ge 0$. Letting $n \to \infty$, we get $\hat{u}_\lambda \le v_\lambda$. This completes the proof of Corollary 3.6. □

3.2. C^1 versus $W^{1,p}$ local minimizers of the energy

Let u_λ be the solution to (P) given by Proposition 3.4. The main result in this paragraph is

Proposition 3.7. *For* $0 < \lambda < \Lambda$, u_λ *is a local minimizer of* $\underline{E}_\lambda$ *in* $W_0^{1,p}(\Omega)$.

Proof. We observe first that u_λ is a local minimizer in the C^1-topology. Indeed, taking advantage of the strong comparison principle shown in Theorem 2.3 and the definition of u_λ, we have that for $\nu > 0$ small enough,

$$\|u - u_\lambda\|_{C^1(\bar{\Omega})} \le \nu \Rightarrow \underline{u}_\lambda \le u \le \bar{u}. \tag{3.23}$$

where $\bar{u}$ is defined in the proof of Proposition 3.4. From (3.23), we get that

$$\|u - u_\lambda\|_{C^1(\bar{\Omega})} \le \nu \Rightarrow \underline{E}_\lambda(u_\lambda) = \tilde{E}_\lambda(u_\lambda) \le \tilde{E}_\lambda(u) = \underline{E}_\lambda(u).$$

Now, let us show that u_λ is a local minimizer of $\underline{E}_\lambda$ in $W_0^{1,p}(\Omega)$. Suppose not and we derive a contradiction. First, we deal with the subcritical case, *i.e.* $q < p^* - 1$. In this case, $\underline{E}_\lambda$ is weakly lower semicontinuous on $W_0^{1,p}(\Omega)$ and achieves its minimum on bounded subsets of $W_0^{1,p}(\Omega)$. Hence, if u_λ is not a local minimum for $\underline{E}_\lambda$, for every $\epsilon > 0$ we obtain v_ϵ such that $0 < \|v_\epsilon\|_{W_0^{1,p}(\Omega)} \le \epsilon$ and

$$\underline{E}_\lambda(u_\lambda + v_\epsilon) < \underline{E}_\lambda(u_\lambda),\ \underline{E}_\lambda(u_\lambda + v_\epsilon) = \inf_{\|v\|_{W_0^{1,p}(\Omega)} \le \epsilon} \underline{E}_\lambda(u_\lambda + v). \tag{3.24}$$

By the Lagrange multiplier rule (see Phelps [36]), we obtain $\mu_\epsilon \leq 0$ such that

$$\langle \underline{E}'_\lambda(u_\lambda + v_\epsilon), h\rangle = \mu_\epsilon \int_\Omega |\nabla v_\epsilon|^{p-2}\nabla v_\epsilon.\nabla h \,\mathrm{d}x, \quad \forall h \in W_0^{1,p}(\Omega).$$

That is, in the weak sense,

$$-\Delta_p(u_\lambda + v_\epsilon) - f_\lambda(x, u_\lambda + v_\epsilon) = -\mu_\epsilon \Delta_p v_\epsilon \tag{3.25}$$

where f_λ is defined in (3.4). Define the maps $A_\epsilon : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}^N$ and $\tilde{h}_\lambda : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}$ as

$$\begin{aligned} A_\epsilon(x, w) &= |\nabla u_\lambda(x) + w|^{p-2}(\nabla u_\lambda(x) + w) - |\nabla u_\lambda(x)|^{p-2}\nabla u_\lambda(x) \\ &\quad - \mu_\epsilon |w|^{p-2} w \quad \text{with} \\ \tilde{h}_\lambda(x, s) &\overset{\text{def}}{=} f_\lambda(x, u_\lambda(x) + s) - f_\lambda(x, u_\lambda(x)) \\ &= \lambda(\max\{u_\lambda(x) + s, \underline{u}_\lambda(x)\})^{-\delta} - \lambda u_\lambda(x)^{-\delta} \\ &\quad + (\max\{u_\lambda(x) + s, \underline{u}_\lambda(x)\})^q - u_\lambda(x)^q. \end{aligned}$$

Then (3.25) can be written as

$$\begin{cases} -\nabla \cdot (A_\epsilon(x, \nabla v_\epsilon)) = \tilde{h}_\lambda(x, v_\epsilon) \text{ in } \Omega; \\ v_\epsilon = 0 \text{ on } \partial\Omega. \end{cases} \tag{3.26}$$

Using similar arguments as in García Azorero, Peral, and Manfredi [21] (Section 2, page 5 in the subcritical case and pages 19-20 in the critical case), the fact that the singular terms in (3.26) are non increasing and arguments in the proof of Theorem A1 in García Azorero and Peral [20], we obtain that $\sup_\epsilon \|v_\epsilon\|_{L^\infty(\Omega)} < \infty$. Let us show now that $\sup_\epsilon \|\frac{v_\epsilon}{d}\|_{L^\infty(\Omega)} < \infty$. For this, we just need to estimate v_ϵ near the boundary. Set $\bar{v}_\epsilon$ the unique solution to

$$\begin{cases} -\Delta_p(u_\lambda + \bar{v}_\epsilon) + \mu_\epsilon \Delta_p \bar{v}_\epsilon = \dfrac{\lambda}{(\max\{\underline{u}_\lambda, u_\lambda + \bar{v}_\epsilon\})^\delta} \text{ in } \Omega; \\ \bar{v}_\epsilon = 0 \text{ on } \partial\Omega. \end{cases} \tag{3.27}$$

Observing that $(\eta - 1)u_\lambda$ is a subsolution to (3.27) for $\eta > 0$ small enough and Ku_λ is a supersolution for $K > 0$ large, we get $(\eta - 1)u_\lambda \leq \bar{v}_\epsilon \leq Ku_\lambda$, η and K could be chosen independently of ϵ. Furthermore, $\bar{v_\epsilon} \leq v_\epsilon$. Indeed,

$$\begin{aligned} 0 &\leq \int_{\bar{v}_\epsilon - v_\epsilon > 0} (-\Delta_p(u_\lambda + \bar{v}_\epsilon) + \Delta_p(u_\lambda + v_\epsilon))(\bar{v}_\epsilon - v_\epsilon)\,\mathrm{d}x \\ &\quad + \mu_\epsilon \int_{\bar{v}_\epsilon - v_\epsilon > 0} (-\Delta_p \bar{v}_\epsilon + \Delta_p v_\epsilon)(\bar{v}_\epsilon - v_\epsilon)\,\mathrm{d}x \\ &\leq \int_{\bar{v}_\epsilon - v_\epsilon > 0} \left(\frac{\lambda}{\max\{\underline{u}_\lambda, u_\lambda + \bar{v}_\epsilon\}^\delta} - \frac{\lambda}{\max\{\underline{u}_\lambda, u_\lambda + v_\epsilon\}^\delta}\right)(\bar{v}_\epsilon - v_\epsilon)\,\mathrm{d}x \leq 0 \end{aligned}$$

from which it follows that $\bar{v_\epsilon} \le v_\epsilon$ and $\eta u_\lambda \le u_\lambda + v_\epsilon$. Finally, using the weak comparison principle in a small neighborhood of $\partial\Omega$ and for K large enough (independent of ϵ), we get $v_\epsilon \le K u_\lambda$ near the boundary. Hence, we have $\sup_\epsilon \|\frac{v_\epsilon}{d}\|_{L^\infty(\Omega)} < \infty$. Now, using Theorem B.1 in Appendix B, it follows that for some $0 < \beta < 1$, $\sup_\epsilon \|v_\epsilon\|_{C^{1,\beta}(\overline{\Omega})} < \infty$. From Ascoli-Arzela theorem, we get then $v_\epsilon \to 0$ as $\epsilon \to 0^+$ in $C^1(\overline{\Omega})$ which contradicts the definition of v_ϵ since u_λ is a C^1-minimizer of $\underline{E}_\lambda$, $u_\lambda > 0$ in Ω and $\frac{\partial u_\lambda}{\partial \nu} < 0$ on $\partial\Omega$. This completes the proof of Proposition 3.7 in the subcritical case. Now, we deal with the critical case, *i.e.* $q = p^* - 1$. Following some ideas contained in Brezis and Nirenberg [12], we use the truncated nonlinearity $f_{\lambda,j} : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}^+$ defined by

$$f_{\lambda,j}(x,s) \overset{\text{def}}{=} \lambda \max\{\underline{u}_\lambda(x), s\}^{-\delta} + \min\left\{\max\{\underline{u}_\lambda(x), s\}^q, j^q\right\}.$$

Let $F_{\lambda,j}(x,s) = \int_0^s f_{\lambda,j}(x,t)\,\mathrm{d}t$. Define $\underline{E}_{\lambda,j} : W_0^{1,p}(\Omega) \to \mathbb{R}$ by

$$\underline{E}_{\lambda,j}(u) \overset{\text{def}}{=} \frac{1}{p}\int_\Omega |\nabla u|^p\,\mathrm{d}x - \int_\Omega F_{\lambda,j}(x,u)\,\mathrm{d}x. \tag{3.28}$$

Then, $\underline{E}_{\lambda,j}$ is weakly lower semicontinuous on $W_0^{1,p}(\Omega)$ and

$$\forall w \in W_0^{1,p}(\Omega), \quad \underline{E}_{\lambda,j}(w) \to \underline{E}_\lambda(w) \quad \text{as } j \to \infty. \tag{3.29}$$

Now, suppose that u_λ is not a local minimizer of $\underline{E}_\lambda$ in $W_0^{1,p}(\Omega)$. Then, for any $\epsilon > 0$ small enough, there exists $v_\epsilon \in W_0^{1,p}(\Omega)$ such that $\|v_\epsilon\|_{W_0^{1,p}(\Omega)} \le \epsilon$ and

$$\underline{E}_\lambda(u_\lambda + v_\epsilon) < \underline{E}_\lambda(u_\lambda). \tag{3.30}$$

From (3.29) and (3.30), there exists $j(\epsilon) \in \mathbb{N}$ such that $j(\epsilon) \to \infty$ as $\epsilon \to 0^+$ and satisfying

$$\underline{E}_{\lambda,j(\epsilon)}(u_\lambda + v_\epsilon) < \underline{E}_\lambda(u_\lambda) = \underline{E}_{\lambda,j(\epsilon)}(u_\lambda). \tag{3.31}$$

Therefore, for any $\epsilon > 0$ small enough, there exists $w_\epsilon \in W_0^{1,p}(\Omega)$ such that $\|w_\epsilon\|_{W_0^{1,p}(\Omega)} \le \epsilon$ and satisfying

$$\underline{E}_{\lambda,j(\epsilon)}(u_\lambda + w_\epsilon) = \min_{\|v\|_{W_0^{1,p}(\Omega)} \le \epsilon} \underline{E}_{\lambda,j(\epsilon)}(u_\lambda + v) \le \underline{E}_{\lambda,j(\epsilon)}(u_\lambda + v_\epsilon). \tag{3.32}$$

As in the subcritical case, we can prove that there exists $\mu_\epsilon \le 0$ such that

$$-\Delta_p(u_\lambda + w_\epsilon) - f_{\lambda,j(\epsilon)}(x, u_\lambda + w_\epsilon) = -\mu_\epsilon \Delta_p w_\epsilon. \tag{3.33}$$

As above, we get that $\sup_\epsilon \|w_\epsilon\|_{C^{1,\beta}(\overline{\Omega})} < \infty$ for some $0 < \beta < 1$. Hence, $w_\epsilon \to 0$ in $C^1(\overline{\Omega})$ as $\epsilon \to 0^+$. Together with (3.31) and (3.32), it implies that for $\epsilon > 0$ small enough

$$\underline{E}_{\lambda,j(\epsilon)}(u_\lambda + w_\epsilon) = \underline{E}_\lambda(u_\lambda + w_\epsilon) < \underline{E}_\lambda(u_\lambda)$$

which contradicts the fact that u_λ is a C^1 local minimizer of $\underline{E}_\lambda$. The proof of Proposition 3.7 is now complete. □

4. Existence of a second weak solution for $0 < \lambda < \Lambda$

Now, we are able to show the existence of a second solution using the Moutain Pass lemma. As in Paragraph 3.1, since the functional E_λ is not C^1, we use the cut--off functional $\underline{E}_\lambda$ defined in (3.5). We recall that from Proposition 3.7, u_λ, given by Proposition 3.4, is a local minimizer of $\underline{E}_\lambda$. Moreover, from Theorem 2.3, any critical point v_λ of $\underline{E}_\lambda$ satisfies $v_\lambda > \underline{u}_\lambda$ in Ω and hence solves (P). Therefore, to prove the existence of a second solution it is enough to show that $\underline{E}_\lambda$ has a critical point v_λ different from u_λ. We first define a generalized notion of Palais Smale sequence for $\underline{E}_\lambda$:

Definition 4.1. Let $\mathcal{F} \subset W_0^{1,p}(\Omega)$ be a closed set. We say that a sequence $\{v_n\}_{n=1}^\infty \subset W_0^{1,p}(\Omega)$ is a Palais Smale sequence for $\underline{E}_\lambda$ at the level c around $\mathcal{F}$ (a $(PS_{\mathcal{F},c})$ for short) if

$$\lim_{n\to\infty} \operatorname{dist}(v_n, \mathcal{F}) = 0, \qquad \lim_{n\to\infty} \underline{E}_\lambda(v_n) = c, \qquad \lim_{n\to\infty} \|\underline{E}'_\lambda(v_n)\|_{W^{-1,p'}(\Omega)} = 0.$$

We have the following compactness result for $(PS_{\mathcal{F},c})$ sequences for $\underline{E}_\lambda$:

Lemma 4.2. *Let $\mathcal{F} \subset W_0^{1,p}(\Omega)$ be a closed set, $c \in \mathbb{R}$. Let $\{v_n\}_{n=1}^\infty \subset W_0^{1,p}(\Omega)$ be a $(PS_{\mathcal{F},c})$ sequence for $\underline{E}_\lambda$. Then $\{v_n\}_{n=1}^\infty$ is bounded in $W_0^{1,p}(\Omega)$ and there exists a subsequence of $\{v_n\}_{n=1}^\infty$, we still denote by $\{v_n\}_{n=1}^\infty$, such that $v_n \rightharpoonup v_\lambda$ in $W_0^{1,p}(\Omega)$ where v_λ is a weak solution to* (P).

Proof. From Definition 4.1, there exists $K > 0$ such that

$$\begin{aligned}
&\frac{1}{p}\int_\Omega |\nabla v_n|^p \,\mathrm{d}x \\
&\quad - \int_{v_n > \underline{u}_\lambda} \left[\left(\frac{\lambda}{1-\delta}\, v_n^{1-\delta} + \frac{v_n^{q+1}}{q+1}\right) - \left(\frac{\lambda}{1-\delta}\, \underline{u}_\lambda^{1-\delta} + \frac{\underline{u}_\lambda^{q+1}}{q+1}\right)\right] \mathrm{d}x \\
&\quad - \int_{v_n \le \underline{u}_\lambda} v_n(\lambda \underline{u}_\lambda^{-\delta} + \underline{u}_\lambda^q)\,\mathrm{d}x \le K
\end{aligned}$$

from which it follows that

$$\frac{1}{p}\int_\Omega |\nabla v_n|^p \,\mathrm{d}x - \int_{v_n > \underline{u}_\lambda} \left(\frac{\lambda}{1-\delta}\, v_n^{1-\delta} + \frac{v_n^{q+1}}{q+1}\right) \mathrm{d}x \le K. \tag{4.1}$$

Again from Definition 4.1 we have

$$\begin{aligned}
\int_\Omega |\nabla v_n|^p \,\mathrm{d}x = &\int_{v_n > \underline{u}_\lambda} (\lambda v_n^{1-\delta} + v_n^{q+1})\,\mathrm{d}x + \int_{v_n \le \underline{u}_\lambda} (\lambda \underline{u}_\lambda^{-\delta} + \underline{u}_\lambda^q) v_n \,\mathrm{d}x \\
&+ o_n(1)\, \|v_n\|_{W_0^{1,p}(\Omega)}\,.
\end{aligned} \tag{4.2}$$

From (4.1) and (4.2) we get

$$\begin{aligned} &\|v_n\|^p_{W_0^{1,p}(\Omega)} + O_n(\|v_n\|_{W_0^{1,p}(\Omega)}) \\ &\geq \int_{v_n > \underline{u}_\lambda} v_n^{q+1}\,\mathrm{d}x \geq \frac{q+1}{p}\,\|v_n\|^p_{W_0^{1,p}(\Omega)} - K. \end{aligned} \tag{4.3}$$

From (4.3), it follows that $\{v_n\}_{n=1}^\infty$ is bounded in $W_0^{1,p}(\Omega)$ and there exists v_λ such that a suitable subsequence satisfies $v_n \rightharpoonup v_\lambda$ in $W_0^{1,p}(\Omega)$. Let $\phi \in W_0^{1,p}(\Omega)$. From Definition 4.1, we get

$$\int_\Omega |\nabla v_n|^{p-2}\nabla v_n \nabla\phi\,\mathrm{d}x = \int_\Omega f_\lambda(x, v_n)\phi\,\mathrm{d}x + o_n(1). \tag{4.4}$$

Since $f_\lambda(x, v_n) \leq \underline{u}_\lambda^{-\delta} + (\max(\underline{u}_\lambda, v_n))^q$, $q \leq p^* - 1$, $v_n \rightharpoonup v_\lambda$ in $W_0^{1,p}(\Omega)$ and doing $n \to \infty$ in (4.4), we get

$$\int_\Omega |\nabla v_\lambda|^{p-2}\nabla v_\lambda \nabla\phi\,\mathrm{d}x = \int_\Omega f_\lambda(x, v_\lambda)\phi\,\mathrm{d}x. \tag{4.5}$$

From Theorem 2.3 and (4.5), it follows that v_λ is a weak solution to (P). This completes the proof of Lemma 4.2. □

We observe that from Proposition 3.4 and the fact that $\lim_{t\to\infty} \underline{E}_\lambda(t\phi) = -\infty$ for $0 \leq \phi \in W_0^{1,p}(\Omega)\backslash\{0\}$, $\underline{E}_\lambda$ has a Moutain Pass geometry close to u_λ. Hence we may fix $e \in W_0^{1,p}(\Omega)\backslash\{0\}$ such that $\underline{E}_\lambda(e) < \underline{E}_\lambda(u_\lambda)$. Let $R_0 = \|e - u_\lambda\|_{W_0^{1,p}(\Omega)}$, $l_0 > 0$ small enough such that u_λ is a minimizer of $\underline{E}_\lambda$ on $\overline{\mathcal{B}_{l_0}(u_\lambda)}$. Set

$$\Gamma \stackrel{\text{def}}{=} \{\eta \in C([0,1], W_0^{1,p}(\Omega)) \,|\, \eta(0) = u_\lambda\,,\ \eta(1) = e\}$$

and define the mountain pass level

$$\gamma_0 = \inf_{\eta\in\Gamma} \max_{t\in[0,1]} E_\lambda(\eta(t)).$$

We distinguish between the following two cases:

(P1) "Zero altitude case"
$\inf\left\{\underline{E}_\lambda(u) \colon u \in W_0^{1,p}(\Omega) \text{ and } \|u - u_\lambda\|_{W_0^{1,p}(\Omega)} = l\right\} \leq \underline{E}_\lambda(u_\lambda)$
for all $l < R_0$;

(P2) there exists $l_1 < R_0$ such that
$\inf\left\{\underline{E}_\lambda(u) \colon u \in W_0^{1,p}(\Omega) \text{ and } \|u - u_\lambda\|_{W_0^{1,p}(\Omega)} = l\right\} > \underline{E}_\lambda(u_\lambda)$.

Note that (P1) (respectively (P2)) implies that $\gamma_0 = \underline{E}_\lambda(u_\lambda)$ (respectively $\gamma_0 > \underline{E}_\lambda(u_\lambda)$). In case where (P1) occurs, we can construct a $(PS_{\mathcal{F},\gamma_0})$ sequence with $\mathcal{F} = \partial\mathcal{B}_l(u_\lambda)$, $l \leq l_0$, and get at least a second weak solution to (P). More precisely, we have the following result:

Proposition 4.3. *Let* $p \in (1, \infty)$, $q \in (p-1, p^*-1]$, $\delta \in (0,1)$ *and* $\lambda \in (0, \Lambda)$. *Suppose that* (P1) *holds. Then, there exists a weak solution* v_λ *of* (P) *such that* $v_\lambda \neq u_\lambda$.

Proof. From Theorem (1) in Ghoussoub and Preiss [25], for $l \leq l_0$ we get the existence of a $(PS_{\mathcal{F},\gamma_0})$ sequence, $\{v_k\}_{k=1}^\infty$. From Lemma 4.2, $\{v_k\}_{k=1}^\infty$ is bounded and up to a subsequence weakly convergent in $W_0^{1,p}(\Omega)$ to v_λ, a weak solution to (P). To prove that $u_\lambda \neq v_\lambda$, it is sufficient to prove that $v_k \to v_\lambda$ strongly in $W_0^{1,p}(\Omega)$ as $k \to \infty$. Since $v_k \rightharpoonup v_\lambda$ as $k \to \infty$, applying Theorem 2.1 in Boccardo and Murat [9] as in the proof of Lemma 3.3 and from Remark 2.1 in [9], we get the following result from Brezis-Lieb (see [10]): As $k \to \infty$,

$$\begin{aligned} \|v_k\|_{W_0^{1,p}(\Omega)} &= \|v_k - v_\lambda\|_{W_0^{1,p}(\Omega)} + \|v_\lambda\|_{W_0^{1,p}(\Omega)} + o_k(1) \quad \text{and} \\ \|v_k\|_{L^{q+1}(\Omega)} &= \|v_k - v_\lambda\|_{L^{q+1}(\Omega)} + \|v_\lambda\|_{L^{q+1}(\Omega)} + o_k(1). \end{aligned} \tag{4.6}$$

By Sobolev imbedding theorem, we have also:

$$\int_{v_k \geq \underline{u}_\lambda} |v_k^{1-\delta} - v_\lambda^{1-\delta}|\,\mathrm{d}x = o_k(1) \text{ as } k \to \infty.$$

Since v_λ is a weak solution to (P), we have:

$$\|v_\lambda\|^p_{W_0^{1,p}(\Omega)} - \|v_\lambda\|^{q+1}_{L^{q+1}(\Omega)} - \lambda \int_\Omega v_\lambda^{1-\delta} = 0. \tag{4.7}$$

Therefore, as $k \to \infty$

$$\begin{aligned} \int_\Omega |\nabla v_k|^{p-2}\nabla v_k \nabla(v_k - v_\lambda)\,\mathrm{d}x = \lambda \int_{v_k \geq \underline{u}_\lambda} & v_k^{-\delta}(v_k - v_\lambda)\,\mathrm{d}x \\ &+ \int_\Omega v_k^q (v_k - v_\lambda)\,\mathrm{d}x + o_k(1). \end{aligned} \tag{4.8}$$

It follows from (4.6), (4.8) and (4.7) that

$$\int_\Omega |\nabla v_k - \nabla v_\lambda|^p\,\mathrm{d}x = \int_\Omega |v_k - v_\lambda|^{q+1}\,\mathrm{d}x + o_k(1) \text{ as } k \to \infty. \tag{4.9}$$

Now, we consider two cases:

(i) $\underline{E}_\lambda(u_\lambda) \neq \underline{E}_\lambda(v_\lambda)$,
(ii) $\underline{E}_\lambda(u_\lambda) = \underline{E}_\lambda(v_\lambda)$.

In case (i), we are done. If (ii) holds, then from (4.6), we get

$$\underline{E}_\lambda(v_k - v_\lambda) = \underline{E}_\lambda(v_k) - \underline{E}_\lambda(v_\lambda) + o_k(1) = o_k(1) \text{ as } k \to \infty.$$

Thus,

$$\frac{1}{p}\|v_k - v_\lambda\|^p_{W_0^{1,p}(\Omega)} - \frac{1}{q+1}\|v_k - v_\lambda\|^{q+1}_{L^{q+1}(\Omega)} \le o_k(1) \text{ as } k \to \infty. \tag{4.10}$$

Then, from (4.9) and (4.10), we obtain $\|v_k - v_\lambda\|_{W_0^{1,p}(\Omega)} \to 0$ as $k \to \infty$. Hence $\|u_\lambda - v_\lambda\|_{W_0^{1,p}(\Omega)} = l$ and $u_\lambda \neq v_\lambda$. This completes the proof of Proposition 4.3. □

In case where (P2) occurs, we have the following result:

Proposition 4.4. *Let the pair (p, q) satisfy either $p \in (1, \infty)$ and $q \in (p-1, p^* - 1)$, or else $p \in \left(\frac{2N}{N+2}, 2\right] \cup \left(\frac{3N}{N+3}, 3\right)$ and $q = p^* - 1$. Let $\lambda \in (0, \Lambda)$ and suppose that* (P2) *holds. Then there exists a weak solution v_λ such that $u_\lambda \neq v_\lambda$.*

Proof. We give the proof only in the second case (critical case) *i.e.* $p \in (\frac{2N}{N+2}, 2] \cup (\frac{3N}{N+3}, 3)$ and $q = p^* - 1$. The first case (subcritical case) *i.e.* $p \in (1, \infty)$ and $q \in (p-1, p^*-1)$ follows from Lemma 4.2 and Lemma C.1 in Appendix C. First, without loss of generality, we can assume that u_λ has the minimal energy among all weak solutions (if not, we would have already found our second solution). Let $\mathcal{F} = W_0^{1,p}(\Omega)$. By Lemma 4.2, all $(PS_{\mathcal{F},\gamma_0})$ sequences are bounded in $W_0^{1,p}(\Omega)$. Again we need to prove the compactness of the $(PS_{\mathcal{F},\gamma_0})$ sequences in $W_0^{1,p}(\Omega)$. For that, we show that γ_0 is strictly below the first critical level where the Palais-Smale condition fails. Following the ideas in Brezis and Nirenberg [11], we use the test functions

$$U_\epsilon(x) = \frac{C_N \epsilon^{\frac{N-p}{p(p-1)}}}{\left(\epsilon^{p/(p-1)} + |x-y|^{p/(p-1)}\right)^{\frac{N-p}{p}}} \phi(x)$$

where $\epsilon > 0$, C_N is a normalization constant, $y \in \Omega$, and $\phi \in C_c^\infty(\Omega)$ is a cut-off function such that $\phi = 1$ in a neighborhood of y. Then, we prove the following statement:

Claim. There exists $\epsilon_0 > 0$ and $R_0 \ge 1$ such that $\forall \epsilon \in (0, \epsilon_0)$

$$\underline{E}_\lambda(u_\lambda + RU_\epsilon) = E_\lambda(u_\lambda + RU_\epsilon) < E_\lambda(u_\lambda) \; \forall R \ge R_0,$$
$$\underline{E}_\lambda(u_\lambda + tR_0U_\epsilon) = (E_\lambda(u_\lambda + tR_0U_\epsilon) < E_\lambda(u_\lambda) + \frac{1}{N}S^{\frac{N}{p}} \; \forall t \in [0, 1].$$

Proof of the claim. The first inequality shows that Γ is non empty. The proof is a direct consequence of the fact that $q > p-1$ and R large. Let us prove the second inequality. We use the approach in García Azorero and Peral [20] where the following estimates are proved (see pages 946 and 949):

$$\begin{aligned}\int_\Omega |\nabla u_\lambda + tR_0U_\epsilon|^p\,\mathrm{d}x \le \int_\Omega |\nabla u_\lambda|^p\,\mathrm{d}x + (tR_0)^p\int_\Omega |\nabla U_\epsilon|^p\,\mathrm{d}x\\ + ptR_0\int_\Omega |\nabla u_\lambda|^{p-2}\nabla u_\lambda \nabla U_\epsilon\,\mathrm{d}x + O(\epsilon^\beta)\end{aligned} \tag{4.11}$$

with $\beta > \frac{(N-p)}{p}$ and

$$\begin{aligned}\int_\Omega (u_\lambda + tR_0U_\epsilon)^{q+1}\,\mathrm{d}x \ge \int_\Omega u_\lambda^{q+1}\,\mathrm{d}x + (tR_0)^{q+1}\int_\Omega U_\epsilon^{q+1}\,\mathrm{d}x\\ + (q+1)tR_0\int_\Omega u_\lambda^q U_\epsilon\,\mathrm{d}x + (q+1)(tR_0)^q\int_\Omega u_\lambda U_\epsilon^q\,\mathrm{d}x + O(\epsilon^\gamma)\end{aligned} \tag{4.12}$$

with $\gamma > \frac{(N-p)}{p}$. Then,

$$\begin{aligned}E_\lambda(u_\lambda + tR_0U_\epsilon) &= \frac{1}{p}\int_\Omega |\nabla u_\lambda + tR_0U_\epsilon|^p\,\mathrm{d}x\\ &- \frac{\lambda}{1-\delta}\int_\Omega (u_\lambda + tR_0U_\epsilon)^{1-\delta}\,\mathrm{d}x - \frac{1}{q+1}\int_\Omega (u_\lambda + tR_0U_\epsilon)^{q+1}\,\mathrm{d}x\\ &\le \int_\Omega |\nabla u_\lambda|^p\,\mathrm{d}x + \frac{(tR_0)^p}{p}\int_\Omega |\nabla U_\epsilon|^p\,\mathrm{d}x + tR_0\int_\Omega (\lambda u_\lambda^{-\delta} + u_\lambda^q)U_\epsilon\,\mathrm{d}x\\ &-\frac{\lambda}{1-\delta}\int_\Omega (u_\lambda + tR_0U_\epsilon)^{1-\delta}\,\mathrm{d}x - \frac{1}{q+1}\int_\Omega (u_\lambda + tR_0U_\epsilon)^{q+1}\,\mathrm{d}x + o(\epsilon^{\frac{N-p}{p}})\\ &\le E_\lambda(u_\lambda) + (tR_0)^p\int_\Omega |\nabla U_\epsilon|^p\,\mathrm{d}x - \frac{(tR_0)^{q+1}}{q+1}\int_\Omega U_\epsilon^{q+1}\,\mathrm{d}x\\ &- (tR_0)^q\int_\Omega u_\lambda U_\epsilon^q\,\mathrm{d}x + tR_0\int_\Omega u_\lambda^q U_\epsilon\,\mathrm{d}x + \frac{\lambda}{1-\delta}\int_\Omega u_\lambda^{1-\delta}\,\mathrm{d}x\\ &- \frac{\lambda}{1-\delta}\int_\Omega (u_\lambda + tR_0U_\epsilon)^{1-\delta}\,\mathrm{d}x + tR_0\int_\Omega \lambda u_\lambda^{-\delta}U_\epsilon\,\mathrm{d}x + o(\epsilon^{\frac{N-p}{p}})\,.\end{aligned} \tag{4.13}$$

Now, we estimate the last three terms as follows:

$$\begin{aligned}&\frac{\lambda}{1-\delta}\int_\Omega u_\lambda^{1-\delta}\,\mathrm{d}x - \frac{\lambda}{1-\delta}\int_\Omega (u_\lambda + tR_0U_\epsilon)^{1-\delta}\,\mathrm{d}x + tR_0\int_\Omega \lambda u_\lambda^{-\delta}U_\epsilon\,\mathrm{d}x\\ &\le K\int_{B_\mu(y)} U_\epsilon\,\mathrm{d}x\end{aligned}$$

for $\mu > 0$. Moreover,

$$\int_{B_\mu(y)} U_\epsilon\,\mathrm{d}x \le \begin{cases} \epsilon^{N-\frac{(N-p)}{p}}\big[O(1) + O\big(\epsilon^{\frac{N-p}{p-1}-N}\big)\big] & \text{if } p \ne \dfrac{2N}{N+1},\\ O\big(\epsilon^{N-\frac{(N-p)}{p}}\log\epsilon\big) & \text{if } p = \dfrac{2N}{N+1}.\end{cases} \tag{4.14}$$

From (4.14) and $p \in (\frac{2N}{N+2}, 2)$, we get

$$\int_{B_\mu(y)} U_\epsilon \,dx = o(\epsilon^{\frac{N-p}{p}}). \tag{4.15}$$

Now, assume that $p \in (\frac{3N}{N+3}, 3)$. In this case, using the Taylor expansion, we estimate the last three terms in (4.13) as follows:

$$\begin{aligned}&\frac{\lambda}{1-\delta}\int_\Omega u_\lambda^{1-\delta}\,dx - \frac{\lambda}{1-\delta}\int_\Omega (u_\lambda + tR_0U_\epsilon)^{1-\delta}\,dx + tR_0\int_\Omega \lambda u_\lambda^{-\delta}U_\epsilon\,dx\\ &\le K\int_{B_\mu(y)} U_\epsilon^2\,dx\end{aligned}$$

for $\mu > 0$. Moreover,

$$\int_{B_\mu(y)} U_\epsilon^2\,dx \le \begin{cases} \epsilon^{N-\frac{2(N-p)}{p}}\big[O(1)+O\big(\epsilon^{\frac{2(N-p)}{p-1}-N}\big)\big] \text{ if } p \ne \dfrac{3N}{N+2}, \\ O\big(\epsilon^{N-\frac{2(N-p)}{p}}\log\epsilon\big) \text{ if } p = \dfrac{3N}{N+2}. \end{cases} \tag{4.16}$$

From (4.16) and $p \in (\frac{3N}{N+3}, 3)$, we get

$$\int_{B_\mu(y)} U_\epsilon^2 \,dx = o(\epsilon^{\frac{N-p}{p}}). \tag{4.17}$$

Thus, From (4.13), (4.15) and (4.17), it follows that

$$\begin{aligned}E_\lambda(u_\lambda + tR_0U_\epsilon) \le E_\lambda(u_\lambda) + (tR_0)^p\int_\Omega |\nabla U_\epsilon|^p\,dx - \frac{(tR_0)^{q+1}}{q+1}\int_\Omega U_\epsilon^{q+1}\,dx\\ - (tR_0)^q\int_\Omega u_\lambda U_\epsilon^q\,dx + o(\epsilon^{\frac{N-p}{p}}).\end{aligned}$$

for $p \in (\frac{2N}{N+2}, 2) \cup (\frac{3N}{N+3}, 3)$. The case $p = 2$ is done in Haitao [29] and in Hirano, Saccon and Shioji [31]. Arguing as in García Azorero and Peral [20] (see page 947), we get for $p \in (\frac{2N}{N+2}, 2] \cup (\frac{3N}{N+3}, 3)$:

$$\sup_{t\in\mathbb{R}^+} E_\lambda(u_\lambda + tR_0U_\epsilon) < E_\lambda(u_\lambda) + \frac{1}{N}S^{\frac{N}{p}}$$

which completes the proof of the claim Now, the compactness of $\{v_k\}_{k=1}^\infty$ implies that $E_\lambda(v_\lambda) = \gamma_0 > E_\lambda(u_\lambda)$. Therefore $v_\lambda \ne u_\lambda$. □

Thus, the proof of Theorem 2.1 follows from Propositions 4.3 and 4.4. Now, Theorem 2.2 follows from the subsequent regularity results given in Appendices A and B.

A. Appendix

We start with an important technical tool which enables us to estimate the singularity in the Gâteaux derivative of the energy functional $E_\lambda \colon W_0^{1,p}(\Omega) \to \mathbb{R}$ defined in (1.1).

Lemma A.1. *Let* $0 < \delta < 1$. *Then there exists a constant* $C_\delta > 0$ *such that the inequality*

$$\int_0^1 |\mathbf{a} + s\mathbf{b}|^{-\delta}\, \mathrm{d}s \le C_\delta \left(\max_{0\le s\le 1} |\mathbf{a} + s\mathbf{b}| \right)^{-\delta} \tag{A.1}$$

holds true for all $\mathbf{a}, \mathbf{b} \in \mathbb{R}^N$ *with* $|\mathbf{a}| + |\mathbf{b}| > 0$.

An elementary proof of this lemma can be found in Takáč [40, Lemma A.1, page 233].

We continue by showing the Gâteaux-differentiability of the energy functional E_λ at a point $u \in W_0^{1,p}(\Omega)$ satisfying $u \ge \varepsilon\varphi_1$ in Ω with a constant $\varepsilon > 0$.

Lemma A.2. *Let* $0 < \delta < 1$, $1 < p < \infty$, *and* $p - 1 < q \le p^* - 1$. *Assume that* $u, v \in W_0^{1,p}(\Omega)$ *and* u *satisfies* $u \ge \varepsilon\varphi_1$ *in* Ω *with a constant* $\varepsilon > 0$. *Then we have*

$$\begin{aligned} &\lim_{t\to 0} \frac{1}{t} \left(E_\lambda(u + tv) - E_\lambda(u) \right) \\ &= \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v\, \mathrm{d}x - \lambda \int_\Omega u^{-\delta} v\, \mathrm{d}x - \int_\Omega u^q v\, \mathrm{d}x. \end{aligned} \tag{A.2}$$

Proof. We show the result only for the singular term $\int_\Omega u^{-\delta} v\, \mathrm{d}x$; the other two terms are treated in a standard way. So let

$$F(u) = \frac{1}{1-\delta} \int_\Omega (u^+)^{1-\delta}\, \mathrm{d}x \quad \text{for } u \in W_0^{1,p}(\Omega).$$

For $\xi \in \mathbb{R} \setminus \{0\}$ we define

$$z(\xi) = \frac{1}{1-\delta} \frac{\mathrm{d}}{\mathrm{d}\xi} (\xi^+)^{1-\delta} = \begin{cases} \xi^{-\delta} & \text{if } \xi > 0; \\ 0 & \text{if } \xi < 0. \end{cases}$$

Consequently,

$$\frac{1}{t} \left(F(u + tv) - F(u) \right) = \int_\Omega \left(\int_0^1 z(u + stv)\, \mathrm{d}s \right) v\, \mathrm{d}x. \tag{A.3}$$

Notice that for almost every $x \in \Omega$ we have $u(x) > 0$ and

$$\int_0^1 z(u(x) + stv(x))\, \mathrm{d}s \longrightarrow z(u(x)) = u(x)^{-\delta} \quad \text{as } t \to 0.$$

Moreover, the integral on the left-hand side (with nonnegative integrand) is dominated by

$$\begin{aligned}\int_0^1 z(u(x)+stv(x))\,\mathrm{d}s &\le \int_0^1 |u(x)+stv(x)|^{-\delta}\,\mathrm{d}s \\ &\le C_\delta \left(\max_{0\le s\le 1} |u(x)+stv(x)|\right)^{-\delta} \\ &\le C_\delta\, u(x)^{-\delta} \le C_\delta\,(\varepsilon\varphi_1(x))^{-\delta} = C_{\delta,\varepsilon}\,\varphi_1(x)^{-\delta}\end{aligned}$$

with a constant $C_{\delta,\varepsilon} > 0$ independent of $x \in \Omega$. Here, we have used the estimate (A.1) from Lemma A.1 above. Finally, we have $v\varphi_1^{-\delta} \in L^1(\Omega)$, by $v \in W_0^{1,p}(\Omega)$ and Hardy's inequality. Hence, we are allowed to invoke the Lebesgue dominated convergence theorem in (A.3) from which the lemma follows by letting $t \to 0$. □

Corollary A.3. *Let $0 < \delta < 1$, $1 < p < \infty$, and $p-1 < q \le p^* - 1$. Then the energy functional $E_\lambda\colon W_0^{1,p}(\Omega) \to \mathbb{R}$ is Gâteaux-differentiable at every point $u \in W_0^{1,p}(\Omega)$ that satisfies $u \ge \varepsilon\varphi_1$ in Ω with a constant $\varepsilon > 0$. Its Gâteaux derivative $E'_\lambda(u)$ at u is given by*

$$\langle E'_\lambda(u),\ v\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla v\,\mathrm{d}x - \lambda\int_\Omega u^{-\delta} v\,\mathrm{d}x - \int_\Omega u^q v\,\mathrm{d}x \tag{A.4}$$

for $v \in W_0^{1,p}(\Omega)$.

We continue by proving the C^1-differentiability of the cut off energy functional:

Lemma A.4. *Let $0 < \delta < 1$, $1 < p < \infty$, $p-1 < q < \infty$, and $w \in W_0^{1,p}(\Omega)$ such that $w \ge \epsilon\varphi_1$ with some $\epsilon > 0$. Setting for $x \in \Omega$*

$$f_\lambda(x,s) = \begin{cases} \lambda s^{-\delta} + s^q & \text{if } s \ge w(x), \\ \lambda w(x)^{-\delta} + w(x)^q & \text{if } s < w(x), \end{cases}$$

$F_\lambda(x,s) = \int_0^s f_\lambda(x,t)\,\mathrm{d}t$ and for $u \in W_0^{1,p}(\Omega)$

$$\overline{E}_\lambda(u) = \frac{1}{p}\int_\Omega |\nabla u|^p\,\mathrm{d}x - \int_\Omega F_\lambda(x,u)\,\mathrm{d}x,$$

we have that $\overline{E}_\lambda$ belongs to $C^1(W_0^{1,p}(\Omega),\mathbb{R})$.

Proof. As in Lemma A.2, we concentrate on the singular term, the others being standard. Let

$$h(x,s) = \begin{cases} s^{-\delta} & \text{if } s \geq w(x), \\ w(x)^{-\delta} & \text{if } s < w(x), \end{cases}$$

$H(x,s) = \int_0^s h(x,t)\,\mathrm{d}t$, and $S(u) = \int_\Omega H(x,u)\mathrm{d}x$. Proceeding as in Lemma A.2, we obtain that for all $u \in W_0^{1,p}(\Omega)$, $S(u)$ has a Gâteaux derivative $S'(u)$ given by

$$\langle S'(u), v\rangle = \int_\Omega (\max\{u(x), w(x)\})^{-\delta} v(x)\,\mathrm{d}x.$$

Let $u_k \in W_0^{1,p}(\Omega)$, $u_k \to u_0$. Then

$$\begin{aligned} |\langle S'(u_k) - S'(u_0), v\rangle| &= \Big|\int_\Omega \big((\max\{u_k(x), w(x)\})^{-\delta} v(x) \\ &\qquad -(\max\{u(x), w(x)\})^{-\delta} v(x)\big)\,\mathrm{d}x\Big| \\ &\leq 2\int_\Omega w^{-\delta}|v|\,\mathrm{d}x \\ &\leq 2\epsilon^{-\delta}\int_\Omega \varphi_1^{-\delta}|v|\,\mathrm{d}x \end{aligned}$$

for all $v \in W_0^{1,p}(\Omega)$. Again, as in Lemma A.2, we use Hardy's inequality to deduce that $\varphi_1^{-\delta} v \in L^1$, so that by Lesbegue's dominated convergence theorem we conclude that the Gâteaux derivative of S is continuous which implies that $S \in C^1(W_0^{1,p}(\Omega), \mathbb{R})$. □

Next, we give some regularity results for weak solutions to problem (P). We start with the following lemma which allows for test functions ϕ in equation (2.1) to be taken in $W_0^{1,p}(\Omega)$ rather than only in $C_c^\infty(\Omega)$ ($\subset W_0^{1,p}(\Omega)$).

Lemma A.5. *Each positive weak solution u of problem* (P) *satisfies $u \geq \epsilon_\lambda \phi_1$ a.e. in Ω, where $\epsilon_\lambda > 0$ is a constant independent of u. Moreover, for every function $w \in W_0^{1,p}(\Omega)$ we have $u^{-\delta} w \in L^1(\Omega)$ and*

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla w\,\mathrm{d}x = \lambda\int_\Omega u^{-\delta} w\,\mathrm{d}x + \int_\Omega u^q w\,\mathrm{d}x. \tag{A.5}$$

Proof. Let u be a positive weak solution of (P). Recall that u is required to satisfy $\operatorname{ess\,inf}_K u > 0$ over every compact set $K \subset \Omega$.

First, we establish the inequality

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla w\,\mathrm{d}x \geq \lambda\int_\Omega u^{-\delta} w\,\mathrm{d}x + \int_\Omega u^q w\,\mathrm{d}x \tag{A.6}$$

for every $w \in W_0^{1,p}(\Omega)$ satisfying $w \geq 0$ a.e. in Ω. Given $0 \leq w \in W_0^{1,p}(\Omega)$, there exists a sequence $\{w_k\}_{k=1}^{\infty} \subset C_c^{\infty}(\Omega)$ such that $w_k \geq 0$ in Ω and $w_k \to w$ strongly in $W_0^{1,p}(\Omega)$ as $k \to \infty$. Since $p < q+1 \leq p^*$, this entails $w_k \to w$ strongly also in $L^{q+1}(\Omega)$ as $k \to \infty$. Moreover, we can find a subsequence, denoted again by $\{w_k\}_{k=1}^{\infty}$, such that $w_k \to w$ almost everywhere in Ω as $k \to \infty$. In equation (A.5) we now replace w by w_k and apply Fatou's lemma to the integral $\int_\Omega u^{-\delta} w_k \, dx$ as $k \to \infty$, thus arriving at the desired inequality (A.6).

In particular, inequality (A.6) implies

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla w \, dx \geq \lambda \int_\Omega u^{-\delta} w \, dx \tag{A.7}$$

whenever $0 \leq w \in W_0^{1,p}(\Omega)$. Now we are ready to compare u with the unique weak solution $\underline{u}_\lambda$ of problem (3.1) obtained in Lemma 3.1. We apply the weak comparison principle (*cf.* the proof of Theorem 2.3) to (the weak formulation of) problem (3.1) (with $\underline{u}_\lambda$ in place of u) and to inequality (A.7) (with u), thus obtaining $u \geq \underline{u}_\lambda$ a.e. in Ω. This guarantees $u \geq \epsilon_\lambda \phi_1$ a.e. in Ω.

Next, there are constants $0 < \ell < L < \infty$ such that $\ell \, d(x) \leq \phi_1(x) \leq L \, d(x)$ for all $x \in \Omega$. It follows that $u \geq \epsilon_\lambda \ell \, d$ a.e. in Ω. Now, instead of using Fatou's lemma in the limiting process above, we apply Hardy's inequality to the integral $\int_\Omega u^{-\delta} w_k \, dx$ as $k \to \infty$, thus arriving at the desired equality (A.5) for every $w \in W_0^{1,p}(\Omega)$ satisfying $w \geq 0$ a.e. in Ω.

Finally, we make use of the polar decomposition $w = w^+ - w^-$ of an arbitrary function $w \in W_0^{1,p}(\Omega)$, where $w^+ = \max\{w, 0\}$ and $w^- = \max\{-w, 0\}$ satisfy $w^+, w^- \in W_0^{1,p}(\Omega)$ and $\nabla w = \nabla w^+ - \nabla w^-$. Since we have already verified equation (A.5) for w^+ and w^-, the desired equality (A.5) holds also for every $w \in W_0^{1,p}(\Omega)$. □

Lemma A.6. *Each positive weak solution u of* (P) *belongs to* $L^\infty(\Omega)$.

Proof. First, we show that each positive weak solution u of (P) satisfies

$$\int_\Omega |\nabla (u-1)^+|^{p-2} \nabla (u-1)^+ \cdot \nabla w \, dx \leq \int_\Omega (\lambda + u^q) w \, dx \tag{A.8}$$

for every $w \in C_c^{\infty}(\Omega)$ with $w \geq 0$. Indeed, let $\psi : \mathbb{R} \to [0,1]$ be a C^1 cut-off function such that $\psi(s) = 0$ if $s \leq 0$, $\psi'(s) \geq 0$ if $0 \leq s \leq 1$, and $\psi(s) = 1$ if $s \geq 1$. Given any $\epsilon > 0$, define $\psi_\epsilon(t) \stackrel{\text{def}}{=} \psi((t-1)/\epsilon)$ for $t \in \mathbb{R}$. Hence, $\psi_\epsilon \circ u \in W_0^{1,p}(\Omega)$ with $\nabla(\psi_\epsilon \circ u) = (\psi'_\epsilon \circ u) \nabla u$. Using the weak form of problem (P), equation (2.1), with the test function $\phi = (\psi_\epsilon \circ u) w$, where $w \in C_c^{\infty}(\Omega)$ satisfies $w \geq 0$, we obtain

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla[(\psi_\epsilon \circ u) w] \, dx = \int_\Omega (\lambda u^{-\delta} + u^q)(\psi_\epsilon \circ u) w \, dx.$$

Hence,

$$\begin{aligned}&\int_\Omega |\nabla u|^p\,(\psi_\epsilon' \circ u) w\,\mathrm{d}x + \int_\Omega |\nabla u|^{p-2}(\nabla u\cdot\nabla w)(\psi_\epsilon \circ u)\,\mathrm{d}x\\ &= \int_\Omega (\lambda u^{-\delta} + u^q)(\psi_\epsilon\circ u) w\,\mathrm{d}x\end{aligned}$$

with $\psi_\epsilon' \circ u \geq 0$, which yields

$$\int_\Omega |\nabla u|^{p-2}(\nabla u\cdot\nabla w)(\psi_\epsilon \circ u)\,\mathrm{d}x \leq \int_\Omega (\lambda u^{-\delta} + u^q)(\psi_\epsilon\circ u) w\,\mathrm{d}x.$$

Letting $\epsilon \to 0+$ we arrive at (A.8). Finally, the L^∞ bound and regularity of u are obtained directly from equation (A.8) as follows: if $q < p^* - 1$, one applies Theorem A.1 from Anane [6], and if $q = p^* - 1$, the bootstrapping arguments from the proof of Theorem A.1, pages 950–953, in García Azorero and Peral [20] yield the desired result. In both references [6, 20] the bootstrapping arguments use the technique due to Serrin [39] (proof of Theorem 1). □

Finally, we are ready to bound any weak solution u of problem (P) by a positive scalar multiple of the eigenfunction ϕ_1 also from above. This result complements the corresponding bound from below, $u \geq \epsilon_\lambda \phi_1$ a.e. in Ω, stated in the first part of Lemma A.5 above. Equivalently, these lower and upper bounds for u/ϕ_1 can be reformulated as follows, using the distance function d in place of ϕ_1:

Lemma A.7. *Each positive weak solution u of problem* (P) *satisfies $c_\lambda\, d \leq u \leq K_\lambda\, d$ a.e. in Ω, where $0 < c_\lambda \leq K_\lambda < \infty$ are some constants independent of u.*

Proof. Let $u \in W_0^{1,p}(\Omega)$ be a positive weak solution of problem (P). It follows from the first part of Lemma A.5 and its proof that $u(x) \geq \underline{u}_\lambda(x) \geq \epsilon_\lambda\phi_1(x) \geq \epsilon_\lambda \ell\, d(x)$ for a.e. $x \in \Omega$. Hence, we can take $c_\lambda = \epsilon_\lambda \ell > 0$ to get $u \geq c_\lambda\, d$ a.e. in Ω.

Next, we take advantage of the inequality $u \geq c_\lambda\, d$ to derive also $u \leq K_\lambda\, d$. Recall that $u \in L^\infty(\Omega)$, by Lemma A.6 above. First, we apply the estimate

$$u^q = \frac{u^{q+\delta}}{u^\delta} \leq \frac{\|u\|_{L^\infty(\Omega)}^{q+\delta}}{u^\delta} \quad \text{a.e. in } \Omega$$

to the right-hand side of the equation in problem (P) to conclude that

$$\begin{cases} -\Delta_p u \leq \left(\lambda + \|u\|_{L^\infty(\Omega)}^{q+\delta}\right) u^{-\delta} & \text{in } \Omega;\\ u|_{\partial\Omega} = 0, \quad u > 0 & \text{in } \Omega. \end{cases} \tag{A.9}$$

After the substitution

$$v = \left(1 + \lambda^{-1}\|u\|_{L^\infty(\Omega)}^{q+\delta}\right)^{-1/(p-1+\delta)} u,$$

inequality (A.9) is equivalent with

$$\begin{cases} -\Delta_p v \leq \lambda v^{-\delta} & \text{in } \Omega; \\ v|_{\partial\Omega} = 0, \quad v > 0 & \text{in } \Omega. \end{cases} \tag{A.10}$$

Now, in analogy with the proof of Lemma A.5, we apply the weak comparison principle to problem (3.1) (with $\underline{u}_\lambda$ in place of u) and to inequality (A.10) (with v), thus arriving at $v \leq \underline{u}_\lambda$ a.e. in Ω. Thus, it remains to verify $\underline{u}_\lambda \leq c'_\lambda\, d$ a.e. in Ω, where $0 < c'_\lambda < \infty$ is a constant. This will imply $u \leq K_\lambda\, d$ a.e. in Ω with

$$K_\lambda = c'_\lambda \left(1 + \lambda^{-1} \|u\|^{q+\delta}_{L^\infty(\Omega)}\right)^{1/(p-1+\delta)} .$$

Thanks to $\ell\, d(x) \leq \phi_1(x) \leq L\, d(x)$ for all $x \in \Omega$, with some constants $0 < \ell < L < \infty$, the inequality $\underline{u}_\lambda \leq c'_\lambda\, d$ in Ω is equivalent to $\underline{u}_\lambda \leq c''_\lambda \phi_1$ in Ω, where $0 < c''_\lambda < \infty$ is a constant. We now construct a supersolution w to problem (3.1) of the form $w = \beta \cdot \Theta_\alpha \circ \phi_1$ in Ω. Here, $\alpha, \beta > 0$ are suitable numbers and $\Theta_\alpha \colon [0, R_\alpha) \to \mathbb{R}_+$ is a C^1 function (where $0 < R_\alpha < \infty$ and $\mathbb{R}_+ = [0, \infty)$) that satisfies the initial value problem

$$\begin{cases} -\dfrac{\mathrm{d}}{\mathrm{d}r}\left(|\Theta'_\alpha(r)|^{p-2}\,\Theta'_\alpha(r)\right) = \Theta_\alpha(r)^{-\delta}, \quad 0 < r < R_\alpha; \\ \Theta_\alpha(0) = 0, \quad \Theta'_\alpha(0) = \alpha > 0. \end{cases} \tag{A.11}$$

The endpoint R_α is defined to be the supremum of all numbers $s \in (0, \infty)$ such that $\Theta'_\alpha(r) > 0$ holds for all $r \in [0, s)$. We will see that $0 < R_\alpha < \infty$ together with $\Theta'_\alpha(r) \searrow 0$ as $r \nearrow R_\alpha$.

Making use of the transformation

$$\begin{cases} \Theta_\alpha(r) = \alpha^{\frac{p}{1-\delta}} \cdot \Theta_1(\alpha^{-\frac{p}{p-1+\delta}}\, r), \quad 0 \leq r \leq R_\alpha; \\ R_\alpha = \alpha^{\frac{p}{p-1+\delta}}\, R_1, \end{cases} \tag{A.12}$$

we conclude that it suffices to treat the case $\alpha = 1$. Problem (A.11) with $\alpha = 1$ has the first integral

$$\begin{cases} -\dfrac{p-1}{p}\,|\Theta'_1(r)|^p - \dfrac{1}{1-\delta}\,\Theta_1(r)^{1-\delta} + C = 0, \quad 0 \leq r < R_1; \\ \Theta_1(0) = 0, \quad \Theta'_1(0) = 1 > 0, \end{cases} \tag{A.13}$$

where the constant C is given by $C = (p-1)/p$. There exists precisely one C^1 function $\Theta_1 \colon [0, R_1) \to \mathbb{R}_+$ that satisfies (A.13) together with $\Theta'_1(r) > 0$ for all $r \in [0, R_1)$; it is determined from

$$\int_0^{\Theta_1(r)} \left(1 - \frac{p}{(p-1)(1-\delta)}\,\theta^{1-\delta}\right)^{-1/p} \mathrm{d}\theta = r, \quad 0 \leq r < R_1, \tag{A.14}$$

where

$$R_1 = \int_0^{[(p-1)(1-\delta)/p]^{1/(1-\delta)}} \left(1 - \frac{p}{(p-1)(1-\delta)}\,\theta^{1-\delta}\right)^{-1/p} \mathrm{d}\theta$$
$$= \left(\frac{(p-1)(1-\delta)}{p}\right)^{1/(1-\delta)} \int_0^1 (1-t^{1-\delta})^{-1/p}\,\mathrm{d}t < \infty \tag{A.15}$$

is the maximal number such that $\Theta_1'(r) > 0$ for all $r \in [0, R_1)$.

Let us first fix $\alpha > 0$ large enough, such that $R_\alpha > M \stackrel{\text{def}}{=} \max_{\overline{\Omega}} \phi_1$. In the following calculations we make use of equations (2.2) and (A.11) for ϕ_1 and Θ_α, respectively. The function $w(x) = \beta \cdot \Theta_\alpha(\phi_1(x))$ of $x \in \Omega$ satisfies

$$\nabla w(x) = \beta \cdot \Theta_\alpha'(\phi_1(x))\, \nabla\phi_1(x),$$
$$|\nabla w(x)|^{p-2}\nabla w(x) = \beta^{p-1}\left[\Theta_\alpha'(\phi_1(x))\right]^{p-1} |\nabla\phi_1(x)|^{p-2}\nabla\phi_1(x),$$

whence

$$\begin{aligned}
-\Delta_p w &= -\beta^{p-1}\left[\left((\Theta_\alpha')^{p-1}\right)' \circ \phi_1\right] |\nabla\phi_1|^p \\
&\quad + \beta^{p-1}\left[\left((\Theta_\alpha')^{p-1}\right) \circ \phi_1\right] (-\Delta_p\phi_1) \\
&= \beta^{p-1}\,(\Theta_\alpha \circ \phi_1)^{-\delta}\, |\nabla\phi_1|^p \\
&\quad + \beta^{p-1}\lambda_1\left[\left((\Theta_\alpha')^{p-1}\right) \circ \phi_1\right] \cdot \phi_1^{p-1} \\
&= \beta^{p-1+\delta}\, |\nabla\phi_1|^p\, w^{-\delta} \\
&\quad + \beta^{p-1}\lambda_1\left[\left((\Theta_\alpha')^{p-1}\right) \circ \phi_1\right] \cdot \phi_1^{p-1}\,(\beta \cdot \Theta_\alpha \circ \phi_1)^\delta\, w^{-\delta} \\
&= \beta^{p-1+\delta}\left\{|\nabla\phi_1|^p + \lambda_1\left[\left((\Theta_\alpha')^{p-1}\right) \circ \phi_1\right] \cdot \phi_1^{p-1}(\Theta_\alpha \circ \phi_1)^\delta\right\} w^{-\delta}.
\end{aligned} \tag{A.16}$$

Recall $R_\alpha > M = \max_{\overline{\Omega}} \phi_1$. The function Θ_α being strictly increasing with strictly decreasing derivative Θ_α' on the interval $[0, R_\alpha]$, and $\Theta_\alpha(0) = 0$, $\Theta_\alpha'(0) = \alpha > \Theta_\alpha'(R_\alpha) = 0$, we can estimate

$$\left((\Theta_\alpha')^{p-1}\right) \circ \phi_1 \geq \Theta_\alpha'(M)^{p-1} > 0,$$
$$\Theta_\alpha \circ \phi_1 \geq \Theta_\alpha'(M)\,\phi_1.$$

We combine these inequalities to estimate the second summand in the curly brackets at the end of equation (A.16) above, thus obtaining

$$-\Delta_p w \geq \beta^{p-1+\delta}\left\{|\nabla\phi_1|^p + \lambda_1\,(\Theta_\alpha'(M)\,\phi_1)^{p-1+\delta}\right\} w^{-\delta}. \tag{A.17}$$

Moreover, we have $w \in C^1(\overline{\Omega})$ together with $w = 0$ on $\partial\Omega$, $w > 0$ in Ω, and $\frac{\partial w}{\partial \nu} < 0$ on $\partial\Omega$. These claims follow from $\phi_1 \in C^1(\overline{\Omega})$ combined with the strong maximum and boundary point principles $\phi_1 > 0$ in Ω and $\frac{\partial \phi_1}{\partial \nu} < 0$ on $\partial\Omega$ (see Vázquez [43, Theorem 5, page 200]). The same arguments render

$$\gamma \stackrel{\text{def}}{=} \min_{\overline{\Omega}} \left\{ |\nabla \phi_1|^p + \lambda_1 \left(\Theta'_\alpha(M)\, \phi_1 \right)^{p-1+\delta} \right\} > 0.$$

We choose the number $\beta > 0$ large enough, such that $\beta^{p-1+\delta}\gamma \geq \lambda$. In particular, inequality (A.17) yields

$$-\Delta_p w \geq \lambda w^{-\delta} \quad \text{in } \Omega. \tag{A.18}$$

Finally, we apply the weak comparison principle to problem (3.1) (with $\underline{u}_\lambda$ in place of u) and to inequality (A.18) (with w satisfying $w = 0$ on $\partial\Omega$), thus arriving at $w \geq \underline{u}_\lambda$ a.e. in Ω. We have thus verified

$$v \leq \underline{u}_\lambda \leq w = \beta \cdot \Theta_\alpha \circ \phi_1 \leq \alpha\beta\, \phi_1 \leq c'_\lambda\, d \quad \text{a.e. in } \Omega,$$

where $c'_\lambda \in (0, \infty)$ is a constant, as desired.

The proof of Lemma A.7 is now complete. □

B. Appendix

Regularity of weak solutions to various types of degenerate elliptic partial differential equations is a broadly developed subject, with a number of general methods and results. However, when a method is applied to a particular equation, this often needs to be done in a way specific to this equation. In this appendix we consider the following quasilinear elliptic boundary value problem,

$$-\nabla \cdot (\mathbf{a}(x, \nabla u)) = f(x) \text{ in } \Omega; \qquad u = 0 \text{ on } \partial\Omega, \tag{B.1}$$

in a setting that is closely related to Lieberman's in [33, Theorem 1, page 1203]. We assume that Ω is a (nonempty) bounded domain in $\mathbb{R}^N$ whose boundary $\partial\Omega$ is a compact C^2 manifold. We denote by $x = (x_1, \dots, x_N)$ a generic point in Ω and by u the unknown function of x, where $u \in W_0^{1,p}(\Omega)$ for $p \in (1, \infty)$. The quasilinear elliptic operator $(x, u) \mapsto \nabla \cdot (\mathbf{a}(x, \nabla u))$ is defined by

$$\nabla \cdot (\mathbf{a}(x, \nabla u)) \stackrel{\text{def}}{=} \sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i(x, \nabla u(x)) \quad \text{for } x \in \Omega \text{ and } u \in W_0^{1,p}(\Omega) \tag{B.2}$$

with values in $W^{-1,p'}(\Omega)$, the dual space of $W_0^{1,p}(\Omega)$, where $\frac{1}{p} + \frac{1}{p'} = 1$. The components a_i of the vector field $\mathbf{a}\colon \Omega \times \mathbb{R}^N \to \mathbb{R}^N$, $\mathbf{a} = (a_1, \dots, a_N)$, are functions of x and $\eta = \nabla u \in \mathbb{R}^N$, such that $a_i \in C^0(\Omega \times \mathbb{R}^N)$ and $\partial a_i / \partial \eta_j \in C^0(\Omega \times (\mathbb{R}^N \setminus \{0\}))$. We assume that $\mathbf{a}$ satisfies the following *ellipticity* and *growth conditions*:

(**H1**) There exist some constants $\kappa \in [0, 1]$, $\gamma, \Gamma \in (0, \infty)$, and $\alpha \in (0, 1)$, such that

$$a_i(x, 0) = 0; \quad i = 1, \dots, N, \tag{B.3}$$

$$\sum_{i,j=1}^{N} \frac{\partial a_i}{\partial \eta_j}(x, \eta) \cdot \xi_i \xi_j \geq \gamma \cdot (\kappa + |\eta|)^{p-2} \cdot |\xi|^2, \tag{B.4}$$

$$\sum_{i,j=1}^{N} \left| \frac{\partial a_i}{\partial \eta_j}(x, \eta) \right| \leq \Gamma \cdot (\kappa + |\eta|)^{p-2}, \tag{B.5}$$

$$\sum_{i=1}^{N} |a_i(x, \eta) - a_i(y, \eta)| \leq \Gamma \cdot (1 + |\eta|)^p \cdot |x - y|^\alpha, \tag{B.6}$$

for all $x, y \in \Omega$, all $\eta \in \mathbb{R}^N \setminus \{0\}$, and all $\xi \in \mathbb{R}^N$.

We remark that conditions (B.3) through (B.6) are motivated by the elliptic boundary value problem

$$-\Delta_p u = f(x) \text{ in } \Omega; \qquad u = 0 \text{ on } \partial\Omega, \tag{B.7}$$

with the p-Laplacian defined by $\Delta_p u \overset{\text{def}}{=} \nabla \cdot (|\nabla u|^{p-2} \nabla u)$.

Finally, we impose the following *growth condition* on the function $f \in L^\infty_{\text{loc}}(\Omega)$:

(**H2**) There exist constants c and δ, $0 < c < \infty$ and $0 < \delta < 1$, such that

$$0 \leq f(x) \leq c\, d(x)^{-\delta} \quad \text{holds for almost all } x \in \Omega. \tag{B.8}$$

It is readily seen that

$$|d(x) - d(y)| \leq |x - y| \quad \text{for } x, y \in \overline{\Omega}.$$

Since the boundary $\partial\Omega$ is of class C^2, d is a C^2 function in a neighborhood of $\partial\Omega$. More precisely, we have $d \in C^2(\Gamma_\mu)$ where $\Gamma_\mu \overset{\text{def}}{=} \{x \in \overline{\Omega} \colon d(x) < \mu\}$, for some $\mu > 0$, by Gilbarg and Trudinger [28, Lemma 14.16, page 355].

We will show the following analogue of a well-known regularity result for problem (B.1) due to Lieberman [33, Theorem 1, page 1203] (regularity near the boundary). Interior regularity was established earlier independently by DiBenedetto [19, Theorem 2, page 829] and Tolksdorf [42, Theorem 1, page 127].

Theorem B.1. *Assume that* $\mathbf{a}(x, \eta)$ *satisfies the structural hypotheses* (B.3) *through* (B.6)*, and* $f(x)$ *satisfies the growth hypothesis* (B.8)*. Let* $u \in W_0^{1,p}(\Omega)$ *be the* (*unique*) *weak solution of problem* (B.1)*. In addition, assume*

$$0 \leq u(x) \leq C\, d(x) \quad \textit{for almost all } x \in \Omega, \tag{B.9}$$

where C is a constant, $0 \le C < \infty$. Then there exist constants β and M, $0 < \beta < \alpha$ and $0 \le M < \infty$, depending solely on Ω, N, p, on the constants γ, Γ, α in (B.4) *through* (B.6)*, on the constants c, δ in* (B.8)*, and on the constant C in* (B.9)*, but* not *on $\kappa \in [0, 1]$, such that u satisfies $u \in C^{1,\beta}(\overline{\Omega})$ and*

$$\|u\|_{C^{1,\beta}(\overline{\Omega})} \le M. \tag{B.10}$$

It is well-known that the existence and uniqueness of a weak solution in $W_0^{1,p}(\Omega)$ are guaranteed by the Minty-Browder theorem for (nonlinear) monotone operators; see *e.g.* Deimling [16, Theorem 12.1, page 117].

We need to modify the proof of Theorem 1 from Lieberman's work [33]. In what follows we employ Lemma 5, page 1211, from [33] as it stands there, but adapt the remaining part of the proof of Theorem 1, pages 1212–1213, to our setting, in particular, equations (3.5) through (3.8).

Proof of Theorem B.1. We "flatten" the boundary $\partial\Omega$ locally by a C^2 diffeomorphism Φ. Such a local transformation of coordinates, $\tilde{x} = \Phi(x)$, leaves all structural conditions for a_i unchanged. The same remark is valid also for f and u in inequalities (B.8) and (B.9). In particular, we can adjust this transformation (by rotation and translation of coordinate axes) in order to achieve $d(\tilde{x}) = \tilde{x}_N$ for all $\tilde{x} \in \mathbb{R}^N$ from an open ball centered at the origin and such that $\tilde{x}_N \ge 0$. Therefore, writing

$$x = (x_1, \dots, x_N) \in \mathbb{R}^N, \ x' = (x_1, \dots, x_{N-1}) \in \mathbb{R}^{N-1}, \text{ and } x = (x', x_N),$$

let us consider only an open ball

$$B_r(y) \stackrel{\text{def}}{=} \{x \in \mathbb{R}^N : |x - y| < r\} \quad \text{for some } y \in \mathbb{R}^N \text{ and } 0 < r < \infty$$

and the corresponding open half-ball

$$B_r^+(y) \stackrel{\text{def}}{=} \{x \in B_r(y) : x_N - y_N > 0\}$$

with the flat boundary portion

$$B_r^0(y) \stackrel{\text{def}}{=} \{x \in B_r(y) : x_N - y_N = 0\}.$$

Finally, let us introduce the half-sphere

$$\begin{aligned} S_r^+(y) &\stackrel{\text{def}}{=} \partial B_r^+(y) \setminus B_r^0(y) \\ &= \{x \in \mathbb{R}^N : |x - y| = r \text{ and } x_N - y_N \ge 0\}. \end{aligned}$$

This means that we have replaced a general domain Ω by an open half-ball; we fix and normalize this half-ball to be $B_1^+(0) \subset \mathbb{R}^N$.

We recall that the vector field

$$\mathbf{a}\colon (x, \nabla u) \mapsto \mathbf{a}(x, \nabla u)\colon B_1^+(0) \times \mathbb{R}^N \to \mathbb{R}^N$$

satisfies the structural hypotheses (B.3) through (B.6), and the function $f\colon B_1^+(0) \to \mathbb{R}^N$ verifies (B.8), *i.e.*,

$$0 \le f(x) \le c\, x_N^{-\delta} \quad \text{holds for almost all } x = (x', x_N) \in B_1^+(0). \tag{B.11}$$

Hypothesis (B.9) for u reads

$$0 \le u(x) \le C\, x_N \quad \text{for almost all } x \in B_1^+(0). \tag{B.12}$$

Following Lieberman's proof of $C^{1,\beta}$ regularity near the boundary in [33], we will prove that the (unique) weak solution $u \in W^{1,p}(B_1^+(0))$ of the partial differential equation

$$-\nabla \cdot \mathbf{a}(x, \nabla u) = f(x) \quad \text{in } B_1^+(0); \qquad u = 0 \quad \text{on } B_1^0(0), \tag{B.13}$$

which is assumed to obey (B.12), satisfies $u \in C^{1,\beta}(\overline{B_{1/2}^+(0)})$ for some $\beta \in (0, \alpha)$. We do not specify the boundary data of u on the half-sphere $S_1^+(0)$, but assume $u \in W^{1,p}(B_1^+(0))$ instead. The method of proof is based on a standard perturbation argument using the Dirichlet boundary value problem

$$\begin{cases} -\nabla \cdot \mathbf{a}(x, \nabla v) = 0 \quad \text{in } B_R^+(y); \\ v = 0 \quad \text{on } B_R^0(y), \qquad v = u \quad \text{on } S_R^+(y), \end{cases} \tag{B.14}$$

for any $y \in B_{1/2}^+(0)$ and any $0 < R < 1/2$; notice that $B_R^+(y) \subset B_1^+(0)$. This problem possesses a unique weak solution $v \equiv v_R$ in $W^{1,p}(B_R^+(y))$. We will estimate an expression for a Campanato norm of the difference $u - v$ in $B_R^+(y)$ depending on the radius R ($0 < R < 1/2$). Using the equivalence of Campanato and Hölder norms, we will thus be able to conclude that u is in $C^{1,\beta}(\overline{B_{1/2}^+(0)})$ for some $\beta \in (0, \alpha)$.

In order to establish the desired estimate for the Campanato expression for $u - v$ in $B_R^+(y)$, it suffices to consider the "normalized" case $y = 0 \in \mathbb{R}^N$ and $0 < R < 1$. In other words, the Dirichlet boundary value problem (B.14) becomes

$$\begin{cases} -\nabla \cdot \mathbf{a}(x, \nabla v) = 0 \quad \text{in } B_R^+(0); \\ v = 0 \quad \text{on } B_R^0(0), \qquad v = u \quad \text{on } S_R^+(0), \end{cases} \tag{B.15}$$

with a unique weak solution $v \in W^{1,p}(B_R^+(0))$, for any $0 < R < 1$.

First of all, we have $0 \le v \le u$ in $B_R^0(0)$, by the weak comparison principle. Hypothesis (B.12) on u thus forces

$$0 \le u(x) - v(x) \le C\, x_N \quad \text{for all } x = (x', x_N) \in B_R^+(0). \tag{B.16}$$

Subtracting equation (B.15) from (B.13), multiplying the difference by $u - v$, and finally integrating over $B_R^+(0)$, we arrive at

$$\begin{aligned} &\int_{B_R^+(0)} [\mathbf{a}(x, \nabla u) - \mathbf{a}(x, \nabla v)] \cdot \nabla(u - v)\, \mathrm{d}x \\ &= \int_{B_R^+(0)} f(x)\,(u - v)\, \mathrm{d}x \le c\, C \int_{B_R^+(0)} x_N^{1-\delta}\, \mathrm{d}x = c_1\, R^{N+1-\delta}, \end{aligned} \tag{B.17}$$

for any $0 < R < 1$, by (B.11) and (B.16). The constant $c_1 = c\, C\, c_0 \ge 0$ has been obtained using

$$\begin{aligned} &\int_{B_R^+(0)} x_N^{1-\delta}\, \mathrm{d}x = R^{N+1-\delta} \int_{B_1^+(0)} z_N^{1-\delta}\, \mathrm{d}z \\ &= R^{N+1-\delta}\, \omega_{N-1} \int_0^1 (1 - z_N^2)^{(N-1)/2}\, z_N^{1-\delta}\, \mathrm{d}z_N \equiv c_0\, R^{N+1-\delta}. \end{aligned}$$

We estimate the left-hand side of inequality (B.17) from below as follows, applying ellipticity condition (B.4). For almost every $x \in B_R^+(0)$ we have

$$\begin{aligned} &[\mathbf{a}(x, \nabla u) - \mathbf{a}(x, \nabla v)] \cdot \nabla(u - v) \\ &= \left[\left(\int_0^1 \nabla\mathbf{a}\,(x, \nabla(v + \theta(u - v)))\ \mathrm{d}\theta \right) \nabla(u - v) \right] \cdot \nabla(u - v) \\ &\ge \gamma \left(\int_0^1 |\nabla(v + \theta(u - v))|^{p-2}\ \mathrm{d}\theta \right) |\nabla(u - v)|^2. \end{aligned} \tag{B.18}$$

If $2 \le p < \infty$, we obtain immediately

$$[\mathbf{a}(x, \nabla u) - \mathbf{a}(x, \nabla v)] \cdot \nabla(u - v) \ge \gamma\, \kappa_p\, |\nabla(u - v)|^p \tag{B.19}$$

where $\kappa_p > 0$ is the constant from the inequality

$$\kappa_p\, |\mathbf{w}|^{p-2} \le \int_0^1 |\mathbf{v} + \theta\mathbf{w}|^{p-2}\, \mathrm{d}\theta \quad \text{for all } \mathbf{v}, \mathbf{w} \in \mathbb{R}^N,$$

$$\kappa_p \stackrel{\text{def}}{=} \min_{\substack{\mathbf{v}, \mathbf{w} \in \mathbb{R}^N \\ |\mathbf{w}|=1}} \int_0^1 |\mathbf{v} + \theta\mathbf{w}|^{p-2}\, \mathrm{d}\theta > 0.$$

We combine (B.17), (B.18), and (B.19) to get

$$\int_{B_R^+(0)} |\nabla(u - v)|^p\, \mathrm{d}x \le c_2\, R^{N+1-\delta} \tag{B.20}$$

with the constant $c_2 = (\gamma\, \kappa_p)^{-1}\, c_1 \ge 0$ independent from $0 < R < 1$.

If $1 < p < 2$, we use the (trivial) inequality

$$(|\mathbf{v}| + |\mathbf{w}|)^{p-2} \le \int_0^1 |\mathbf{v} + \theta \mathbf{w}|^{p-2}\, \mathrm{d}\theta \qquad \text{for all } \mathbf{v}, \mathbf{w} \in \mathbb{R}^N$$

to obtain

$$[\mathbf{a}(x, \nabla u) - \mathbf{a}(x, \nabla v)] \cdot \nabla(u-v) \ge \gamma\, (|\nabla v| + |\nabla(u-v)|)^{p-2}\, |\nabla(u-v)|^2. \quad \text{(B.21)}$$

Next, by Hölder's inequality, we have

$$\begin{aligned} &\int_{B_R^+(0)} |\nabla(u-v)|^p\, \mathrm{d}x \\ &\le \left(\int_{B_R^+(0)} (|\nabla v| + |\nabla(u-v)|)^{p-2}\, |\nabla(u-v)|^2\, \mathrm{d}x \right)^{p/2} \\ &\quad \times \left(\int_{B_R^+(0)} (|\nabla v| + |\nabla(u-v)|)^p\, \mathrm{d}x \right)^{(2-p)/2}, \end{aligned}$$

and then, applying (B.21) followed by Minkowski's inequality,

$$\begin{aligned} &\int_{B_R^+(0)} |\nabla(u-v)|^p\, \mathrm{d}x \\ &\le \gamma^{-p/2} \left(\int_{B_R^+(0)} [\mathbf{a}(x, \nabla u) - \mathbf{a}(x, \nabla v)] \cdot \nabla(u-v)\, \mathrm{d}x \right)^{p/2} \\ &\quad \times \left(\int_{B_R^+(0)} (|\nabla v| + |\nabla(u-v)|)^p\, \mathrm{d}x \right)^{(2-p)/2} \\ &\le \left(\frac{c_1 R^{N+1-\delta}}{\gamma} \right)^{p/2} \left(\int_{B_R^+(0)} (|\nabla v| + |\nabla(u-v)|)^p\, \mathrm{d}x \right)^{(2-p)/2} \\ &\le \left(\frac{c_1 R^{N+1-\delta}}{\gamma} \right)^{p/2} \left[\left(\int_{B_R^+(0)} |\nabla v|^p \mathrm{d}x \right)^{1/p} + \left(\int_{B_R^+(0)} |\nabla(u-v)|^p \mathrm{d}x \right)^{1/p} \right]^{(2-p)p/2}. \end{aligned}$$

With the notation

$$J(u-v;\, R) = \int_{B_R^+(0)} |\nabla(u-v)|^p\, \mathrm{d}x \qquad \text{and} \qquad J(v;\, R) = \int_{B_R^+(0)} |\nabla v|^p\, \mathrm{d}x,$$

this inequality simplifies to

$$\begin{aligned} J(u-v;\, R)^{\frac{2}{(2-p)p}} &= J(u-v;\, R)^{\frac{1}{p} + \frac{1}{2-p}} \\ &\le \left(c_3\, R^{N+1-\delta} \right)^{1/(2-p)} \left(J(u-v;\, R)^{1/p} + J(v;\, R)^{1/p} \right) \end{aligned}$$

whenever $0 < R < 1$, where we have introduced $c_3 = \gamma^{-1} c_1 \geq 0$. Substituting

$$J(u-v;\, R) = c_3\, R^{N+1-\delta}\, \tilde{J}(u-v;\, R) \quad \text{and} \quad J(v;\, R) = c_3\, R^{N+1-\delta}\, \tilde{J}(v;\, R)$$

in the last inequality, we obtain

$$\tilde{J}(u-v;\, R)^{\frac{1}{p}+\frac{1}{2-p}} \leq \tilde{J}(u-v;\, R)^{1/p} + \tilde{J}(v;\, R)^{1/p}$$

whenever $0 < R < 1$. Examining the alternatives

$$\tilde{J}(u-v;\, R) \geq \tilde{J}(v;\, R)^{1/p} \quad \text{and} \quad \tilde{J}(u-v;\, R) \leq \tilde{J}(v;\, R)^{1/p},$$

from this inequality we deduce further

$$\tilde{J}(u-v;\, R) \leq \max\left\{2^{2-p},\ 2^{(2-p)p/2}\, \tilde{J}(v;\, R)^{(2-p)/2}\right\},$$

whenever $0 < R < 1$, and consequently

$$\begin{aligned} &J(u-v;\, R) \\ &\leq c_3\, R^{N+1-\delta} \cdot \max\left\{2^{2-p},\ 2^{(2-p)p/2} \left(c_3\, R^{N+1-\delta}\right)^{-(2-p)/2} J(v;\, R)^{(2-p)/2}\right\} \\ &= c_3\, R^{N+\frac{(1-\delta)p}{2}} \cdot \max\left\{2^{2-p}\, R^{\frac{(1-\delta)(2-p)}{2}},\ 2^{(2-p)p/2} \left(c_3^{-1}\, R^{-N}\, J(v;\, R)\right)^{(2-p)/2}\right\} \end{aligned}$$

which yields

$$R^{-N}\, J(u-v;\, R) \leq c_4\, R^{(1-\delta)p/2} \cdot \max\left\{1,\ \left(R^{-N}\, J(v;\, R)\right)^{(2-p)/2}\right\} \tag{B.22}$$

where $c_4 \geq 0$ is a constant independent from $0 < R < 1$.

Applying certain estimates on suitable norms of v from [33, Lemma 5, page 1211], Lieberman has derived the following inequality for $J(v;\, \cdot)\colon (0,1) \to [0,\infty)$, see [33, Inequality (3.6), page 1212]:

$$J(v;\, r) \leq C_0 \left\{R^N + (r/R)^N\, J(v;\, R)\right\} \quad \text{for all } 0 < r < R \leq R_0, \tag{B.23}$$

where $C_0 \geq 0$ and $0 < R_0 < 1$ are constants independent from both r and R. By Lemma B.2 below, this implies

$$\sup_{0<R\leq R_0} R^{-N+\eta}\, J(v;\, R) = \sup_{0<R\leq R_0} \frac{1}{R^{N-\eta}} \int_{B_R^+(0)} |\nabla v|^p\, \mathrm{d}x \equiv C(\eta) < \infty \tag{B.24}$$

for any number $0 < \eta < N$. Finally, we apply this inequality to (B.22), thus arriving at

$$J(u-v;\, R) \leq c_5\, R^{N+\mu(1-\delta)} \tag{B.25}$$

where $c_5 \equiv c_5(R_0) \geq 0$ is a constant independent from $0 < R \leq R_0$, $\mu = \frac{1}{2}\left(p - \frac{(2-p)\eta}{1-\delta}\right)$ satisfies $0 < \mu < p/2$, and η needs to be taken such that $0 < \eta < \frac{p}{2-p}(1-\delta)$ in order to guarantee $\mu > 0$.

Inequality (B.24) holds for any $1 < p < \infty$. We summarize inequalities (B.20) and (B.25) to obtain, for any $1 < p < \infty$,

$$J(u - v; R) \leq c_5\, R^{N+\mu(1-\delta)} \tag{B.26}$$

where $\mu = 1$ if $2 \leq p < \infty$, $0 < \mu < p/2$ if $1 < p < 2$, and $c_5 \equiv c_5(R_0) \geq 0$ is a constant independent from $0 < R \leq R_0$.

The proof of regularity of u, *i.e.*, $u \in C^{1,\beta}(\overline{B^+_{1/2}(0)})$, can now be completed exactly as in the work of Lieberman [33, page 1213] or DiBenedetto [19, page 849], again with a help from certain estimates on suitable norms of v obtained in [33, Lemma 5, page 1211]. □

It remains to prove (B.23) $\Longrightarrow$ (B.24). This is an easy consequence of the following lemma. We denote $\mathbb{R}_+ = [0, \infty)$.

Lemma B.2. *Let* $J\colon [0,1] \to \mathbb{R}_+$ *be a function which satisfies the following inequality*

$$J(r) \leq C\left\{R^N + \left[\theta(R) + (r/R)^N\right] J(R)\right\} \tag{B.27}$$

for all $0 \leq r < R \leq 1$, *where* $C \geq 0$ *and* $N > 0$ *are some constants, and* $\theta\colon [0,1] \to \mathbb{R}_+$ *is a monotone decreasing function with* $\theta(R) \searrow 0$ *as* $R \searrow 0$. *Then, for any* $0 < \eta < N$, *there exists a constant* $C(\eta) \geq 0$ *such that*

$$J(R) \leq C(\eta)\, R^{N-\eta} \quad \text{for all } 0 < R \leq 1. \tag{B.28}$$

Proof. Fix any number η with $0 < \eta < N$. We choose $0 < R_0 \leq 1$ and $0 < t_0 \leq 1$ such that

$$C\left[\theta(R_0) + t_0^N\right] \leq \frac{1}{2}\, t_0^{N-\eta}.$$

Hence, by our hypothesis on the function θ, we have

$$C\left[\theta(R) + t^N\right] \leq \frac{1}{2}\, t^{N-\eta}$$

for all t and R with $0 < t \leq t_0$ and $0 < R \leq R_0$. We infer from inequality (B.27) that

$$J(tR) \leq C\, R^N + \frac{1}{2}\, t^{N-\eta}\, J(R)$$

and therefore also

$$\frac{J(tR)}{(tR)^{N-\eta}} \leq \frac{C\, R^\eta}{t^{N-\eta}} + \frac{1}{2}\, \frac{J(R)}{R^{N-\eta}} \tag{B.29}$$

whenever $0 < t \le t_0$ and $0 < R \le R_0$. Replacing R by $t^j R$ for $j = 0, 1, \dots, k-1$ in this estimate, we obtain by induction on $k = 1, 2, \dots$ that

$$\frac{J(t^k R)}{(t^k R)^{N-\eta}} \le \frac{C\, R^\eta}{t^{N-\eta}} \sum_{j=0}^{k-1} 2^{-j} + \frac{1}{2^k} \frac{J(R)}{R^{N-\eta}} \le \frac{2C\, R^\eta}{t^{N-\eta}} + \frac{J(R)}{2R^{N-\eta}}$$

whenever $0 < t \le t_0$ and $0 < R \le R_0$. Fixing $t = t_0$ and $R = R_0$ we arrive at

$$\frac{J(t_0^k R_0)}{(t_0^k R_0)^{N-\eta}} \le C_0(\eta) \stackrel{\text{def}}{=} \frac{2C\, R_0^\eta}{t_0^{N-\eta}} + \frac{1}{R_0^{N-\eta}} \cdot \sup_{0<R\le R_0} J(R) < \infty \tag{B.30}$$

for every $k = 0, 1, 2, \dots$.

Finally, if $t_0^k R_0 \le r < t_0^{k-1} R_0$ for some $k \ge 1$, we first apply inequality (B.27) with $t_0^{k-1} R_0$ in place of R to get

$$J(r) \le C \left\{ (t_0^{k-1} R_0)^N + \left[\theta(t_0^{k-1} R_0) + \left(\frac{r}{t_0^{k-1} R_0} \right)^N \right] J(t_0^{k-1} R_0) \right\}$$

and then combine the result with inequality (B.30) to obtain

$$\begin{aligned} J(r) &\le C \left\{ (t_0^{k-1} R_0)^N + [\theta(t_0^{k-1} R_0) + 1]\, C(\eta)\, (t_0^{k-1} R_0)^{N-\eta} \right\} \\ &\le C \left\{ (r/t_0)^N + [\theta(R_0) + 1]\, C(\eta)\, (r/t_0)^{N-\eta} \right\} \\ &= C\, (r/t_0)^{N-\eta} \left\{ (r/t_0)^\eta + C(\eta)\, [\theta(R_0) + 1] \right\} \\ &\le C \left\{ (R_0/t_0)^\eta + C(\eta)\, [\theta(R_0) + 1] \right\} (r/t_0)^{N-\eta}\,. \end{aligned} \tag{B.31}$$

The desired estimate (B.28) follows immediately. □

C. Appendix

The following result is a standard argument which gives sufficient conditions guaranteeing that a weakly convergent Palais-Smale sequence in $W_0^{1,p}(\Omega)$ is also strongly convergent.

Lemma C.1. *Let $1 < p < \infty$ and let $F\colon W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ be a mapping, such that both functionals $u \mapsto \langle F(u), u\rangle$ and $u \mapsto \|F(u)\|_{W^{-1,p'}(\Omega)}$ are* *weakly lower semicontinuous on* $W_0^{1,p}(\Omega)$*. Define the mapping* $T\colon W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$:

$$T(u) \stackrel{\text{def}}{=} -\Delta_p u + F(u), \quad u \in W_0^{1,p}(\Omega).$$

Finally, assume that $\{u_k\}_{k=1}^{\infty} \subset W_0^{1,p}(\Omega)$ *is a sequence satisfying, as* $k \to \infty$,

(i) $\|T(u_k)\|_{W^{-1,p'}(\Omega)} \to 0$;
(ii) $\|u_k\|_{W_0^{1,p}(\Omega)} \to \ell > 0$.

Then $\{u_k\}_{k=1}^{\infty}$ *possesses a strongly convergent subsequence.*

Proof. Owing to (ii), $\{u_k\}_{k=1}^{\infty}$ possesses a weakly convergent subsequence, denoted again by $\{u_k\}_{k=1}^{\infty}$, $u_k \rightharpoonup u_0$ weakly in $W_0^{1,p}(\Omega)$ as $k \to \infty$. Next, we employ the identity

$$\begin{aligned}\langle T(u) - F(u),\, u\rangle = \langle -\Delta_p u,\, u\rangle &= \int_\Omega |\nabla u|^p \,\mathrm{d}x \\ &= \| - \Delta_p u\|_{W^{-1,p'}(\Omega)}\, \|u\|_{W_0^{1,p}(\Omega)} \\ &= \|T(u) - F(u)\|_{W^{-1,p'}(\Omega)}\, \|u\|_{W_0^{1,p}(\Omega)}\end{aligned}$$

for $u \in W_0^{1,p}(\Omega)$. We take $u = u_k$ and let $k \to \infty$, thus obtaining

$$\begin{aligned}-\langle F(u_0),\, u_0\rangle &\geq -\limsup_{k\to\infty}\langle F(u_k),\, u_k\rangle = \liminf_{k\to\infty}\langle -F(u_k),\, u_k\rangle \\ &= \liminf_{k\to\infty}\langle T(u_k) - F(u_k),\, u_k\rangle = \liminf_{k\to\infty}\int_\Omega |\nabla u_k|^p \,\mathrm{d}x \\ &= \left(\liminf_{k\to\infty}\|T(u_k) - F(u_k)\|_{W^{-1,p'}(\Omega)}\right)\left(\lim_{k\to\infty}\|u_k\|_{W_0^{1,p}(\Omega)}\right) \\ &= \ell\cdot\liminf_{k\to\infty}\|F(u_k)\|_{W^{-1,p'}(\Omega)} \geq \ell\cdot\|F(u_0)\|_{W^{-1,p'}(\Omega)},\end{aligned}$$

by (i). It follows that $\|u_0\|_{W_0^{1,p}(\Omega)} \geq \ell$ provided we can show $F(u_0) \neq 0$ in $W^{-1,p'}(\Omega)$. Indeed, $\|F(u_0)\|_{W^{-1,p'}(\Omega)} = 0$ would force

$$\liminf_{k\to\infty}\|u_k\|^p_{W_0^{1,p}(\Omega)} = \liminf_{k\to\infty}\int_\Omega |\nabla u_k|^p \,\mathrm{d}x = 0,$$

thus contradicting $\ell > 0$ in (ii).

Finally, we combine $u_k \rightharpoonup u_0$ in $W_0^{1,p}(\Omega)$ with the inequalities

$$0 < \ell \leq \|u_0\|_{W_0^{1,p}(\Omega)} \leq \liminf_{k\to\infty} \|u_k\|_{W_0^{1,p}(\Omega)} = \ell$$

to conclude that $u_k \to u_0$ strongly in $W_0^{1,p}(\Omega)$. □

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