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Dirichlet problem with L^p-boundary data in contractible domains of Carnot groups

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Abstract. Let $\mathcal{L}$ be a sub-Laplacian on a stratified Lie group G. In this paper we study the Dirichlet problem for $\mathcal{L}$ with L^p-boundary data, on domains Ω which are contractible with respect to the natural dilations of G. One of the main difficulties we face is the presence of non-regular boundary points for the usual Dirichlet problem for $\mathcal{L}$. A potential theory approach is followed. The main results are applied to study a suitable notion of Hardy spaces.

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1. Introduction

In this paper, we deal with a boundary value problem (in a suitable L^p sense) for sub-Laplacians on Carnot groups, *i.e.*, on stratified Lie groups. In recent years, the interest in these operators is rapidly growing: they appear both in theoretical and application settings, mainly involving partial differential equations of sub-elliptic type, the theory of several complex variables, mathematical models of crystal material and human vision, see [15, 18, 21, 25, 32, 34, 35, 36, 37, 41, 42].

As it is well known, a sub-Laplacian $\mathcal{L}$ on a Carnot group G is a linear second order operator with nonnegative characteristic form, which is elliptic only in the "trivial" case when G is the Euclidean group and $\mathcal{L}$ is (up to a linear change of coordinates) the classical Laplace operator. Then due to the "degeneracy" of $\mathcal{L}$, given a bounded open set $\Omega \subset G$, the Dirichlet problem

$$\begin{cases} \mathcal{L}u = 0 & \text{in } \Omega, \\ u|_{\partial\Omega} = \varphi & \varphi \in C(\partial\Omega), \end{cases} \tag{1.1}$$

may not be solvable in a classical sense even if $\partial\Omega$ (the boundary of Ω) is a C^∞ embedded manifold (see [26, 30, 31]).

However, the operator $\mathcal{L}$ has the following redeeming feature. Denoting by $\mathcal{T}$ the topology of G, the map

$$\mathcal{T} \ni \Omega \mapsto \mathcal{H}^{\mathcal{L}}(\Omega) := \{u \in C^\infty(\Omega) \ : \ \mathcal{L}u = 0\}$$

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is a harmonic sheaf endowing G with a structure of $\mathfrak{B}$-harmonic space of elliptic type (we directly refer to Section 2 for the definitions, notation and basic results recalled in this introduction). Then the Dirichlet problem (1.1) has a *generalized solution* in the sense of Perron-Wiener-Brelot, for every $\varphi \in C(\partial\Omega)$. We shall denote this generalized solution by H_φ^Ω. This function is *$\mathcal{L}$-harmonic in* Ω, *i.e.*,

$$H_\varphi^\Omega \in C^\infty(\Omega) \quad \text{and} \quad \mathcal{L}(H_\varphi^\Omega)(x) = 0 \ \text{ for every } x \in \Omega.$$

However, given a point $y \in \partial\Omega$, it is well known that we do not have, in general,

$$\lim_{x \to y} H_\varphi^\Omega(x) = \varphi(y). \tag{1.2}$$

The point $y \in \partial\Omega$ is called $\mathcal{L}$-regular if (1.2) holds for every $\varphi \in C(\partial\Omega)$. If we set

$$\partial_{\mathrm{reg}}\Omega := \{y \in \partial\Omega \ : \ \ y \text{ is } \mathcal{L}\text{-regular}\},$$

then the Dirichlet problem (1.1) has a *classical* solution for every $\varphi \in C(\partial\Omega)$ (*i.e.*, there exists $u \in C^\infty(\Omega) \cap C(\overline{\Omega})$ such that $\mathcal{L}u = 0$ in Ω and $u|_{\partial\Omega} = \varphi$) if and only if

$$\partial_{\mathrm{reg}}\Omega = \partial\Omega. \tag{1.3}$$

In this case the only solution of (1.1) is given by the cited Perron-Wiener-Brelot function H_φ^Ω (in short, PWB function).

The aim of this paper is to show that H_φ^Ω provides the *unique solution of* (1.1)*, in a suitable L^p-sense*, if Ω is any δ_λ-contractible domain, without assuming neither (1.3) nor any other regularity assumption on the boundary. Here $\{\delta_\lambda\}_{\lambda>0}$ is the dilation group of G. We shall say that a bounded open set Ω containing the origin is *δ_λ-contractible* if $\delta_\lambda(\partial\Omega) \subset \Omega$ for every $\lambda \in]0, 1[$ (see Section 4). Moreover, our solvability result does not require the continuity of the boundary datum φ: it holds only assuming $\varphi \in L^p(\partial\Omega)$ with respect to the $\mathcal{L}$-harmonic measures related to Ω.

In order to clearly state our main result, we need some more notation and definitions. For any $x \in \Omega$, let μ_x^Ω be the $\mathcal{L}$-harmonic measure related to Ω and x, *i.e.*, the Radon measure supported on $\partial\Omega$ such that

$$H_\varphi^\Omega(x) = \int_{\partial\Omega} \varphi(y)\, \mathrm{d}\mu_x^\Omega(y) \tag{1.4}$$

for every $\varphi \in C(\partial\Omega)$. From potential theory in abstract harmonic spaces (see e.g., [19]) we know that the function

$$\Omega \ni x \mapsto \int_{\partial\Omega} \varphi(y)\, \mathrm{d}\mu_x^\Omega(y) \tag{1.5}$$

is $\mathcal{L}$-harmonic in Ω if and only if the function φ belongs to $L^1(\partial\Omega, \mu_x^\Omega)$ for every $x \in \Omega$. When φ satisfies this condition, we still denote by H_φ^Ω the function defined in (1.4) (and (1.5)). As a consequence of the Harnack inequality for $\mathcal{L}$, when Ω is connected, the summability of φ with respect to a fixed $\mathcal{L}$-harmonic measure $\mu_{x_0}^{\Omega}$ implies the summability of φ with respect to every μ_x^{Ω}, $x \in \Omega$ (see Proposition 3.5). We would like to explicitly remark that the boundary of a general δ_λ-contractible domain *may contain $\mathcal{L}$-irregular boundary points* even if it is a C^∞ manifold (as showed by the counter-examples in [26], comprising suitable Euclidean balls). The balls related to any δ_λ-homogeneous norm centered at the origin are simple examples of δ_λ-contractible domains.

For a δ_λ-contractible domain Ω there is a natural way to define the L^p-trace on the boundary for any continuous function $u : \Omega \to \mathbb{R}$. If we denote by μ the $\mathcal{L}$-harmonic measure related to Ω and the origin (*i.e.*, $\mu := \mu_0^{\Omega}$) we say that $\varphi \in L^p(\partial\Omega, \mu)$ $(1 \le p \le \infty)$ is the L^p-trace of u on $\partial\Omega$ if

$$u \circ \delta_\lambda \longrightarrow \varphi \quad \text{in } L^p(\partial\Omega, \mu), \qquad \text{as } \lambda \to 1^-. \tag{1.6}$$

When (1.6) holds, we shall write $u|_{\partial\Omega} = \varphi$ in L^p.

Since every δ_λ-contractible domain is connected, we would obtain an equivalent definition of L^p-trace on $\partial\Omega$ by replacing μ_0^{Ω} with any $\mathcal{L}$-harmonic measure μ_x^{Ω}, $x \in \Omega$. Moreover, since $L^p(\partial\Omega, \mu) \subseteq L^1(\partial\Omega, \mu)$, any function $\varphi \in L^p(\partial\Omega, \mu)$ is summable with respect to every $\mathcal{L}$-harmonic measure and the PWB function

$$H_\varphi^{\Omega}(x) := \int_{\partial\Omega} \varphi(y)\, \mathrm{d}\mu_x^{\Omega}(y), \qquad x \in \Omega,$$

is well defined and *$\mathcal{L}$-harmonic in Ω.*

As a first goal, we shall show that φ *is the L^p-trace of H_φ^{Ω} on $\partial\Omega$*. More precisely, our main result reads as follows:

Theorem 1.1. *Let $\Omega \subset G$ be a δ_λ-contractible domain and let $1 \le p < \infty$. Then, for every $\varphi \in L^p(\partial\Omega, \mu)$, the boundary value problem*

$$\begin{cases} \mathcal{L}u = 0 & \text{in } \Omega, \\ u|_{\partial\Omega} = \varphi & \text{in } L^p \end{cases} \tag{1.7}$$

has a unique solution given by $u = H_\varphi^{\Omega}$.

When saying that u is a solution of (1.7), we obviously mean that u is $\mathcal{L}$-harmonic in Ω and its L^p-trace on $\partial\Omega$ is the function φ.

A crucial step in proving Theorem 1.1 is the uniqueness part, which will be obtained as an easy corollary of the following *monotonocity* result:

For every $\mathcal{L}$-harmonic function u in Ω, the map

$$]0, 1[\ni \lambda \mapsto \int_{\partial\Omega} |u(\delta_\lambda(y))|^p \, \mathrm{d}\mu(y)$$

is monotone increasing (see Theorem 4.4).

We shall prove this theorem by using the *Poisson-Jensen formula* for $\mathcal{L}$-subharmonic functions recently proved by one of us and C. Cinti in [4] (see also [7] where the Poisson-Jensen formula was demonstrated for $\mathcal{L}$-regular domains).

From our existence and uniqueness theorem, we readily obtain a *L^p-maximum principle*. We first fix the following definition: Let $u : \Omega \to \mathbb{R}$ be a continuous function and let $1 \le p < \infty$. We say that $u|_{\partial\Omega} \ge 0$ in L^p, if u has a nonnegative L^p-trace on $\partial\Omega$. Then, we have the following result (see Corollary 5.9):

Theorem 1.2. *Let Ω be a δ_λ-contractible domain and let $1 \le p < \infty$. If u is $\mathcal{L}$-harmonic in Ω and $u|_{\partial\Omega} \ge 0$ in L^p, then $u \ge 0$ in Ω.*

From Theorem 1.1, by using well established Harmonic Analysis results on stratified Lie groups (see [21]), we easily obtain an existence and uniqueness theorem for the non-homogeneous boundary value problem

$$\begin{cases} \mathcal{L}u = f & \text{in } \Omega, \\ u|_{\partial\Omega} = \varphi & \text{in } L^p \end{cases} \tag{1.8}$$

for every $\varphi \in L^p(\partial\Omega, \mu)$ $(1 < p < \infty)$ and any $f \in L^q(\Omega, \mathrm{d}x)$ with $q > Q/p$ (see Theorem 6.6).

In Section 6.1, we shall also show the following property of the PWB operator:

If $\Omega \subset G$ is a δ_λ-contractible domain, $1 < p < \infty$, and u is a $\mathcal{L}$-harmonic function in Ω such that

$$\| u \|_{h^p} := \sup_{0<\lambda<1} \left(\int_{\partial\Omega} |u \circ \delta_\lambda|^p \, \mathrm{d}\mu \right)^{1/p} < \infty,$$

then there exists a function $\varphi \in L^p(\partial\Omega, \mu)$ such that $u = H^\Omega_\varphi$. As a consequence, if we define the $\mathcal{L}$-Hardy space in Ω as

$$h^p(\Omega, \mu) := \{u \in C^\infty(\Omega) \mid \mathcal{L}u = 0 \text{ in } \Omega \text{ and } \| u \|_{h^p} < \infty\},$$

the map $\varphi \mapsto H^\Omega_\varphi$ is a linear isometry of $L^p(\partial\Omega, \mu)$ onto $h^p(\Omega, \mu)$ (see Theorem 6.5).

Now, some comments are in order. The method of "approximating boundaries" in the study of the Dirichlet problem for the classical Laplace operator with L^p boundary data was seemingly introduced by G. Cimmino in two papers dated 1937 and 1940 [16, 17]. Several years later, by using the same method, boundary value problems for second order elliptic operators with L^2 boundary data were studied by many authors. We refer to the monograph [14] by J. Chabrowski for a comprehensive survey and a wide bibliography on this subject. We want to stress that the Chabrowski's monograph places a special emphasis on the global weighted summability properties of the gradient of the solutions.[1] In the sub-Laplacian context, this matter presents great difficulties and seems to require new and ad hoc approaches. We plan to take into account this problem in a forthcoming paper.

[1] The otherwise rather complete list of references in [14] does not include Cimmino's works. We should say, however, that Cimmino did not prove any summability of the gradient of the solutions.

The Dirichlet problem for $\mathcal{L}$ could also be attacked with variational techniques, since the $\mathcal{L}$-harmonic functions are the critical points of the functional

$$J(u) := \int_\Omega |\nabla_{\mathcal{L}} u|^2 \, \mathrm{d}x, \quad \nabla_{\mathcal{L}} u = (X_1 u, \dots, X_m u).$$

However, this approach requires a characterization of the *trace spaces* of the functions with finite G-Dirichlet Integral, *i.e.*, of the functions u such that $J(u) < \infty$. Once again, this problem presents very high difficulties, it requires striking properties of the boundary of Ω (see e.g., [24, 38]) and does not allow to consider general L^p boundary data. Indeed, in general, the PWB function H_φ^Ω does not have finite G-Dirichlet integral.[2]

A yet different approach to the Dirichlet problem is the one presented in the survey [33] for the classical elliptic operators, which is based on powerful and deep Harmonic Analysis techniques. This approach takes into account the *pointwise* boundary behavior of the solution, then it is essentially different in spirit to the present one. Moreover, once again, these techniques may hardly be extended to the sub-elliptic contexts. Some results in this direction are contained in the papers [3, 11, 12, 13, 20].

To end the introduction, we give a detailed plan of the paper. In Section 2, we fix the main notation and we recall some prerequisites of potential theory, namely Theorems A to D. In Section 3, we prove some new potential-theoretic results in the framework of Carnot groups. In Section 4, we introduce the notion of *δ_λ-contractible domain* and we prove the basic geometric and analytic properties of such domains (some of which have an interest in their own; see e.g., Theorem 4.4). In Section 5, we introduce the notion of Dirichlet problem $(\mathrm{DP})_\varphi$ with L^p boundary data φ's on δ_λ-contractible domains. We provide existence and uniqueness theorems for this problem (Theorems 5.2 and 5.3).

In Section 6.1, we define the *Hardy-space $h^p(\Omega)$* related to a sub-Laplacian $\mathcal{L}$ and a δ_λ-contractible domain Ω. The main result is Theorem 6.5, which, when $1 < p < \infty$, identifies $h^p(\Omega)$ with the space $L^p(\partial\Omega, \mu)$. In Section 6.2, we show our solvability result for the non-homogeneous Dirichlet problem (Theorem 6.6). Finally, for the sake of completeness, in the Appendix, we give the proof of the main geometric and analytic properties of δ_λ-contractible domains.

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[2] This is well known since the celebrated Hadamard counterexample related to the classical Laplace operator. We would like to mention that the first example of a harmonic function on the unit disc in the plane, continuous up to the boundary and with non-finite Dirichlet Integral was given by F. Prym thirty years before the re-discovery by Hadamard, see the historical paper by M. von Rentlen [40].

2. Notation and basic potential theory

In this section, we collect the notation used throughout the paper and some basic potential-theoretic results on Carnot groups.

We begin with the relevant definition. A Carnot (or stratified) group is a connected and simply connected Lie group G whose Lie algebra $\mathfrak{g}$ admits a stratification, *i.e.*, a direct sum decomposition $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r$ with

$$[\mathfrak{g}_1, \mathfrak{g}_i] = \mathfrak{g}_{i+1} \text{ (for } i \le r-1), \quad [\mathfrak{g}_1, \mathfrak{g}_r] = \{0\}. \tag{2.1}$$

Let $X_1, \ldots, X_m$ be a basis for $\mathfrak{g}_1$. The operator

$$\mathcal{L} = \sum_{i=1}^{m} X_i^2$$

is called a sub-Laplacian on G. By means of the natural identification of G with its Lie algebra via the exponential map, it is non restrictive to suppose that $G = \mathbb{R}^N$ is equipped with a family of group-morphisms (called *dilations*) $\{\delta_\lambda\}_{\lambda>0}$, of the form

$$\delta_\lambda(x_1, \ldots, x_N) = (\lambda^{\alpha_1} x_1, \ldots, \lambda^{\alpha_N} x_N), \tag{2.2}$$

where $1 = \alpha_1 \le \cdots \le \alpha_N = r$. Moreover, $X_1, \ldots, X_m$ are δ_λ-homogeneous of degree one so that $\mathcal{L}$ is δ_λ-homogeneous of degree two, *i.e.*,

$$\mathcal{L}(u \circ \delta_\lambda) = \lambda^2\, (\mathcal{L}u) \circ \delta_\lambda \quad \text{for every } u \in C^\infty(G, \mathbb{R}). \tag{2.3}$$

The integer $Q = \sum_{i=1}^r i \dim(\mathfrak{g}_i)$ (see (2.1)) is called the homogeneous dimension of G. We shall always assume that $Q \ge 3$ (the case $Q = 2$ gives back the well-known case $G = (\mathbb{R}^2, +)$, $\mathcal{L} =$ elliptic constant-coefficient second order operator).

The stratification condition (2.1) ensures that the Lie algebra generated by the vector fields $X_1, \ldots, X_m$ coincides with $\mathfrak{g}$ and is therefore everywhere of rank N. Consequently, by a well known theorem of Hörmander [29], $\mathcal{L}$ is hypoelliptic, *i.e.*, any distributional solution u to $\mathcal{L}u = f \in C^\infty(\Omega)$ in any open set $\Omega \subseteq \mathbb{R}^N$ is actually a $C^\infty(\Omega)$ function. We call *$\mathcal{L}$-harmonic* in Ω every smooth function $u : \Omega \to \mathbb{R}$ such that $\mathcal{L}u = 0$. We shall denote by $\mathcal{H}(\Omega)$ the space of the $\mathcal{L}$-harmonic functions in Ω.

Furthermore (with respect to the cited logarithmic coordinates on G) $\mathcal{L}$ can be written as

$$\mathcal{L} = \operatorname{div}\big(A(x)\, \nabla\big), \quad (\text{here } \nabla = (\partial_{x_1}, \ldots, \partial_{x_N})),$$

where $A(x)$ is a positive semi-definite matrix and its $(1, 1)$-entry is a non-vanishing constant. This ensures that the weak maximum principle for $\mathcal{L}$ holds, *i.e.*, if $\Omega \subset \mathbb{R}^N$ is a bounded open set, and $u \in C^2(\Omega)$ satisfies $\mathcal{L}u \ge 0$ in Ω and $\limsup u \le 0$ on $\partial\Omega$, then $u \le 0$ in Ω.

A consequence of the weak maximum principle is that the Dirichlet problem

$$\text{(DP):} \qquad \mathcal{L}u = 0 \text{ in } \Omega, \quad u|_{\partial\Omega} = \varphi$$

has at most one solution $u \in C^\infty(\Omega) \cap C(\overline{\Omega})$, for every bounded open set $\Omega \subset G$ and every $\varphi \in C(\partial\Omega)$. Moreover $u \geq 0$ in Ω whenever $\varphi \geq 0$ on $\partial\Omega$. A bounded open set $\Omega \subset G$ is said to be *$\mathcal{L}$-regular* if (DP) admits a solution for every $\varphi \in C(\partial\Omega)$. We denote by $H_\varphi^\Omega \in \mathcal{H}(\Omega) \cap C(\overline{\Omega})$ such (unique) solution. Hence, if Ω is $\mathcal{L}$-regular, for every fixed $x \in \Omega$ the map $\varphi \mapsto H_\varphi^\Omega(x)$ defines a positive linear functional on $C(\partial\Omega)$. Hence, there exists a Radon measure μ_x^Ω supported in $\partial\Omega$, such that

$$H_\varphi^\Omega(x) = \int_{\partial\Omega} \varphi(y)\,\mathrm{d}\mu_x^\Omega(y), \quad \text{for every } \varphi \in C(\partial\Omega). \tag{2.4}$$

We call μ_x^Ω the *$\mathcal{L}$-harmonic measure* related to (the $\mathcal{L}$-regular set) Ω and x.

If Ω is an open subset of G, we say that an upper semi-continuous function $u : \Omega \to [-\infty, \infty)$ is *$\mathcal{L}$-subharmonic* in Ω, if $u > -\infty$ in a dense subset of Ω and for every open $\mathcal{L}$-regular set $V \subset \overline{V} \subset \Omega$ and for every $x \in V$, it holds $u(x) \leq \int_{\partial V} u(y)\,\mathrm{d}\mu_x^V(y)$. The set of $\mathcal{L}$-subharmonic functions on Ω will be denoted by $\underline{S}(\Omega)$. We call the functions in $-\underline{S}(\Omega) =: \overline{S}(\Omega)$, the *$\mathcal{L}$-superharmonic* functions in Ω. A set $Z \subset \mathbb{R}^N$ is said *$\mathcal{L}$-polar* if there exists $u \in \overline{S}(\mathbb{R}^N)$ such that $u \equiv \infty$ in Z. The following Ascoli-Phragmén-Lindelöf type maximum principle holds (see [4, Corollary 4.3]):

(Maximum Principle). *Let $\Omega \subset G$ be open and bounded and let $P \subset \partial\Omega$ be a $\mathcal{L}$-polar set. If $u \in \underline{S}(\Omega)$ is bounded above and satisfies*

$$\limsup_{x \to y} u(x) \leq 0 \qquad \textit{for every } y \in \partial\Omega \setminus P,$$

then $u \leq 0$ in Ω.

We now recall a noteworthy result on the fundamental solution for $\mathcal{L}$. Indeed, there exists a homogeneous norm d on G such that

$$\Gamma(x, y) = d^{2-Q}(y^{-1} \circ x), \quad x,\ y \in G \tag{2.5}$$

is a fundamental solution for $\mathcal{L}$ (see [21, Theorem 2.1] and [23]; see also [8, Theorem 2.2]). We call *homogeneous norm* on G any function $d : G \to [0, \infty)$ such that: $d \in C^\infty(G \setminus \{0\}) \cap C(G)$, $d(\delta_\lambda(x)) = \lambda\, d(x)$, $d(x^{-1}) = d(x)$, $d(x) = 0$ iff $x = 0$. It can be easily seen that the map $(x, y) \mapsto d(y^{-1} \circ x)$ is a quasi-distance on G. Throughout the paper, we shall denote by $B_d(x, r)$ the d-ball of center x and radius r, *i.e.*, the set

$$B_d(x, r) = \{y \in G \,|\, d(x^{-1} \circ y) < r\}.$$

The map $G \supseteq \Omega \mapsto \mathcal{H}(\Omega)$ is a *sheaf of functions* endowing G with a structure of *$\mathfrak{B}$-harmonic elliptic space* in the sense of [19]. Indeed, we have:

- The $\mathcal{L}$-regular sets form a basis for the Euclidean topology, see [10, Corollaire 5.2].

– the sheaf $\mathcal{H}$ satisfies the Brelot convergence property, *i.e.*, the pointwise limit of any increasing sequence of $\mathcal{L}$-harmonic functions on a open connected set is $\mathcal{L}$-harmonic whenever it is finite at a point (see [10, Théorème 8.2]).
– For every $x,\ y \in G$, $x \neq y$, there exists $u \in \underline{\mathcal{S}}(\Omega)$, $u \leq 0$ such that $u(x) \neq u(y)$. Indeed, it is enough to choose as u the function defined by $z \mapsto u(z) := -\Gamma(x^{-1} \circ z)$.

A basic potential theory for a class of operators containing the sub-Laplacians $\mathcal{L}$ can be found in [28].

The following *Representation formula for* $\mathcal{L}$ holds (see e.g., [6, Section 3]): If $\Omega \subseteq G$ is an open set and $u \in C^2(\Omega, \mathbb{R})$, then

$$u(x) = M_r(u)(x) - N_r(\mathcal{L}u)(x), \qquad \text{for every} \quad \overline{B_d(x,r)} \subset \Omega,$$

where (with reference to (2.5)) if $K = \sum_{i=1}^m (X_i d)^2$ we have

$$M_r(u)(x) = \frac{Q(Q-2)}{r^Q} \int_{B_d(x,r)} K(x^{-1} \circ y)\, u(y)\, \mathrm{d}y,$$
$$N_r(w)(x) = \frac{Q}{r^Q} \int_0^r \rho^{Q-1} \left(\int_{B_d(x,\rho)} \left(\Gamma(x^{-1} \circ y) - \rho^{2-Q} \right) w(y)\, \mathrm{d}y \right) \mathrm{d}\rho.$$

We briefly recall the relevance of the mean-integral operator M_r within the potential theory for $\mathcal{L}$. If $\Omega \subseteq G$ is an open set, we say that an upper semicontinuous function $u : \Omega \to [-\infty, \infty)$ is $\mathcal{L}$*-submean* (respectively, $\mathcal{L}$*-supermean*) if for every $x \in \Omega$ there exists $r_x > 0$ such that $u(x) \leq M_r(u)(x)$ (respectively, $u(x) \geq M_r(u)(x)$) for $0 < r < r_x$. In the following lemma, we collect some results on M_r proved in [7], which we shall use in the sequel.

Theorem A. *Let $\Omega \subseteq G$ be open and $u : \Omega \to [-\infty, \infty)$ be an upper semicontinuous function. Then, we have:*

(i) *if $u \in C^2(\Omega)$, then $u \in \underline{\mathcal{S}}(\Omega)$ iff $\mathcal{L}u \geq 0$ on Ω.*
(ii) *$u \in \underline{\mathcal{S}}(\Omega)$ iff u is $\mathcal{L}$-submean on Ω and u is finite at least at a single point in every connected component of Ω.*
(iii) *$u \in \underline{\mathcal{S}}(\Omega)$ iff $u \in L^1_{\mathrm{loc}}(\Omega)$, $u(x) = \lim_{r \to 0^+} M_r(u)(x)$ for every $x \in \Omega$ and $\mathcal{L}u \geq 0$ in Ω, in the weak sense of distributions.*

Theorem A follows by collecting Theorem 4.1, Corollary 4.1, Theorem 4.3 in [7].

In particular (by Theorem A-(iii)), for every $u \in \underline{\mathcal{S}}(\Omega)$ there exists a Radon measure μ_u on Ω such that $\int_\Omega u\, \mathcal{L}\varphi = \int_\Omega \varphi\, \mathrm{d}\mu_u$, for every $\varphi \in C_0^\infty(\Omega)$. We say that μ_u is the $\mathcal{L}$*-Riesz measure* related to $u \in \underline{\mathcal{S}}(\Omega)$. Analogously, if $u \in \overline{\mathcal{S}}(\Omega)$, we agree to denote by μ_u the $\mathcal{L}$-Riesz measure related to the $\mathcal{L}$-subharmonic function $-u$.

We end this section by recalling the Poisson-Jensen formula for $\mathcal{L}$, a proof of which can be found in [4, Theorem 3.5] (see also a weaker version in [7, Theorem 5.1]). We shall make a crucial use of this formula in the proof of Theorem 4.4. First we need the relevant definitions.

Throughout the paper, we say that $\Omega \subseteq G$ is a *domain* if it is an open and connected set. We then recall the classical notion of (irr-)regular point $x \in \partial\Omega$ for the Dirichlet problem related to $\mathcal{L}$. Let Ω be a bounded domain and let $\varphi : \partial\Omega \to [-\infty, \infty]$. We let

$$\mathcal{U}(\varphi) = \Big\{u \in \underline{\mathcal{S}}(\Omega) \cup \{-\infty\} \ : \ \sup_{\Omega} u < \infty \text{ and } \limsup_{x\to\zeta} u(x) \le \varphi(\zeta), \ \forall \zeta \in \partial\Omega\Big\}$$

and we set

$$H_\varphi^\Omega(x) := \sup_{u\in\mathcal{U}(\varphi)} u(x), \qquad \text{for } x \in \Omega. \tag{2.6}$$

We say that H_φ^Ω is the *Perron-Wiener-Brelot (lower-)solution of the generalized Dirichlet problem* (PWB solution, in short) related to the set Ω and the function φ. Since G is a $\mathfrak{B}$-harmonic space, we know that H_φ^Ω is $\mathcal{L}$-harmonic in Ω (unless it is identically $\pm\infty$). A point $\zeta \in \partial\Omega$ is called regular for the Dirichlet problem related to $\mathcal{L}$ (in short, *$\mathcal{L}$-regular point*) if, for every bounded φ on $\partial\Omega$ continuous at ζ, it holds $\lim_{x\to\zeta} H_\varphi^\Omega(x) = \varphi(\zeta)$. We denote by $\partial_{\text{reg}}\Omega$ the set of the $\mathcal{L}$-regular points and we call $\partial_{\text{irr}}\Omega := \partial\Omega \setminus \partial_{\text{reg}}\Omega$ the set of the *$\mathcal{L}$-irregular points*. It is a standard matter to prove that, if Ω is $\mathcal{L}$-regular and $\varphi \in C(\partial\Omega)$, then $\partial_{\text{reg}}\Omega = \partial\Omega$ and H_φ^Ω coincides with the function in (2.4). We have the following important result:

Theorem B. *Let $\Omega \subset G$ be bounded. Then $\partial_{\text{irr}}\Omega$ is a $\mathcal{L}$-polar subset of $\partial\Omega$.*

Theorem B follows by combining Theorem 3.1 and Remark 3.2 in [4].

For any fixed $x \in \Omega$, the functional $C(\partial\Omega) \ni \varphi \mapsto H_\varphi^\Omega(x)$ is linear and positive, so that there exists a measure, still denoted by μ_x^Ω, such that (2.4) holds. We agree to call μ_x^Ω the *$\mathcal{L}$-harmonic measure* related to Ω and x. When Ω is $\mathcal{L}$-regular, μ_x^Ω coincides with the formerly defined one. The following crucial result holds:

Theorem C. (i) *Let $\Omega \subset G$ be a bounded domain. Let E be a $\mathcal{L}$-polar subset of $\partial\Omega$. Then, for every $x \in \Omega$, $\mu_x^\Omega(E) = 0$. In particular, $\mu_x^\Omega(\partial_{\text{irr}}\Omega) = 0$.*

(ii) *The representation formula*

$$H_\varphi^\Omega(x) = \int_{\partial\Omega} \varphi \, \mathrm{d}\mu_x^\Omega, \qquad x \in \Omega \tag{2.7}$$

holds for every upper semicontinuous or lower semi-continuous boundary functions φ and for any φ which is μ_x^Ω-integrable on $\partial\Omega$ for at least one $x \in \Omega$ (or, equivalently, for every $x \in \Omega$). In particular, (2.7) *holds for any $\varphi \in L^p(\partial\Omega, \mu_x^\Omega)$, for all $p \in [1, \infty]$ and all $x \in \Omega$.*

The first part of Theorem C is contained in [4, Theorem 3.3]; the second one follows from [19] (for the classical setting, see [1, Chapter 6] and [27, Chapter 8]).

Our last needed prerequisite is the Poisson-Jensen formula for $\mathcal{L}$. Let $\Omega \subseteq G$ be open. We define the *$\mathcal{L}$-Green's function of* Ω as:

$$G_\Omega(x, y) = \Gamma(x^{-1} \circ y) - h_x(y), \qquad x, \ y \in \Omega, \tag{2.8}$$

where, for any fixed $x \in \Omega$, h_x is the greatest $\mathcal{L}$-harmonic minorant of the function $y \mapsto \Gamma(x^{-1} \circ y)$ on Ω. The existence of h_x is ensured by the positivity of Γ and by general results on $\mathfrak{B}$-harmonic elliptic spaces. Other characterizations of G_Ω are recalled in Section 3.

Let now $\Omega \subset G$ be a bounded domain and let $x \in \Omega$ be fixed. We define the *extended $\mathcal{L}$-Green's function $\overline{G}_\Omega(x, \cdot)$ for* Ω. For $y \in G$, $\overline{G}_\Omega(x, y)$ is given by:

$$G_\Omega(x, y) \text{ if } y \in \Omega, \qquad 0 \text{ if } y \in G \setminus \overline{\Omega}, \qquad \limsup_{\Omega \ni z \to y} G_\Omega(x, z) \text{ if } y \in \partial\Omega. \quad (2.9)$$

We then have the following remarkable result (see [4, Theorem 3.5]):

Theorem D (Extended formula of Poisson-Jensen). Let Ω be a bounded domain in $\mathbb{R}^N$. Suppose u is $\mathcal{L}$-subharmonic on a neighborhood of $\overline{\Omega}$. Then, for all $x \in \Omega$,

$$u(x) = \int_{\partial_{reg}\Omega} u(y)\, \mathrm{d}\mu_x^\Omega(y) - \int_{\Omega \cup \partial_{irr}\Omega} \overline{G}_\Omega(x, y)\, \mathrm{d}\mu_u(y). \qquad (2.10)$$

μ_u is the $\mathcal{L}$-Riesz measure of u and $\overline{G}_\Omega$ is the extended $\mathcal{L}$-Green's function for Ω.

We shall make a crucial use of Theorem D in the proof of Theorem 4.4.

3. Potential theory results for sub-Laplacians: further results

In this section, we develop some new aspects of potential theory for sub-Laplacians on stratified groups. The relevant counterparts in the classical Laplace's operator are well known, but in the setting of Carnot groups, they are not explicitly present in literature.

To begin with, we collect some results on the $\mathcal{L}$-Green's function. We recall that the homogeneous dimension of $\mathbb{R}^N$, as a Carnot group, is strictly greater than 2.

Proposition 3.1. *Let $\Omega \subseteq \mathbb{R}^N$ be open and let $x \in \Omega$ be fixed. Let us denote by h_x the greatest $\mathcal{L}$-harmonic minorant of $\Gamma(x^{-1} \circ \cdot)$. Then the following facts hold:*

(i) *The $\mathcal{L}$-Green's function $G_\Omega(x, y) = \Gamma(x^{-1} \circ y) - h_x(y)$ is a symmetric function,* i.e., *$G_\Omega(x, y) = G_\Omega(y, x)$ for all $x,\ y \in \Omega$. Moreover, G_Ω is continuous (in the extended sense) on $\Omega \times \Omega$ and the greatest $\mathcal{L}$-harmonic minorant of $G(x, \cdot)$ is the null function.*

(ii) *It holds*
$$h_x = \sup\big\{u \in \underline{S}(\Omega) \,:\, u \le \Gamma(x^{-1} \circ \cdot) \ \text{ on } \Omega\big\}.$$

(iii) *If Ω is a bounded domain, then $h_x = H^\Omega_{\Gamma(x^{-1}\circ\cdot)}$, in the sense of* (2.6)*, or equivalently*

$$h_x = \sup\big\{u \in \underline{S}(\Omega) \,:\, \limsup_{z \to \zeta} u(z) \le \Gamma(x^{-1} \circ \zeta) \ \text{ for all } \zeta \in \partial\Omega\big\}.$$

(iv) *If Ω is a $\mathcal{L}$-regular domain, then h_x is the solution (in the classical sense) to*

$$\mathcal{L}u = 0 \text{ in } \Omega \quad \text{and} \quad u = \Gamma(x^{-1} \circ \cdot) \text{ on } \partial\Omega. \tag{3.1}$$

(v) *An equivalent definition of $\mathcal{L}$-Green's function is the following one: G_Ω is a nonnegative function on $\Omega \times \Omega$ such that (for every $x \in \Omega$) $G_\Omega(x, \cdot)$ is the sum of $\Gamma(x^{-1} \circ \cdot)$ plus a $\mathcal{L}$-harmonic function on Ω and, moreover, $G_\Omega(x, \cdot)$ does not exceed any other nonnegative $\mathcal{L}$-superharmonic function on Ω which is the sum of $\Gamma(x^{-1} \circ \cdot)$ plus a $\mathcal{L}$-superharmonic function on Ω.*

(vi) *For any Radon measure μ on Ω, let us set*

$$(G_\Omega * \mu)(x) := \int_\Omega G_\Omega(x, y)\,\mathrm{d}\mu(y), \quad x \in \Omega.$$

Then the greatest $\mathcal{L}$-harmonic minorant of $G_\Omega * \mu$ is the null function (provided $G_\Omega * \mu$ is not identically ∞ on any component of Ω).

We do not give a complete proof of Proposition 3.1 (which, indeed, follows classical ideas), but we limit ourselves to outlining some references for it in the Appendix.

With these results in hands, we can give the proof of the following:

Theorem 3.2 (Riesz-Representation). *Let $\Omega \subseteq \mathbb{R}^N$ be open and let $w \in \overline{\mathcal{S}}(\Omega)$ be equipped with a $\mathcal{L}$-harmonic minorant on Ω. Then, if μ_w is the $\mathcal{L}$-Riesz measure of w on Ω, we have*

$$w = G_\Omega * \mu_w + h \quad \text{on } \Omega,$$

where h is the greatest $\mathcal{L}$-harmonic minorant of w on Ω.

Proof. Let h be the greatest $\mathcal{L}$-harmonic minorant of w on Ω (the existence of h follows from general facts in abstract harmonic spaces, see e.g., [19, Theorem 2.2.1]). Let us denote by Ω_n a sequence of bounded open sets such that $\overline{\Omega_n} \subset \Omega_{n+1}$ and $\cup_n \Omega_n = \Omega$. For every $n \in \mathbb{N}$, we denote by $\mu^{(n)}$ the measure on $\mathbb{R}^N$ defined by $\mu^{(n)}(A) := \mu_w(A \cap \overline{\Omega_n})$. By Riesz's Representation Theorem [4, Theorem 2.4], there exists $h_1^{(n)} \in \mathcal{H}(\Omega)$ such that $w = \Gamma * \mu^{(n)} + h_1^{(n)}$ on Ω_n.

Since $\Gamma(x^{-1} \circ \cdot)$ and $G_\Omega(x, \cdot)$ differ by a $\mathcal{L}$-harmonic function, differentiation under the integral sign shows that there exists $h_2^{(n)} \in \mathcal{H}(\Omega_n)$ such that $w = G_\Omega * \mu^{(n)} + h_2^{(n)}$ on Ω_n. Letting n go to infinity, standard arguments show that this implies the existence of $h_2 \in \mathcal{H}(\Omega)$ such that $w = G_\Omega * \mu + h_2$ on Ω.

In particular, h_2 is a $\mathcal{L}$-harmonic minorant of w on Ω, so that $h_2 \leq h$. On the other hand, $G_\Omega * \mu_w + h_2 - h = w - h \geq 0$ on Ω, *i.e.*, $h - h_2$ is a $\mathcal{L}$-harmonic minorant of $G_\Omega * \mu_w$. Since the greatest $\mathcal{L}$-harmonic minorant of $G_\Omega * \mu_w$ is the null function (see Proposition 3.1-(vi)), this gives $h - h_2 \leq 0$. Consequently, $h \equiv h_2$. □

In the proof of our monotonicity theorem (see Theorem 4.4 in Section 4), we shall make use of the following proposition, concerning with the negligeability of $\mathcal{L}$-polar sets with respect to the $\mathcal{L}$-harmonic measures (in the classical context of Laplace's operator, see [1, Theorem 5.1.9]):

Proposition 3.3. *Let $O \subseteq \mathbb{R}^N$ be an open set and let $u \in \underline{\mathcal{S}}(O)$ be locally bounded. If μ_u denotes the $\mathcal{L}$-Riesz measure of u on O, then $\mu_u(Z) = 0$ for every Borel $\mathcal{L}$-polar set $Z \subseteq O$.*

Proof. We only need to show that $\mu_u(Z \cap B) = 0$ for every d-ball B with $\overline{B} \subset O$.

First, we claim that there exists a compactly supported Radon measure ν on $\mathbb{R}^N$ such that $G_B * \nu = \infty$ on $Z \cap B$. Indeed, let $w \in \overline{\mathcal{S}}(\mathbb{R}^N)$ be such that $w|_{Z\cap B} \equiv \infty$. The lower semi-continuity of w proves that w is lower bounded on B. Hence, by Theorem 3.2, we have $w = G_B * \mu_w + h$ on B, for a suitable $h \in \mathcal{H}(B)$. Since h is finite-valued, this proves the claim.

Let $\alpha = \inf_B u$ ($\alpha \in \mathbb{R}$ since u is lower semi-continuous). From Theorem 3.2 applied to the function $w = u - \alpha$ (which is nonnegative on B, hence endowed with a $\mathcal{L}$-harmonic minorant on B), we infer the existence of $h \in \mathcal{H}(B)$ such that

$$u(x) - \alpha = (G_B * \mu_u)(x) + h(x), \quad \text{for every } x \in B. \tag{3.2}$$

(We used $\mu_w = \mu_u$.) Since h is the greatest $\mathcal{L}$-harmonic minorant of $w \geq 0$ on B, it holds $h \geq 0$, so that (3.2) gives

$$(G_B * \mu_u)(x) \leq u(x) - \alpha, \quad \text{for every } x \in B. \tag{3.3}$$

As a consequence, we have

$$\begin{aligned} \infty \cdot \mu_u(Z \cap B) &\leq \int_B (G_B * \nu)(y)\, \mathrm{d}\mu_u(y) = \int_B (G_B * \mu_u)(y)\, \mathrm{d}\nu(y) \\ \text{(see (3.3))} \quad &\leq (\sup_{\overline{B}} u - \alpha) \cdot \nu(\mathbb{R}^N) < \infty \quad \text{(by the local boundedness of } u\text{)} \end{aligned}$$

(in the equality, we used Fubini-Tonelli's Theorem jointly with the symmetry of G_B). This gives $\mu_u(Z \cap B) = 0$, ending the proof. □

Proposition 3.4. *Let $\Omega \subseteq \mathbb{R}^N$ be open and $u \in \mathcal{H}(\Omega)$. Then, for every convex function $g : \mathbb{R} \to \mathbb{R}$, we have $g \circ u \in \underline{\mathcal{S}}(\Omega)$. In particular, this holds for $g = |\cdot|^p$ (where $p \in [1, \infty[$).*

Proof. We prove the assertion by making use of the $\mathcal{L}$-subharmonicity test in Theorem A-(ii). Since $g \circ u$ is continuous on Ω, we only have to show that $g \circ u$ is $\mathcal{L}$-submean. Since $u \in \mathcal{H}(\Omega)$, by Theorem A-(ii) we infer that for every $x \in \Omega$, there exists $r_x > 0$ such that

$$u(x) = M_r(u)(x) = \int_{B_d(x,r)} u(y)\, \frac{Q(Q-2)}{r^Q}\, K(x^{-1} \circ y)\, \mathrm{d}y$$

for every $0 < r < r_x$. Now, since $M_r(1) \equiv 1$, the density

$$\mathrm{d}\nu(y) := \frac{Q(Q-2)}{r^Q}\, K(x^{-1} \circ y)\, \mathrm{d}y$$

defines on $B_d(x,r)$ a probability measure, whence (by Jensen's inequality) we derive

$$g(u(x)) \leq \int_{B_d(x,r)} g(u(y)) \,\frac{Q(Q-2)}{r^Q}\, K(x^{-1}\circ y)\,\mathrm{d}y = M_r(g\circ u)(x),$$

for every $0 < r < r_x$, *i.e.*, $g \circ u$ is $\mathcal{L}$-submean. □

We shall make a crucial use of the following theorem (a simple consequence of Harnack's Theorem for sub-Laplacians) in the proof of our uniqueness Theorem 5.2.

Proposition 3.5. *Let $\Omega \subseteq \mathbb{R}^N$ be open. For all $x \in \Omega$, there exists $h_x \in L^1(\partial\Omega, \mu_0^\Omega)$ such that*

$$\mathrm{d}\mu_x^\Omega(\xi) = h_x(\xi)\,\mathrm{d}\mu_0^\Omega(\xi).$$

Moreover, if K is a compact subset of Ω, there exists a positive constant $\mathbf{c}(K)$ only depending on K, such that

$$\mathbf{c}(K)^{-1} \leq \|\, h_x; L^\infty(\partial\Omega, \mu_0^\Omega)\,\| \leq \mathbf{c}(K), \qquad \textit{for every } x \in K. \tag{3.4}$$

Here and henceforth, $\|\, u; L^p(\partial\Omega, \mu_0^\Omega)\,\|$, $p \in [1,\infty]$, will denote the usual L^p-norm of a function $u \in L^p(\partial\Omega, \mu_0^\Omega)$.

Proof. In this proof, for brevity, for any given $x \in \Omega$ we agree to set $\mu_x := \mu_x^\Omega$.

Let $E \subseteq \partial\Omega$ be closed. Then χ_E is upper semicontinuous, and we have (by Theorem C-(ii))

$$\mu_x(E) = \int_{\partial\Omega} \chi_E\,\mathrm{d}\mu_x = H_{\chi_E}^\Omega(x). \tag{3.5}$$

Now, $H_{\chi_E}^\Omega$ is a non-negative $\mathcal{L}$-harmonic function on the connected open set Ω, so that we can apply Harnack inequality for $\mathcal{L}$ (see, e.g., [10]). In particular, for every compact set $K \subset \Omega$ there exists $\mathbf{c} = \mathbf{c}(K) > 0$ such that

$$H_{\chi_E}^\Omega(x) \leq \mathbf{c}\, H_{\chi_E}^\Omega(y), \quad \text{for every } x, y \in K.$$

Hence, (3.5) gives $\mu_x(E) \leq \mathbf{c}\,\mu_y(E)$, for every closed $E \subseteq \partial\Omega$. Consequently, since the μ_x's are regular measures, we obtain (also reversing the rôles of x and y)

$$\mathbf{c}^{-1}\mu_y(E) \leq \mu_x(E) \leq \mathbf{c}\,\mu_y(E), \quad \text{for every Borel set } E \subseteq \partial\Omega. \tag{3.6}$$

This proves that $\mu_y \ll \mu_x \ll \mu_y$ for every couple of points $x, y \in \Omega$. Hence, by Lebesgue decomposition theorem, we infer the existence of $h_{x,y} \in L^1(\partial\Omega, \mu_y)$ with

$$\mathrm{d}\mu_x(\xi) = h_{x,y}(\xi)\,\mathrm{d}\mu_y(\xi). \tag{3.7}$$

We claim that

$$\mathbf{c}^{-1} \leq \|\, h_{x,y}(\cdot); L^\infty(\partial\Omega, \mu_y)\,\| \leq \mathbf{c}.$$

Indeed, suppose first to the contrary there exists a Borel set $E \subseteq \partial\Omega$ such that $\mu_y(E) > 0$ and $h_{x,y}(\xi) > \mathbf{c}$, μ_y-almost-everywhere on E. This would give the following contradiction (using (3.6) and (3.7))

$$\mathbf{c} \geq \frac{\mu_x(E)}{\mu_y(E)} = \frac{1}{\mu_y(E)} \int_E h_{x,y}(\xi)\, \mathrm{d}\mu_y(\xi) \gneqq \mathbf{c}\, \frac{\int_E \mathrm{d}\mu_y(\xi)}{\mu_y(E)} = \mathbf{c}.$$

Analogously, suppose to the contrary there exists a Borel set $E \subseteq \partial\Omega$ such that $\mu_y(E) > 0$ and $h_{x,y}(\xi) < \mathbf{c}^{-1}$, μ_y-almost-everywhere on E. This would give another contradiction (using (3.6) and (3.7)):

$$\mathbf{c}^{-1} \leq \frac{\mu_x(E)}{\mu_y(E)} = \frac{1}{\mu_y(E)} \int_E h_{x,y}(\xi)\, \mathrm{d}\mu_y(\xi) \lneqq \mathbf{c}^{-1}\, \frac{\int_E \mathrm{d}\mu_y(\xi)}{\mu_y(E)} = \mathbf{c}^{-1}.$$

Our lemma follows by taking $y = 0$ and setting $h_x := h_{x,0}$. □

The following proposition will be crucial in proving some "smallness" property of $\mathcal{L}$-polar sets in connection with the dilations δ_λ of the group G (see Proposition 4.5 in Section 4).

Proposition 3.6. *If $A \subseteq \mathbb{R}^N$ is an open connected set and $E \subset A$ is $\mathcal{L}$-polar and relatively closed in A, then $A \setminus E$ is connected.*

Proof. With the notation of the assertion, let A_0 be a connected component of $A \setminus E$. We are done if we show that $A_0 = A \setminus E$. Note that, since E is relatively closed in A, then $A \setminus E$ is open, whence A_0 is open too. Since E is $\mathcal{L}$-polar, there exists $u \in \overline{\mathcal{S}}(\mathbb{R}^N)$ such that $u \equiv \infty$ on E. Set

$$v : A \to (-\infty, \infty], \quad v(x) = \begin{cases} u(x) & \text{if } x \in A_0, \\ \infty & \text{if } x \in A \setminus A_0. \end{cases}$$

Note that $\widetilde{A} := (A \setminus E) \setminus A_0$ is an open set since it coincides with the union of the connected components of the open set $A \setminus E$, except for A_0. It is easily seen that v is lower semi-continuous, not identically ∞ and $\mathcal{L}$-supermean on A, since $u|_{A_0}$ is $\mathcal{L}$-superharmonic. Then by Theorem A-(ii), $v \in \overline{\mathcal{S}}(A)$. But $v \equiv \infty$ on the open set $\widetilde{A}$ and this can happen only if $\widetilde{A} = \emptyset$, for $\mathcal{L}$-superharmonic functions are locally integrable by Theorem A-(iii). This proves that A contains no connected components other than A_0, thus completing the proof. □

4. δ_λ-contractible domains

In this section, we collect some preliminary results on the geometry of the domains we are interested in. Throughout the paper, G will be a fixed Carnot group on $\mathbb{R}^N$ with dilation group as in (2.2) and $\mathcal{L}$ will denote a fixed sub-Laplacian on G.

Definition 4.1 (δ_λ-contractible domain). We say that Ω is a *δ_λ-contractible domain* if $\Omega \subset \mathbb{R}^N$ is a bounded open set such that

$$\delta_\lambda(\partial\Omega) \subset \Omega \qquad \text{for every } \lambda \in [0, 1[. \tag{4.1}$$

By Proposition 4.6 at the end of this section (which collects basic geometric properties of δ_λ-contractible domains Ω), it follows that Ω is a connected neighborhood of 0. The d-balls are examples of $\mathcal{L}$-regular δ_λ-contractible domains. However, we explicitly stress that a general δ_λ-contractible domain may possess *$\mathcal{L}$-irregular* boundary points, even in the classical case $\mathcal{L} = \Delta$. Indeed, it is not difficult to construct suitable Lebesgue's spines star-shaped with respect to the origin. This makes the Hardy-space theory on such domains more involved than in the classical case of the unit ball.

Throughout the paper, we shall use the following notation: if A is any subset of G, we let

$$A_\lambda := \delta_\lambda(A) = \{\delta_\lambda(x) \mid x \in A\}.$$

It is easy to see that $\partial(A_\lambda) = (\partial A)_\lambda$, so that we shall simply write ∂A_λ.

When this does not lead to confusion, μ (or μ_0) will denote the *$\mathcal{L}$-harmonic measure for* Ω *at* 0 (see Section 2) *i.e.*,

$$\mu := \mu_0^\Omega.$$

Accordingly, for every $p \in [1, \infty]$, we set

$$L^p(\partial\Omega, \mu) := L^p(\partial\Omega, \mu_0^\Omega).$$

We shall write $\| \, u; L^p(\partial\Omega, \mu) \, \|$ to denote the usual L^p-norm of a function $u \in L^p(\partial\Omega, \mu_0^\Omega)$. Unless otherwise stated, if u is any function on the δ_λ-contractible domain Ω and $0 < \lambda < 1$, the function $u \circ \delta_\lambda$ will be considered as a function defined on $\partial\Omega$.

We now collect some Lemmata on δ_λ-contractible domains which we shall use in the next sections.

Lemma 4.2. *Let Ω be a δ_λ-contractible domain. If $\psi \in C(\Omega, \mathbb{R})$ and $0 < \lambda < 1$ we have*

$$H_\psi^{\Omega_\lambda}(\delta_\lambda(x)) = H_{\psi\circ\delta_\lambda}^{\Omega}(x) \qquad \textit{for every } x \in \Omega. \tag{4.2}$$

Proof. In the left-hand side of (4.2), ψ is meant as $\psi|_{\partial\Omega_\lambda}$. We explicitly note that both sides of (4.2) make sense, for $\psi \in C(\partial\Omega_\lambda, \mathbb{R})$ and $\psi \circ \delta_\lambda \in C(\partial\Omega, \mathbb{R})$.

We need only check that $H_\psi^{\Omega_\lambda} \circ \delta_\lambda$ defines on Ω a bounded $\mathcal{L}$-harmonic function, converging to $\psi \circ \delta_\lambda$ on the $\mathcal{L}$-regular points of $\partial\Omega$, *i.e.*,

$$\lim_{\Omega \ni x \to \xi} H_\psi^{\Omega_\lambda}(\delta_\lambda(x)) = \psi(\delta_\lambda(x)), \quad \text{for all } \xi \in \partial_{\mathrm{reg}}\Omega. \tag{4.3}$$

These facts can be proved as follows:

(i) $H_\psi^{\Omega_\lambda} \circ \delta_\lambda$ is $\mathcal{L}$-harmonic in Ω, since $\mathcal{L}(H_\psi^{\Omega_\lambda} \circ \delta_\lambda) = \lambda^2 (\mathcal{L} H_\psi^{\Omega_\lambda}) \circ \delta_\lambda = 0$ on Ω, for $H_\psi^{\Omega_\lambda} \in \mathcal{H}(\Omega_\lambda)$;

(ii) the boundedness of $H_\psi^{\Omega_\lambda} \circ \delta_\lambda$ on Ω is equivalent to that of $H_\psi^{\Omega_\lambda}$ on Ω_λ, which follows since ψ is bounded on $\partial\Omega_\lambda$);

(iii) finally, (4.3) follows at once from the fact that (see also Theorem B)

$$\lim_{\Omega_\lambda \ni x' \to \xi'} H_\psi^{\Omega_\lambda}(x') = \psi(x'), \quad \text{for all } \xi' \in \partial_{\text{reg}}(\Omega_\lambda)$$

jointly with

$$(\partial_{\text{reg}}\Omega)_\lambda = \partial_{\text{reg}}(\Omega_\lambda),$$

which follows by standard arguments (using again the δ_λ-homogeneity of $\mathcal{L}$).

This ends the proof of the lemma. □

Corollary 4.3. *In the hypotheses of the previous lemma, we have*

$$\int_{\partial\Omega_\lambda} \psi \, \mathrm{d}\mu_0^{\Omega_\lambda} = \int_{\partial\Omega} \psi \circ \delta_\lambda \, \mathrm{d}\mu_0^{\Omega}, \qquad \textit{for every } \lambda \in]0,1[.$$

In particular, for every $1 \le p < \infty$,

$$\left(\int_{\partial\Omega_\lambda} |\psi|^p \, \mathrm{d}\mu_0^{\Omega_\lambda} \right)^{1/p} = \left(\int_{\partial\Omega} |\psi \circ \delta_\lambda|^p \, \mathrm{d}\mu_0^{\Omega} \right)^{1/p}, \tag{4.4}$$

so that, letting $p \to \infty$,

$$\| \psi; L^p(\partial(\Omega_\lambda), \mu_0^{\Omega_\lambda}) \| = \| \psi \circ \delta_\lambda; L^p(\partial\Omega, \mu_0^{\Omega}) \|, \qquad \textit{for every } 1 \le p \le \infty. \tag{4.5}$$

Proof. The corollary follows by taking $x = 0$ in (4.2) and by using the integral representations (2.7) of $H_\psi^{\Omega_\lambda}$ and $H_{\psi\circ\delta_\lambda}^{\Omega}$. □

The following important result will play a central rôle in this paper.

Theorem 4.4 (Monotonicity). *Let* Ω *be a* δ_λ*-contractible domain of* $\mathbb{R}^N$. *Let* u *be a* $\mathcal{L}$*-harmonic function on* Ω. *Then the following map*

$$]0,1[\ni \lambda \mapsto \int_{\partial\Omega} |u \circ \delta_\lambda|^p \, \mathrm{d}\mu_0^{\Omega} \tag{4.6}$$

is monotone increasing, for every fixed $p \in [1, \infty[$. *As a consequence, the map*

$$]0,1[\ni \lambda \mapsto \| u \circ \delta_\lambda; L^p(\partial\Omega, \mu_0^{\Omega}) \|$$

is monotone increasing for every $p \in [1, \infty]$.

Proof. For any $p \in [1, \infty]$, let us denote by $f_p(\lambda)$ the function in (4.6). It is enough to prove the monotonicity of f_p for $1 \le p < \infty$, for this implies the monotonicity of $f_p^{1/p}$ and of $f_\infty = \lim_{p\to\infty} f_p^{1/p}$, which is the second part of the assertion.

Let $u \in \mathcal{H}(\Omega)$, $p \in [1, \infty[$ and $\lambda > 0$ be fixed. Since $|u|^p$ is $\mathcal{L}$-subharmonic on Ω (see Proposition 3.4), and since $\Omega_\lambda \subset \Omega$ (see Proposition 4.6-(ii)), we can apply Poisson-Jensen formula (2.10) for $|u|^p$ on Ω_λ at the point 0:

$$|u|^p(0) = \int_{\partial_{reg}(\Omega_\lambda)} |u|^p(\xi)\, d\mu_0^{\Omega_\lambda}(\xi) - \int_{\Omega_\lambda \cup \partial_{irr}(\Omega_\lambda)} \overline{G}_{\Omega_\lambda}(0, y)\, d\mu_{|u|^p}(y). \tag{4.7}$$

Now, since $|u|^p$ is continuous (whence locally bounded) on Ω and $\partial_{\rm irr}(\Omega_\lambda)$ is a $\mathcal{L}$-polar set (see Theorem B) in $\partial(\Omega_\lambda) = \delta_\lambda(\partial\Omega) \subset \Omega$, by Proposition 3.3 we infer

$$\mu_{|u|^p}(\partial_{\rm irr}(\Omega_\lambda)) = 0. \tag{4.8}$$

Moreover, from the $\mathcal{L}$-polarity of $\partial_{\rm irr}(\Omega_\lambda)$ and Theorem C-(i), we deduce also

$$\mu_0^{\Omega_\lambda}(\partial_{\rm irr}(\Omega_\lambda)) = 0. \tag{4.9}$$

Hence, combining (4.8) and (4.9), (4.7) becomes

$$|u|^p(0) + \int_{\Omega_\lambda} \overline{G}_{\Omega_\lambda}(0, y)\, d\mu_{|u|^p}(y) = \int_{\partial(\Omega_\lambda)} |u|^p(\xi)\, d\mu_0^{\Omega_\lambda}(\xi). \tag{4.10}$$

Since the y dummy-variable appearing in the left-hand of (4.10) belongs to Ω_λ, we have $\overline{G}_{\Omega_\lambda}(0, y) = G_{\Omega_\lambda}(0, y)$ by the definition (2.9). Now, the left-hand side of (4.10) is monotone increasing with respect to $\lambda \in]0, 1[$ since, for every $0 < \lambda_1 < \lambda_2 < 1$ we have $\Omega_{\lambda_1} \subseteq \Omega_{\lambda_2}$ (see Proposition 4.6-(v)), so that

$$G_{\Omega_{\lambda_1}}(0, y) \le G_{\Omega_{\lambda_2}}(0, y).$$

Then, also the right-hand side of (4.10) is monotone increasing with respect to $\lambda \in]0, 1[$. On the other hand, by Lemma 4.3, the right-hand side of (4.10) equals

$$\int_{\partial\Omega} |u(\delta_\lambda(\xi))|^p\, d\mu_0^{\Omega}(\xi),$$

so that our monotonicity theorem follows. □

The following proposition (besides having an interest in its own), concerns with the "smallness" of $\mathcal{L}$-polar sets. Its proof is unexpectedly delicate.

Proposition 4.5. *Let Ω be δ_λ-contractible and $P \subset \partial\Omega$ a $\mathcal{L}$-polar set. Then*

$$\bigcup_{0<\lambda<1} \delta_\lambda(P)$$

has empty interior.

Proof. We argue by contradiction, supposing that there exists an open d-ball, say $B = B_d(x_0, r)$, such that

$$B \subseteq \bigcup_{0<\lambda<1} \delta_\lambda(P). \tag{4.11}$$

It is non-restrictive to suppose that $x_0 \neq 0$. We explicitly remark that the δ_λ-contractibility of Ω, the hypothesis $P \subset \partial\Omega$ and (4.11) imply that $B \subseteq \Omega$. Set

$$A := \bigcup_{\lambda>0} \delta_\lambda(B) \quad \text{and} \quad E := A \cap \partial\Omega.$$

We claim that the following facts hold:

(i) A is an open connected set;
(ii) E is relatively closed in A and E is a subset of P, hence $\mathcal{L}$-polar.
(iii) E disconnects A;

Clearly, Proposition 3.6 is in contradiction with (i)-(ii)-(iii), so that the proof is complete if we demonstrate the above three claimed statements.

(i): Since $\delta_\lambda(B) = \delta_\lambda(B_d(x_0, r)) = B_d(\delta_\lambda(x_0), \lambda\, r)$, A is open; this also proves

$$A = \bigcup_{\lambda>0} B_d(\delta_\lambda(x_0), \lambda\, r),$$

so that A is connected since the d-balls $\delta_\lambda(B)$'s are connected open sets joined together by the connected path $\{\delta_\lambda(x_0) : \lambda > 0\}$.

(ii): E is trivially relatively closed in A by its very definition. We show that the δ_λ-contractibility of Ω ensures that $E \subseteq P$. Suppose by contradiction that there exists a point $\xi_0 \in E \setminus P$. Consequently, there exist $\lambda_0 > 0$ and $b_0 \in B$ such that $\xi_0 = \delta_{\lambda_0}(b_0)$ and $\xi_0 \in \partial\Omega \setminus P$. Thanks to (4.11), there also exist $\lambda_1 \in]0, 1[$ and $z_1 \in P$ such that $b_0 = \delta_{\lambda_1}(z_1)$, so that we infer $z_1 = \delta_{1/\lambda_1}(b_0)$ but this is absurd, for this would imply that $b_0 \in B \subseteq \Omega$ is a point of Ω such that the "δ_λ-ray" $\{\delta_\lambda(b_0) : \lambda > 0\}$ contains two distinct points of $\partial\Omega$, namely z_1 and ξ_0 (which are distinct for $\xi_0 \in \partial\Omega \setminus P$, whereas $z_1 \in P$). But we know from Proposition 4.6-(i) that a δ_λ-ray passing through a point of a δ_λ-contractible open set Ω contains exactly one point of $\partial\Omega$. We reached a contradiction supposing that $E \nsubseteq P$.

(iii): $A \setminus E$ is disconnected for, by the very definition of E, it holds

$$A \setminus E = A \setminus \partial\Omega = (A \cap \Omega) \cup (A \setminus \overline{\Omega}) =: A_1 \cup A_2,$$

and A_1, A_2 are non-empty open sets in A. Indeed, $A \cap \Omega \supseteq B$ and $A \setminus \overline{\Omega}$ contains $\delta_\lambda(x_0)$ for a suitable $\lambda \gg 1$, Ω being bounded. This ends the proof. $\square$

To end with the preliminaries, we state the basic geometric properties of δ_λ-contractible domains in the following proposition, whose proof we postpone to the Appendix.

Proposition 4.6. *Let Ω be a δ_λ-contractible domain. Then, the following statements hold:*

(i) *for every $x \in \Omega$, $x \neq 0$, the set $\{\delta_\lambda(x) : \lambda > 0\} \cap \partial\Omega$ is a singleton. The same is true of $\{\delta_\lambda(x) : \lambda \geq 1\} \cap \partial\Omega$;*
(ii) *for every $0 \leq \lambda < 1$, we have $\delta_\lambda(\overline{\Omega}) \subseteq \Omega$;*
(iii) *we have $\Omega = \bigcup_{0 \leq \lambda < 1} \delta_\lambda(\partial\Omega) = \bigcup_{0 \leq \lambda < 1} \delta_\lambda(\overline{\Omega})$;*
(iv) *Ω is connected;*
(v) *for every $0 \leq \lambda_1 \leq \lambda_2 \leq 1$, we have $\delta_{\lambda_1}(\Omega) \subseteq \delta_{\lambda_2}(\Omega)$.*

5. The L^p-Dirichlet problem

We use the same notation as in Section 4. We begin with the relevant definition.

Definition 5.1. Let Ω be a δ_λ-contractible domain of $\mathbb{R}^N$. Let $\varphi \in L^p(\partial\Omega, \mu)$ be fixed. We say that u is a *L^p-solution of the Dirichlet-problem* $(\mathrm{DP})_\varphi$

$$(\mathrm{DP})_\varphi : \qquad \mathcal{L}u = 0 \ \text{ in } \Omega, \quad u = \varphi \quad \text{in } \partial\Omega$$

if the following conditions hold:

(DP1) $u \in C^\infty(\Omega, \mathbb{R})$ and $\mathcal{L}u = 0$ in Ω;
(DP2) $u \circ \delta_\lambda \to \varphi$ in $L^p(\partial\Omega, \mu)$, as $\lambda \to 1^-$.

Condition (DP2) is well-posed, thanks to the δ_λ-contractibility (4.1) of Ω. It means that

$$\begin{aligned} &\lim_{\lambda \to 1^-} \int_{\partial\Omega} \left| u(\delta_\lambda(\xi)) - \varphi(\xi) \right|^p \mathrm{d}\mu_0^\Omega(\xi) = 0, && \text{if } 1 \leq p < \infty, \\ &\lim_{\lambda \to 1^-} \| \, u \circ \delta_\lambda - \varphi ; L^\infty(\partial\Omega, \mu_0^\Omega) \, \| = 0, && \text{if } p = \infty. \end{aligned} \tag{5.1}$$

We would like to stress that the study of the Dirichlet problem $(\mathrm{DP})_\varphi$ is (not unexpectedly) complicated by the presence of $\mathcal{L}$-irregular points on $\partial\Omega$.

We begin by establishing a uniqueness result for $(\mathrm{DP})_\varphi$:

Theorem 5.2 (Uniqueness of the L^p-solution). *Let Ω be a δ_λ-contractible domain of $\mathbb{R}^N$. Let $p \in [1, \infty]$. For every $\varphi \in L^p(\partial\Omega, \mu)$, there exists at most one L^p-solution of* $(\mathrm{DP})_\varphi$.

Proof. Let $p \in [1, \infty]$. By linearity, the uniqueness will follow if we show that the L^p-solution u to $(\mathrm{DP})_0$ (*i.e.*, with vanishing boundary datum) is identically zero.

To this purpose, let u be a L^p-solution to $(\mathrm{DP})_0$. If $p = \infty$, the uniqueness directly follows from the maximum principle (see also Remark 5.5). Then we suppose $p < \infty$. From (5.1), we see that

$$\int_{\partial\Omega} |u(\delta_\lambda(\xi))|^p \, \mathrm{d}\mu_0(\xi) \longrightarrow 0 \quad \text{as } \lambda \to 1^-.$$

Let now $\rho \in]0, 1[$ be fixed. By the monotonicity theorem 4.4, for every λ such that $\rho < \lambda < 1$ we have

$$0 \le \int_{\partial\Omega} |u(\delta_\rho(\xi))|^p \, \mathrm{d}\mu_0(\xi) \le \int_{\partial\Omega} |u(\delta_\lambda(\xi))|^p \, \mathrm{d}\mu_0(\xi) \longrightarrow 0 \quad \text{as } \lambda \to 1^-.$$

This gives

$$\int_{\partial\Omega} |u(\delta_\rho(\xi))|^p \, \mathrm{d}\mu_0^\Omega(\xi) = 0 \quad \text{for every } 0 < \rho < 1. \tag{5.2}$$

This implies that u is everywhere null in Ω. Indeed, set for brevity

$$v(\xi) := |u(\delta_\rho(\xi))|^p.$$

Note that $v \in C(\partial\Omega, \mathbb{R})$. Then, for every fixed $x \in \Omega$ (by Theorem C-(ii) and Proposition 3.5) we have

$$\begin{aligned} H_v^\Omega(x) &= \int_{\partial\Omega} v(\xi) \, \mathrm{d}\mu_x^\Omega(\xi) = \int_{\partial\Omega} v(\xi) \, h_x(\xi) \, \mathrm{d}\mu_0^\Omega(\xi) \\ &\le \mathbf{c}(\{x\}) \int_{\partial\Omega} v(\xi) \, \mathrm{d}\mu_0^\Omega(\xi) = 0 \ \text{ (by (5.2))}. \end{aligned} \tag{5.3}$$

On the other hand $H_v^\Omega \ge 0$, since $v \ge 0$ on $\partial\Omega$. This gives

$$H_v^\Omega \equiv 0 \ \text{ in } \Omega. \tag{5.4}$$

Let $P \subset \partial\Omega$ be the set of the $\mathcal{L}$-irregular points of $\partial\Omega$. We recall that (Theorem B) P is a $\mathcal{L}$-polar set and

$$H_v^\Omega(x) \longrightarrow v(\xi) \quad \text{as } \Omega \ni x \to \xi$$

for every $\xi \in \partial\Omega \setminus P$. This fact, jointly with (5.4), gives $v = 0$ in $\partial\Omega \setminus P$. Hence, by the very definition of v,

$$u(x) = 0 \text{ for every } x \in \delta_\rho(\partial\Omega \setminus P) = \delta_\rho(\partial\Omega) \setminus \delta_\rho(P), \text{ for every } 0 < \rho < 1. \tag{5.5}$$

Since $\Omega \setminus \{0\} = \cup_{0<\rho<1} \delta_\rho(\partial\Omega)$ (see Proposition 4.6-(iii)) (5.5) implies that

$$u(x) = 0 \ \text{ for every } x \in \Omega \setminus Z, x \ne 0, \tag{5.6}$$

where $Z := \cup_{0<\rho<1} \delta_\rho(P)$. Proposition 4.5 ensures that Z has empty interior, so that (5.6) and the continuity of u on Ω imply that $u \equiv 0$ on Ω. □

Our next task is to establish the following existence result for $(\mathrm{DP})_\varphi$:

Theorem 5.3 (Existence of the L^p-solution). *Let Ω be a δ_λ-contractible domain of $\mathbb{R}^N$. Let $p \in [1, \infty[$. For every $\varphi \in L^p(\partial\Omega, \mu)$, the Perron-Wiener-Brelot function H_φ^Ω is a L^p-solution of* $(\mathrm{DP})_\varphi$.

The hypothesis $p < \infty$ cannot be removed in Theorem 5.3 (neither in Lemma 5.4 below). To see this, see Remark 5.5 below. As for the existence theorem, we need some preliminary results. The following lemma is a version of Theorem 5.3 for continuous boundary data.

Lemma 5.4. *Let Ω be a δ_λ-contractible domain of $\mathbb{R}^N$, $p \in [1, \infty[$ and $\varphi \in C(\partial\Omega, \mathbb{R})$ be fixed. Then, the Perron-Wiener-Brelot function H_φ^Ω is a L^p-solution of* $(\mathrm{DP})_\varphi$.

Proof. We know that H_φ^Ω is $\mathcal{L}$-harmonic in Ω. According to Definition 5.1, we are left with showing that (DP2) holds, *i.e.*,

$$H_\varphi^\Omega \circ \delta_\lambda \longrightarrow \varphi \quad \text{in } L^p(\partial\Omega, \mu), \quad \text{as } \lambda \to 1^-.$$

This follows by dominated convergence in

$$\lim_{\lambda \to 1^-} \int_{\partial\Omega} |H_\varphi^\Omega(\delta_\lambda(\xi)) - \varphi(\xi)|^p \,\mathrm{d}\mu_0^\Omega(\xi) = 0.$$

Indeed, if $M = \max_{\partial\Omega} |\varphi|$, we have $\sup_\Omega |H_\varphi^\Omega| \le M$ by the maximum principle. Moreover,

$$\lim_{\lambda \to 1^-} |H_\varphi^\Omega(\delta_\lambda(\xi)) - \varphi(\xi)| = 0 \text{ for } \mu_0^\Omega\text{-almost every } \xi \in \partial\Omega.$$

This follows since for every $\xi \in \Omega$ we have $\Omega \ni \delta_\lambda(\xi) \longrightarrow \xi$ as $\lambda \to 1^-$ (recall that Ω is δ_λ-contractible) and (see Theorem B)

$$H_\varphi^\Omega(x) \longrightarrow \varphi(\xi) \quad \text{as } \Omega \ni x \to \xi$$

for every $\xi \in \partial\Omega \setminus P$, where P is the $\mathcal{L}$-polar subset of the $\mathcal{L}$-irregular points of $\partial\Omega$ (so that $\mu_0^\Omega(P) = 0$ by Theorem C-(i)). This ends the proof. □

Remark 5.5. The hypothesis $p < \infty$ in Lemma 5.4 cannot be removed (whence the same is true for Theorem 5.3). Indeed, a necessary condition for the above Lemma to hold when $p = \infty$ is that Ω is a $\mathcal{L}$-regular set (which may not be true for an arbitrary δ_λ-contractible domain).

Indeed, given $\varphi \in C(\partial\Omega, \mathbb{R})$, if H_φ^Ω is a L^∞-solution of $(\mathrm{DP})_\varphi$, then

$$\lim_{\Omega \ni x \to y} H_\varphi^\Omega(x) = \varphi(y), \quad \text{for every } y \in \partial\Omega. \tag{5.7}$$

To prove this statement, let $\varepsilon > 0$ and $y \in \partial\Omega$ be fixed. By the continuity of φ, there exists $r_\varepsilon > 0$ such that

$$|\varphi(z) - \varphi(y)| < \varepsilon/2, \quad \text{for every } z \in \partial\Omega \cap B_d(y, r_\varepsilon). \tag{5.8}$$

Moreover, by the definition of L^∞-solution (see (5.1)), there exists $\lambda_\varepsilon \in]0,1[$ with

$$\sup_{z\in\partial\Omega} |H^\Omega_\varphi(\delta_\lambda(z)) - \varphi(z)| < \varepsilon/2, \quad \text{for every } \lambda \in]\lambda_\varepsilon, 1[. \tag{5.9}$$

Let

$$A_\varepsilon := \bigcup_{\lambda>\lambda_\varepsilon} \delta_\lambda(\partial\Omega \cap B_d(y, r_\varepsilon)).$$

It can be proved that A_ε is a neighborhood of y. We now claim that $|H^\Omega_\varphi(x) - \varphi(y)| < \varepsilon$ for every $x \in A_\varepsilon \cap \Omega$, so that (5.7) follows. This claim comes from the following argument. If $x \in A_\varepsilon$, then there exist $\lambda > \lambda_\varepsilon$ and $z \in \partial\Omega \cap B_d(y, r_\varepsilon)$ such that $x = \delta_\lambda(z)$. Being $x \in \Omega$, it must be $\lambda < 1$ (see Proposition 4.6) so that

$$|H^\Omega_\varphi(x) - \varphi(y)| = |H^\Omega_\varphi(\delta_\lambda(z)) - \varphi(y)| \le |H^\Omega_\varphi(\delta_\lambda(z)) - \varphi(z)| + |\varphi(z) - \varphi(y)| < \varepsilon.$$

Here we used (5.8) and (5.9).

Next proposition plays the key rôle in the proof of the Existence Theorem.

Proposition 5.6. *Let Ω be a δ_λ-contractible domain of $\mathbb{R}^N$. Let $p \in [1, \infty]$. Let $\varphi, \varphi_n \in L^p(\partial\Omega, \mu)$, for every $n \in \mathbb{N}$, and suppose that $\varphi_n \to \varphi$ (as $n \to \infty$) in $L^p(\partial\Omega, \mu_0)$. Then, $H^\Omega_{\varphi_n}$ converges to H^Ω_φ uniformly on the compact subsets of Ω.*

Proof. Let $K \subset \Omega$ be a compact set. For brevity, we write μ_x instead of μ^Ω_x. First we suppose $p < \infty$. We have (see Theorem C-(ii) and recall that $\mu_x(\partial\Omega) = 1$ for every $x \in \Omega$)

$$\begin{aligned}
\sup_{x\in K} |H^\Omega_{\varphi_n}(x) - H^\Omega_\varphi(x)| &= \sup_{x\in K} \left| \int_{\partial\Omega} \varphi_n(\xi)\, d\mu_x(\xi) - \int_{\partial\Omega} \varphi(\xi)\, d\mu_x(\xi) \right| \\
&\le \sup_{x\in K} \int_{\partial\Omega} |\varphi_n(\xi) - \varphi(\xi)|\, d\mu_x(\xi) \\
&\le \sup_{x\in K} \left(\int_{\partial\Omega} |\varphi_n(\xi) - \varphi(\xi)|^p d\mu_x(\xi) \right)^{1/p} \cdot \big(\mu_x(\partial\Omega)\big)^{(p-1)/p} \\
\text{(see Proposition 3.5)} &= \sup_{x\in K} \left(\int_{\partial\Omega} |\varphi_n(\xi) - \varphi(\xi)|^p\, h_x(\xi)\, d\mu_0(\xi) \right)^{1/p} \\
\text{(see the estimate (3.4))} &\le \left(\mathbf{c}(K) \int_{\partial\Omega} |\varphi_n(\xi) - \varphi(\xi)|^p\, d\mu_0(\xi) \right)^{1/p} \\
&= \mathbf{c}(K)^{1/p} \; \| \varphi_n - \varphi;\, L^p(\partial\Omega, \mu) \| \longrightarrow 0, \quad \text{as } n \to \infty.
\end{aligned}$$

As for the case $p = \infty$, we have analogously

$$\begin{aligned}
\sup_{x\in K} |H^\Omega_{\varphi_n}(x) - H^\Omega_\varphi(x)| &\le \sup_{x\in K} \int_{\partial\Omega} |\varphi_n(\xi) - \varphi(\xi)|\, h_x(\xi)\, d\mu_0(\xi) \\
&\le \mathbf{c}(K)\mu^\Omega_0(\partial\Omega) \cdot \| \varphi_n - \varphi;\, L^\infty(\partial\Omega, \mu) \| \longrightarrow 0, \ \text{as } n \to \infty.
\end{aligned}$$

This ends the proof. □

Finally, we have the following crucial estimate.

Proposition 5.7. *Let Ω be a δ_λ-contractible domain of $\mathbb{R}^N$. Let $\varphi \in L^p(\partial\Omega, \mu)$ be fixed. Then, if $p \in [1, \infty[$,*

$$\int_{\partial\Omega} |H_\varphi^\Omega \circ \delta_\lambda|^p \, \mathrm{d}\mu_0^\Omega \le \int_{\partial\Omega} |\varphi|^p \, \mathrm{d}\mu_0^\Omega, \quad \textit{for all } 0 < \lambda < 1, \tag{5.10}$$

As a consequence, for every $p \in [1, \infty]$ and every $\varphi \in L^p(\partial\Omega, \mu)$, we have

$$\| \; H_\varphi^\Omega \circ \delta_\lambda; L^p(\partial\Omega, \mu_0^\Omega) \; \| \le \| \; \varphi; L^p(\partial\Omega, \mu_0^\Omega) \; \|, \quad \textit{for all } 0 < \lambda < 1. \tag{5.11}$$

By Lemma 4.3, (5.11) can be rewritten as

$$\| \; H_\varphi^\Omega; L^p(\partial\Omega_\lambda, \mu_0^{\Omega_\lambda}) \; \| \le \| \; \varphi; L^p(\partial\Omega, \mu_0^\Omega) \; \|, \quad \text{for all } 0 < \lambda < 1.$$

Proof. It is enough to prove (5.10) when $p < \infty$. Letting $p \to \infty$ in (5.10), one then obtains (5.11) for $p = \infty$.

Let $1 \le p < \infty$. Let $\varphi_n \in C(\partial\Omega, \mathbb{R})$ be such that $\varphi_n \to \varphi$ (as $n \to \infty$) in $L^p(\partial\Omega, \mu)$. Taking $u = H_{\varphi_n}^\Omega$ in the monotonicity theorem 4.4, it follows that

$$]0, 1[\ni \lambda \mapsto \int_{\partial\Omega} |H_{\varphi_n}^\Omega(\delta_\lambda(\xi))|^p \, \mathrm{d}\mu(\xi)$$

is monotone increasing. In particular, for every $0 < \rho < \lambda < 1$ we have

$$\int_{\partial\Omega} |H_{\varphi_n}^\Omega(\delta_\rho(\xi))|^p \, \mathrm{d}\mu(\xi) \le \int_{\partial\Omega} |H_{\varphi_n}^\Omega(\delta_\lambda(\xi))|^p \, \mathrm{d}\mu(\xi).$$

Now, the right-hand side of this inequality tends to $\int_{\partial\Omega} |\varphi_n|^p \, \mathrm{d}\mu$ as $\lambda \to 1^-$, for $H_{\varphi_n}^\Omega$ is a L^p-solution of $(\mathrm{DP})_{\varphi_n}$ (use Lemma 5.4 and the very definition of L^p-solution). We thus obtain

$$\int_{\partial\Omega} |H_{\varphi_n}^\Omega(\delta_\rho(\xi))|^p \, \mathrm{d}\mu(\xi) \le \int_{\partial\Omega} |\varphi_n|^p \, \mathrm{d}\mu, \quad \text{for every } 0 < \rho < 1. \tag{5.12}$$

Moreover, as $n \to \infty$, the left-hand side of (5.12) tends to

$$\int_{\partial\Omega} |H_\varphi^\Omega(\delta_\rho(\xi))|^2 \, \mathrm{d}\mu(\xi),$$

since $H_{\varphi_n}^\Omega$ converges uniformly to H_φ^Ω on the compact subsets of Ω (see Lemma 5.6; note that here $\rho \in]0, 1[$ is fixed and that $\delta_\rho(\partial\Omega)$ is a compact subset of the δ_λ-contractible domain Ω). Finally, as $n \to \infty$, the right-hand side of (5.12) obviously tends to $\int_{\partial\Omega} |\varphi|^p \, \mathrm{d}\mu$. This proves (5.10), ending the proof. □

We are now in the position to give the

Proof of Theorem 5.3. Since H_φ^Ω is $\mathcal{L}$-harmonic in Ω, we only have to show that (DP2) in Definition 5.1 holds, *i.e.*,

$$H_\varphi^\Omega \circ \delta_\lambda \longrightarrow \varphi \quad \text{in } L^p(\partial\Omega, \mu), \quad \text{as } \lambda \to 1^-. \tag{5.13}$$

Since $p < \infty$, we can fix a sequence $\varphi_n \in C(\partial\Omega, \mathbb{R})$ such that $\lim_{n\to\infty} \varphi_n = \varphi$ in $L^p(\partial\Omega, \mu)$. Let us consider the following family of functions in $L^p(\partial\Omega, \mu)$

$$H_{\varphi_n}^\Omega \circ \delta_\lambda, \qquad \lambda \in]0, 1[, n \in \mathbb{N}.$$

We claim that the following facts hold:

1) *uniformly* with respect to $\lambda \in]0, 1[$

$$\lim_{n\to\infty} H_{\varphi_n}^\Omega \circ \delta_\lambda = H_\varphi^\Omega \circ \delta_\lambda \quad \text{in } L^p(\partial\Omega, \mu);$$

2) for every fixed $n \in \mathbb{N}$,

$$\lim_{\lambda\to 1^-} H_{\varphi_n}^\Omega \circ \delta_\lambda = \varphi_n \quad \text{in } L^p(\partial\Omega, \mu).$$

These results allow to inter-change the two limits with respect to n and λ, so that (5.13) follows. Indeed

$$\begin{aligned} \lim_{\lambda\to 1^-} H_\varphi^\Omega \circ \delta_\lambda &= \lim_{\lambda\to 1^-} \lim_{n\to\infty} H_{\varphi_n}^\Omega \circ \delta_\lambda \\ &= \lim_{n\to\infty} \lim_{\lambda\to 1^-} H_{\varphi_n}^\Omega \circ \delta_\lambda = \lim_{n\to\infty} \varphi_n = \varphi, \quad \text{in } L^p(\partial\Omega, \mu). \end{aligned}$$

The theorem is proved if we demonstrate the claimed (1) and (2). First of all, (2) follows from Lemma 5.4 and the very definition of L^p-solution. We next prove (1). By (5.10) in Proposition 5.7, noticing that $H_\varphi^\Omega - H_{\varphi_n}^\Omega = H_{\varphi-\varphi_n}^\Omega$,

$$\begin{aligned} \| H_\varphi^\Omega \circ \delta_\lambda - H_{\varphi_n}^\Omega \circ \delta_\lambda \|_{L^p(\partial\Omega,\mu)}^p &= \int_{\partial\Omega} |H_{\varphi-\varphi_n}^\Omega(\delta_\lambda(\xi))|^p \, d\mu_0^\Omega(\xi) \\ &\le \int_{\partial\Omega} |\varphi - \varphi_n|^p(\xi) \, d\mu_0^\Omega(\xi) \end{aligned}$$

so that

$$\| H_\varphi^\Omega \circ \delta_\lambda - H_{\varphi_n}^\Omega \circ \delta_\lambda \|_{L^p(\partial\Omega,\mu)}^p \le \| \varphi - \varphi_n \|_{L^p(\partial\Omega,\mu)}^p, \quad \text{for every } 0 < \lambda < 1$$

and the claimed (1) straightforwardly follows. This ends the proof. □

Collecting together Theorems 5.2 and 5.3, we obtain the following result:

Theorem 5.8. *Let $\Omega \subset \mathbb{R}^N$ be a δ_λ-contractible domain. Let also $p \in [1, \infty[$. Then, the problem* $(\mathrm{DP})_\varphi$ *has one and only one L^p-solution, for every boundary datum $\varphi \in L^p(\partial\Omega, \mu)$. This solution is H_φ^Ω, the Perron-Wiener-Brelot solution to the Dirichlet problem related to Ω and φ.*

From Theorem 5.8, we readily obtain a *L^p-maximum principle*, new in the context of Carnot groups. We first fix the following definition: Let $u : \Omega \to \mathbb{R}$ be a continuous function and let $1 \leq p < \infty$. We say that

$$u|_{\partial\Omega} \geq 0 \quad \text{in } L^p,$$

if u has a nonnegative L^p-trace on $\partial\Omega$, *i.e.*, there exists a nonnegative function φ such that $u \circ \delta_\lambda \to \varphi$ in $L^p(\partial\Omega, \mu)$ as $\lambda \to 1^-$. Then, we have the following:

Corollary 5.9 (L^p-Maximum Principle). *Let Ω be a δ_λ-contractible domain and let $1 \leq p < \infty$. If u is $\mathcal{L}$-harmonic in Ω and $u|_{\partial\Omega} \geq 0$ in L^p, then $u \geq 0$ in Ω.*

Proof. Let $\varphi \in L^p(\partial\Omega, \mu)$ be such that such that $u \circ \delta_\lambda \to \varphi$ in $L^p(\partial\Omega, \mu)$ as $\lambda \to 1^-$. Then, by Theorem 5.8, $u = H_\varphi^\Omega$, which is nonnegative since $\varphi \geq 0$. □

6. Some applications

The aim of this section is to provide some applications of the results of the preceding sections. First, we shall give the notion of *$\mathcal{L}$-Hardy space*, proving related basic results. Then, we shall study the *non-homogeneous* Dirichlet problem related to (1.7).

6.1. $\mathcal{L}$-Hardy spaces on a homogeneous Carnot group

In this section, we define and study the main properties of $\mathcal{L}$-Hardy spaces on a homogeneous Carnot group. (For the theory in the classical case of Laplace's operator, see e.g., [2]. For stratified groups, see also [22].) We begin with the relevant definition:

Definition 6.1. Let $\Omega \subset \mathbb{R}^N$ be a δ_λ-contractible domain. Let also $p \in [1, \infty]$. For any smooth function u on Ω, we set

$$\| u \|_{h^p} := \sup_{0<\lambda<1} \| u \circ \delta_\lambda ; L^p(\partial\Omega, \mu_0^\Omega) \| .$$

We define the *$\mathcal{L}$-Hardy space* $h^p(\Omega, \mu_0^\Omega)$ as

$$h^p(\Omega, \mu_0^\Omega) := \{u \in C^\infty(\Omega) \mid \mathcal{L}u = 0 \ \text{ in } \Omega \text{ and } \ \| u \|_{h^p} < \infty \}.$$

In the sequel, we shall also write $h^p(\Omega)$ instead of $h^p(\Omega, \mu_0^\Omega)$.

In the rest of the section, Ω will always denote a δ_λ-contractible domain in $\mathbb{R}^N$.

Remark 6.2. With the notation of Definition 6.1, it is trivially seen that $\| \cdot \|_{h^p}$ defines a norm on $h^p(\Omega)$. After Theorem 6.5, it will be apparent that $\big(h^p(\Omega), \| \cdot \|_{h^p}\big)$ is a Banach space, whenever $p \in]1, \infty[$.

Remark 6.3. Let $1 \le p < \infty$ and let $\varphi \in L^p(\partial\Omega, \mu)$ be fixed. From Theorem 5.8 and Proposition 5.7, we infer that the L^p-solution to $(\mathrm{DP})_\varphi$ (say, u) belongs to $h^p(\Omega)$. The main task of this section is to prove that, viceversa, when $1 < p < \infty$, every function u in $h^p(\Omega)$ is the L^p-solution to $(\mathrm{DP})_\varphi$, for a suitable $\varphi \in L^p(\partial\Omega, \mu)$, *i.e.*, the following map

$$L^p(\partial\Omega, \mu) \ni \varphi \mapsto u \in h^p(\Omega)$$

(which we shall call *the solving map*) is onto. This is done in Theorem 6.5 below. We explicitly remark that, by the Existence Theorem 5.3, the solving map coincides with

$$L^p(\partial\Omega, \mu) \ni \varphi \mapsto H_\varphi^\Omega \in h^p(\Omega).$$

First we prove a useful fact on the h^p-norm.

Lemma 6.4. *Let $p \in [1, \infty[$ and $\varphi \in L^p(\Omega, \mu)$. Then, we have*

$$\| H_\varphi^\Omega \|_{h^p(\Omega)} = \| \varphi \|_{L^p(\partial\Omega,\mu)} . \tag{6.1}$$

Proof. From the estimate (5.11) in Proposition 5.7 and the very definition of h^p-norm, we infer that "$\le$" holds in (6.1). Viceversa, writing for brevity L^p instead of $L^p(\partial\Omega, \mu)$ and keeping in mind (5.11), we have

$$\begin{aligned} \| \varphi \|_{L^p} &\le \| \varphi - H_\varphi^\Omega \circ \delta_\lambda \|_{L^p} + \| H_\varphi^\Omega \circ \delta_\lambda \|_{L^p} \\ &\le \| \varphi - H_\varphi^\Omega \circ \delta_\lambda \|_{L^p} + \| H_\varphi^\Omega \|_{h^p}, \qquad \text{for every } 0 < \lambda < 1. \end{aligned} \tag{6.2}$$

Letting $\lambda \to 1^-$, from Theorem 5.3 (see (5.13)) we infer that the far right-hand side goes to $\| H_\varphi^\Omega \|_{h^p}$. Then, "$\ge$" holds in (6.1). □

We are now in the position to prove the main result of this section.

Theorem 6.5 (Surjectivity of the solving map). *Suppose $p \in]1, \infty[$. The map*

$$L^p(\partial\Omega, \mu) \ni \varphi \mapsto H_\varphi^\Omega \in h^p(\Omega)$$

is a linear bijective isometry.

Proof. The linearity of the map $\varphi \mapsto H_\varphi^\Omega$ is a well-known fact. Moreover, in Lemma 6.4 we have proved that such a map preserves the relevant norms, so that it is an isometry, whence (by linearity) it is also injective.

We are thus left with proving that the solving map is surjective. Let $u \in h^p(\Omega)$. By the very definition of h^p, we have $u \in \mathcal{H}(\Omega)$ and

$$\| u \|_{h^p} = \sup_{0<\lambda<1} \| u \circ \delta_\lambda ; L^p(\partial\Omega, \mu) \| < \infty,$$

i.e., the family $\{u \circ \delta_\lambda\}_{0<\lambda<1}$ is bounded in $L^p(\partial\Omega, \mu)$. As a consequence, there exists $\varphi \in L^p(\partial\Omega, \mu)$ and a sequence $\{\lambda_k\}_k$ in $]0, 1[$, $\lambda_k \to 1^-$ as $k \to \infty$, such that $u \circ \delta_{\lambda_k} \rightharpoonup \varphi$ in the weak* topology, *i.e.*, letting $1/p + 1/p' = 1$,

$$\int_{\partial\Omega} g(\xi)(u \circ \delta_{\lambda_k})(\xi)\, \mathrm{d}\mu_0(\xi) \overset{k\to\infty}{\longrightarrow} \int_{\partial\Omega} g(\xi)\, \varphi(\xi)\, \mathrm{d}\mu_0(\xi), \ \forall\ g \in L^{p'}(\partial\Omega, \mu_0). \quad (6.3)$$

We claim that, if φ is as above, then

$$u = H_\varphi^\Omega. \quad (6.4)$$

To see this, we use Proposition 3.5. With the notation therein, for every $x \in \Omega$, we know that there exists $h_x \in L^\infty(\partial\Omega, \mu_0)$ such that

$$\mathrm{d}\mu_x^\Omega(\xi) = h_x(\xi)\, \mathrm{d}\mu_0^\Omega(\xi). \quad (6.5)$$

Then, we can take $g(\xi) := h_x(\xi)$ in (6.3). This gives

$$\lim_{k\to\infty} \int_{\partial\Omega} (u \circ \delta_{\lambda_k})(\xi)\, h_x(\xi)\, \mathrm{d}\mu_0(\xi) = \int_{\partial\Omega} \varphi(\xi)\, h_x(\xi)\, \mathrm{d}\mu_0(\xi). \quad (6.6)$$

On the other hand, by (6.5) and Theorem C-(ii), we have

$$\int_{\partial\Omega} \varphi(\xi)\, h_x(\xi)\, \mathrm{d}\mu_0(\xi) = \int_{\partial\Omega} \varphi(\xi)\, \mathrm{d}\mu_x(\xi) = H_\varphi^\Omega(x).$$

Moreover

$$\int_{\partial\Omega} (u \circ \delta_{\lambda_k})(\xi)\, h_x(\xi)\, \mathrm{d}\mu_0^\Omega(\xi) = \int_{\partial\Omega} (u \circ \delta_{\lambda_k})(\xi)\, \mathrm{d}\mu_x^\Omega(\xi)$$
$$= H^\Omega_{u\circ\delta_{\lambda_k}}(x) = H_u^{\delta_{\lambda_k}(\Omega)}(\delta_{\lambda_k}(x)) = u(\delta_{\lambda_k}(x)) \longrightarrow u(x) \quad \text{as } k \to \infty.$$

Here we used (6.5) for the first equality, Theorem C-(ii) for the second equality, (4.2) for the third one. As for the fourth equality, we observe that

$$H_u^{\delta_{\lambda_k}(\Omega)} = u,$$

for u is $\mathcal{L}$-harmonic on $\Omega \supset \delta_{\lambda_k}(\Omega)$ and is continuous through $\partial\big(\delta_{\lambda_k}(\Omega)\big)$. Then, (6.4) holds and the proof is complete. □

6.2. Non-homogeneous Dirichlet problem

In this section, we use the results in Section 5, together with some estimates by Folland of the Γ-potential in order to prove the following theorem.

Theorem 6.6. *Let $\Omega \subset G$ be a δ_λ-contractible domain and let $1 < p < \infty$. For every $\varphi \in L^p(\partial\Omega, \mu)$ and $f \in L^q(\Omega, \mathrm{d}x)$, $q > Q/p$, the problem*

$$\begin{cases} \mathcal{L}u = f & \text{in } \Omega, \\ u|_{\partial\Omega} = \varphi & \text{in } L^p \end{cases} \tag{6.7}$$

has a unique solution $u \in S^{2,q}_{\mathrm{loc}}(\Omega)$, where

$$S^{2,q}_{\mathrm{loc}}(\Omega) := \{u \in L^q_{\mathrm{loc}}(\Omega) : \; X_i X_j u \in L^q_{\mathrm{loc}}(\Omega) \;\; \text{for } i, j = 1, \dots, m\}.$$

Proof. The uniqueness of u follows from the one related to the homogeneous equation $\mathcal{L}u = 0$. To prove the existence, we first recall some results from [21]. Let Γ be the fundamental solution of $\mathcal{L}$ in (2.5). Given a function $f \in L^q(G, \mathrm{d}x)$ with $q > Q/p$, the convolution

$$v(x) := (\Gamma * f)(x) := \int_G \Gamma(y^{-1} \circ x)\, f(y)\, \mathrm{d}y, \quad x \in G \tag{6.8}$$

is well defined and locally Hölder-continuous on G. Moreover, for every $i, j = 1, \dots, m$, $X_i X_j u$ exists in the weak sense of distributions and belongs to $L^q(G, \mathrm{d}x)$. Furthermore,

$$\mathcal{L}v = -f \quad \text{in } G, \tag{6.9}$$

both in the distributional sense and point-wise almost everywhere. Then, since Ω is bounded, given $f \in L^q(\Omega, \mathrm{d}x)$, if we agree to continue f to all G by setting $f = 0$ in $G \setminus \Omega$, the function v in (6.9) is locally Hölder-continuous, belongs to $S^{2,q}(\Omega)$ and solves the equation

$$\mathcal{L}v = -f \quad \text{in } \Omega$$

point-wise almost everywhere and in $\mathcal{D}'(\Omega)$. We denote by $S^{2,q}(\Omega)$ the space

$$S^{2,q}(\Omega) := \{u \in L^q(\Omega) : \; X_i X_j u \in L^q(\Omega) \;\; \text{for } i, j = 1, \dots, m\}.$$

Let us now consider the solution w to the boundary value problem

$$\begin{cases} \mathcal{L}w = 0 & \text{in } \Omega, \\ w|_{\partial\Omega} = \varphi - v|_{\partial\Omega} & \text{in } L^p. \end{cases}$$

We know, by Theorem 5.8, that this solution exists. Then, we get the conclusion of the theorem by setting $u := w - v$. □

7. Appendix: geometric properties of δ_λ-contractible domains

The aim of this Appendix is twofold: First, we sketch a proof of Proposition 3.1, collecting the needed properties of the $\mathcal{L}$-Green's function. Second, we provide some geometric properties of δ_λ-contractible domains. In particular, we give the proof of Proposition 4.6, which collects the basic geometric properties of δ_λ-contractible domains that we used throughout the paper.

Proof of Proposition 3.1. We only sketch the proof and give references for it.

(i): The main tool in studying the $\mathcal{L}$-Green's function is the so-called *Perron's regularization* u_B: Let $u \in \overline{\mathcal{S}}(\Omega)$ and let B be a d-ball with $\overline{B} \subset \Omega$. For any $x \in \Omega$, we define $u_B(x)$ as $\int_{\partial B} u \, d\mu_x^B$ if $x \in B$, and $u_B = u$ elsewhere. Then, we have $u_B \in \overline{\mathcal{S}}(\Omega)$, $u \geq u_B$ and u_B is $\mathcal{L}$-harmonic on B. A set $\mathcal{F} \subseteq \overline{\mathcal{S}}(\Omega)$ is called a *Perron-family* if $u, v \in \mathcal{F}$ implies $\min\{u, v\} \in \mathcal{F}$ and $u_B \in \mathcal{F}$, for every d-ball B compactly contained in Ω. If $\mathcal{F}$ is a Perron-family, then $\inf_{u \in \mathcal{F}} u$ is $\mathcal{L}$-harmonic or $-\infty$ on any component of Ω.

Let $\Omega \subseteq \mathbb{R}^N$ be open and let $x \in \Omega$ be fixed. Let us denote by h_x the greatest $\mathcal{L}$-harmonic minorant of $\Gamma(x^{-1} \circ \cdot)$ (note that $\mathcal{F} = \{h \in \mathcal{H}(\Omega),\ h \leq \Gamma(x^{-1} \circ \cdot)\}$ is a Perron-family). We set $G_\Omega(x, y) = \Gamma(x^{-1} \circ y) - h_x(y)$. The symmetry of G_Ω follows by studying the sequence of iterated Perron's regularized $\{G_j\}_j$, where $G_1 = (G_\Omega(x, \cdot))_{B_1}$, $G_j = (G_{j-1})_{B_j}$ where $\{B_j\}_j$ is a sequence of d-balls properly chosen (follow the arguments in [27, Theorem 5.24]). The continuity of G_Ω on $\Omega \times \Omega$ is not immediate and it may be seen as a consequence of Harnack's inequality (by retracing the arguments in [27, Theorem 5.26]; for Harnack-type inequalities see, e.g., [6, Theorem 1.1]). The fact that the greatest $\mathcal{L}$-harmonic minorant of $G(x, \cdot)$ is the null function follows from $\inf_{y \in \Omega} G_\Omega(x, y) = 0$.

(ii): We now consider the family $\mathcal{F} := \{w \in \overline{\mathcal{S}}(\Omega) :\ w \geq -\Gamma(x^{-1} \circ \cdot)\}$. Then we can see that $\mathcal{F}$ is a Perron-family. Hence, if w denotes the inf over $\mathcal{F}$, it is easily proved that $-w$ coincides with h_x.

(iii): This follows from (ii), from definition (2.6) and the maximum principle.

(iv): An application of the maximum principle for $\mathcal{L}$ and the symmetry of G_Ω.

(v): G_Ω as defined in (i) fulfills the requisites of the definition in (v) of Proposition 3.1 (to prove this, we need to use similar arguments to those in the above proof of (ii)). Viceversa, if we define G_Ω according to the definition in (v), it can be proved that $G_\Omega(x, y) = \Gamma(x^{-1} \circ y) - h_x(y)$ by retracing the arguments of Perron-regularization as in (i) above.

(vi): This follows from the fact that the greatest $\mathcal{L}$-harmonic minorant of $G(x, \cdot)$ is the null function and by arguing as in [1, Theorem 4.2.6]; the argument is not immediate and it requires once again the use of Harnack's inequality. □

Proof of Proposition 4.6. (i): Let $x \in \Omega \setminus \{0\}$ be fixed. Set $D := \{\delta_\lambda(x) : \lambda > 0\}$. Then D is connected, for $D = \Psi(]0, \infty[)$, where $\Psi(\lambda) = \delta_\lambda(x)$ is continuous with respect to λ (see (2.2)). First we note that $D \cap \partial\Omega \neq \emptyset$. Otherwise, we would have $D = (D \cap \Omega) \cup (D \cap \mathrm{Est}(\Omega)) =: D_1 \cup D_2$, with D_1, D_2 nonempty, disjoint open sets in D. (Indeed, $x \in D_1$ and $\delta_\rho(x) \in D_2$, for a suitable $\rho \gg 1$, being $\overline{\Omega}$ bounded).

This would mean that D is disconnected, which is not.

Moreover, we show that $D \cap \partial\Omega$ cannot contain two distinct points. Indeed, suppose by contradiction there exist $0 < \mu < \lambda$ with $\delta_\lambda(x), \delta_\mu(x) \in \partial\Omega$. This gives

$$\partial\Omega \ni \delta_\mu(x) = \delta_{\mu/\lambda}\big(\delta_\lambda(x)\big) \in \Omega$$

which is a contradiction (here we used the fact that $\delta_\lambda(x) \in \partial\Omega$, $0 < \mu/\lambda < 1$ and hypothesis (4.1)). This implies that $D \cap \partial\Omega$ is a singleton. The same proof works if we replace D by $\{\delta_\lambda(x) : \lambda \geq 1\}$ so that (i) is proved.

(ii): From (i) we infer that for every $x \in \Omega$, it holds

$$\{\delta_\lambda(x) : 0 \leq \lambda \leq 1\} \cap \partial\Omega = \emptyset. \tag{7.1}$$

Let us prove $\delta_\lambda(\Omega) \subseteq \Omega$ for every $0 < \lambda < 1$. Suppose by contradiction there exist $x \in \Omega$ and $\lambda_0 \in]0, 1[$ such that $\delta_{\lambda_0}(x) \notin \Omega$. Since $\delta_{\lambda_0}(x) \in \partial\Omega$ cannot hold (by (7.1)), this would give $\delta_{\lambda_0}(x) \in \mathrm{Est}(\Omega)$. As a consequence, set $D := \{\delta_\lambda(x) : 0 \leq \lambda < 1\}$, we have $D = (D \cap \Omega) \cup (D \cap \mathrm{Est}(\Omega)) =: D_1 \cup D_2$, with D_1, D_2 nonempty, disjoint open sets in D. (Indeed, $0 \in D_1$ and $\delta_{\lambda_0}(x) \in D_2$.) This would mean that D is disconnected, which is not. Finally, $\delta_\lambda(\overline{\Omega}) \subseteq \Omega$ follows from $\delta_\lambda(\Omega) \subseteq \Omega$ and hypothesis (4.1).

(iii): According to (ii), we need only show that $\Omega \subseteq \bigcup_{0 \leq \lambda \leq 1} \delta_\lambda(\partial\Omega)$. Suppose by contradiction there exists $x_0 \in \Omega$ such that

$$x_0 \notin \delta_\lambda(\partial\Omega) \quad \text{for any } \lambda \in [0, 1[. \tag{7.2}$$

Clearly, $x_0 \neq 0$. Set $D := \{\delta_\lambda(x_0) : \lambda \geq 1\}$. We have $D \cap \partial\Omega = \emptyset$. (Indeed, if there existed $\xi = \delta_\rho(x_0) \in D \cap \partial\Omega$ for some $\rho > 1$, this would give $x_0 = \delta_{1/\rho}(\xi) \in \delta_{1/\rho}(\partial\Omega)$, contradicting (7.2), since $1/\rho \in]0, 1[$.) Consequently, $D = (D \cap \Omega) \cup (D \cap \mathrm{Est}(\Omega)) =: D_1 \cup D_2$, with D_1, D_2 nonempty, disjoint open sets in D. (Indeed, $x_0 \in D_1$ and $\delta_\lambda(x_0) \in D_2$, for some $\lambda \gg 1$.) This means that D is disconnected, which is not.

(iv): We prove that $\Omega = \Omega'$, where Ω' is the connected component of Ω containing 0. Let Ω'' be a connected component different from Ω'. Suppose by contradiction $x \in \Omega''$. Then x can be joined to 0 by the continuous path $D := \{\delta_\lambda(x) : 0 \leq \lambda \leq 1\}$, which (from (ii)) is completely contained in Ω. Hence x belongs to Ω'. This gives a contradiction, whence $\Omega'' = \emptyset$.

(v): We may suppose $0 \leq \lambda_1 < \lambda_2 \leq 1$. Let $x \in \delta_{\lambda_1}(\Omega)$, *i.e.*, $x = \delta_{\lambda_1}(\omega)$ (for some $\omega \in \Omega$) so that $x = \delta_{\lambda_1}(\omega) = \delta_{\lambda_2}(\delta_{\lambda_1/\lambda_2}(\omega)) \in \delta_{\lambda_2}(\Omega)$, for $\lambda_1/\lambda_2 < 1$ and (ii) holds. This ends the proof. $\square$

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