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The BV-energy of maps into a manifold: relaxation and density results

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Abstract. Let $\mathcal{Y}$ be a smooth compact oriented Riemannian manifold without boundary, and assume that its 1-homology group has no torsion. Weak limits of graphs of smooth maps $u_k : B^n \to \mathcal{Y}$ with equibounded total variation give rise to equivalence classes of Cartesian currents in $\mathrm{cart}^{1,1}(B^n \times \mathcal{Y})$ for which we introduce a natural BV-energy. Assume moreover that the first homotopy group of $\mathcal{Y}$ is commutative. In any dimension n we prove that every element T in $\mathrm{cart}^{1,1}(B^n \times \mathcal{Y})$ can be approximated weakly in the sense of currents by a sequence of graphs of smooth maps $u_k : B^n \to \mathcal{Y}$ with total variation converging to the BV-energy of T. As a consequence, we characterize the lower semicontinuous envelope of functions of bounded variations from B^n into $\mathcal{Y}$.

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In this paper we deal with sequences of smooth maps $u_k : B^n \to \mathcal{Y}$ with equibounded total variation

$$\sup_k \mathcal{E}_{1,1}(u_k) < \infty\,, \qquad \mathcal{E}_{1,1}(u_k) := \int_{B^n} |Du_k|\, dx$$

and their limit points. Here B^n is the unit ball in $\mathbb{R}^n$ and $\mathcal{Y}$ is a smooth oriented Riemannian manifold of dimension $M \geq 1$, isometrically embedded in $\mathbb{R}^N$ for some $N \geq 2$. We shall assume that $\mathcal{Y}$ is compact, connected, without boundary. In addition, we assume that the integral 1-homology group $H_1(\mathcal{Y}) := H_1(\mathcal{Y};\mathbb{Z})$ has no torsion.

Modulo passing to a subsequence the $(n,1)$-currents G_{u_k}, integration over the graphs of u_k of n-forms with at most one vertical differential, converge to a current $T \in \mathrm{cart}^{1,1}(B^n \times \mathcal{Y})$, see Section 2 below. To every $T \in \mathrm{cart}^{1,1}(B^n \times \mathcal{Y})$ it corresponds a function $u_T \in BV(B^n, \mathcal{Y})$, *i.e.*, $u_T \in BV(B^n, \mathbb{R}^N)$ such that $u_T(x) \in \mathcal{Y}$ for $\mathcal{L}^n$-a.e. $x \in B^n$, compare [14, Vol. I, Section 4.2] [14, Vol. II, Section 5.4]. Also, the weak convergence $G_{u_k} \rightharpoonup T$ yields the convergence $u_k \rightharpoonup u_T$ weakly in the BV-sense.

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In order to analyze the weak limit currents, it is relevant first to consider the case $n = 1$. Therefore in Section 1 we study some of the structure properties of 1-dimensional Cartesian currents in $B^1 \times \mathcal{Y}$, *i.e.*, of currents in $\mathrm{cart}(B^1 \times \mathbb{R}^N)$ with support $\mathrm{spt}\, T \subset \overline{B}^1 \times \mathcal{Y}$, compare [14, Vol. I]. In the simple case $\mathcal{Y} = S^1$, the unit circle in $\mathbb{R}^2$, and in any dimension n, for any current $T \in \mathrm{cart}(B^n \times S^1)$ we can find a sequence of smooth maps $\{u_k\} \subset C^1(B^n, S^1)$ such that G_{u_k} weakly converges to T and the area of the graph of the u_k's converges to the mass of T, *i.e.*, $\mathbf{M}(G_{u_k}) \to \mathbf{M}(T)$, see [13] and [14, Vol. II, Section 6.2.2]. However, in case of general target manifolds, and even in dimension $n = 1$, a *gap phenomenon* occurs. More precisely, setting

$$\widetilde{\mathbf{M}}(T) := \inf\left\{ \liminf_{k\to\infty} \mathbf{M}(G_{u_k}) \,|\, \{u_k\} \subset C^1(B^1, \mathcal{Y}),\ G_{u_k} \rightharpoonup T \text{ weakly in } \mathcal{D}_1(B^1 \times \mathcal{Y}) \right\},$$

there exist currents $T \in \mathrm{cart}(B^1 \times \mathcal{Y})$ for which

$$\mathbf{M}(T) < \widetilde{\mathbf{M}}(T)\,,$$

i.e., for every smooth sequence $\{u_k\} \subset C^1(B^1, \mathcal{Y})$ such that $G_{u_k} \rightharpoonup T$ weakly in $\mathcal{D}_1(B^1 \times \mathcal{Y})$ we have that

$$\liminf_{k\to\infty} \mathbf{M}(G_{u_k}) \geq \mathbf{M}(T) + C\,,$$

where $C > 0$ is an absolute constant and, we recall, the mass of G_{u_k} is the area of the graph of u_k

$$\mathbf{M}(G_{u_k}) = \mathcal{A}(u_k) := \int_{B^1} \sqrt{1 + |Du_k|^2}\, dx\,.$$

In order to deal with this gap phenomenon, we introduce the class $\mathrm{cart}^{1,1}(B^1 \times \mathcal{Y})$ of equivalence classes of currents in $\mathrm{cart}(B^1 \times \mathcal{Y})$, where the equivalence relation is given by

$$T \sim \widetilde{T} \iff T(\omega) = \widetilde{T}(\omega) \quad \forall\, \omega \in \mathcal{Z}^{1,1}(B^1 \times \mathcal{Y})\,,$$

see Definition 1.6. Here $\mathcal{Z}^{1,1}(B^1 \times \mathcal{Y})$ denotes the class of smooth forms $\omega \in \mathcal{D}^1(B^1 \times \mathcal{Y})$ such that $d_y\omega^{(1)} = 0$, where $d = d_x + d_y$ denotes the splitting into a horizontal and a vertical differential, and $\omega^{(1)}$ is the component of ω with exactly one vertical differential. In other words $\mathrm{cart}^{1,1}(B^1 \times \mathcal{Y})$ is the class of vertical homological representatives of the elements of $\mathrm{cart}(B^1 \times \mathcal{Y})$. Notice that if $\mathcal{Y} = S^1$, actually $\mathrm{cart}^{1,1}(B^1 \times S^1)$ agrees with the class $\mathrm{cart}(B^1 \times S^1)$. We then introduce on $\mathrm{cart}^{1,1}(B^1 \times \mathcal{Y})$ the following energy

$$\mathcal{A}(T) := \int_{B^1} \sqrt{1 + |\nabla u_T(x)|^2}\, dx + \left| D^C u_T \right| (B^1) + \sum_{x \in J_c(T)} \mathcal{L}_T(x)\,,$$

where ∇u_T and $D^C u_T$ are respectively the absolutely continuous and the Cantor part of the distributional derivative of the underlying function $u_T \in BV(B^1, \mathcal{Y})$, and the countable set $J_c(T)$ is the union

$$J_c(T) := J_{u_T} \cup \{x_i : i = 1, \dots, I\}$$

of the *discontinuity set* J_{u_T} of u_T and of the finite set of points x_i where the mass of T concentrates.

In the above formula, $\mathcal{L}_T(x)$ denotes the *minimal length* $\mathcal{L}(\gamma)$ among all Lipschitz curves $\gamma : [0, 1] \to \mathcal{Y}$, with end points equal to the one-sided approximate limits of u_T on $x \in J_c(T)$, such that their image current $\gamma_\# [\![(0, 1)]\!]$ is equal to the 1-*dimensional restriction* $\widehat{\pi}_\#(T \llcorner \{x\} \times \mathcal{Y})$ of T over the point x. In the case $\mathcal{Y} = S^1$, it turns out that $\mathcal{A}(T)$ agrees with the mass of T, compare [13] and [14, Vol. II, Section 6.2.2].

We will show that the functional $T \mapsto \mathcal{A}(T)$ is lower semicontinuous in $\mathrm{cart}^{1,1}(B^1 \times \mathcal{Y})$, Theorem 1.7, and that for every T there exists a sequence of smooth maps $\{u_k\} \subset C^1(B^1, \mathcal{Y})$ such that $G_{u_k} \rightharpoonup T$ and $\mathbf{M}(G_{u_k}) \to \mathcal{A}(T)$ as $k \to \infty$, Theorem 1.8. As a consequence, we conclude that $\mathcal{A}(T)$ coincides with the *relaxed area functional*

$$\widetilde{\mathcal{A}}(T) := \inf \left\{ \liminf_{k\to\infty} \mathcal{A}(u_k) \mid \{u_k\} \subset C^1(B^1, \mathcal{Y}), \quad G_{u_k} \rightharpoonup T \right\}.$$

In Section 2, we deal with the n-dimensional case, $n \geq 2$, introducing the class $\mathrm{cart}^{1,1}(B^n \times \mathcal{Y})$ of vertical homological representatives. The BV*-energy* of a current $T \in \mathrm{cart}^{1,1}(B^n \times \mathcal{Y})$ is then defined by

$$\mathcal{E}_{1,1}(T) := \int_{B^n} |\nabla u_T(x)|\, dx + |D^C u_T|(B^n) + \int_{J_c(T)} \mathcal{L}_T(x)\, d\mathcal{H}^{n-1}(x),$$

see Definition 2.10, where $J_c(T)$ is the countably $\mathcal{H}^{n-1}$-rectifiable subset of B^n given by the union of the *Jump set* J_{u_T} of u_T and of the $(n-1)$*-rectifiable set of mass-concentration* of T. Finally, the integrand $\mathcal{L}_T(x)$ is defined as above, by taking into account that the 1-dimensional restriction $\widehat{\pi}_\#(T \llcorner \{x\} \times \mathcal{Y})$ of T is well-defined for $\mathcal{H}^{n-1}$-a.e. point $x \in J_c(T)$.

Notice that, if $T = G_u$, where $u : B^n \to \mathcal{Y}$ is smooth or at least in $W^{1,1}$, then $\mathcal{E}_{1,1}(G_u) = \mathcal{E}_{1,1}(u)$. Moreover, in the case $\mathcal{Y} = S^1$, we have $\mathrm{cart}^{1,1}(B^n \times S^1) = \mathrm{cart}(B^n \times S^1)$ and, due to the absence of gap phenomenon, the functional $\mathcal{E}_{1,1}(T)$ agrees with the *parametric variational integral* associated to the total variation integral, see Definition 2.5, and can be dealt with as in [13], see also [14, Vol. II, Section 6.2], [8], [19]. The functional $T \mapsto \mathcal{E}_{1,1}(T)$ turns out to be lower semicontinuous in $\mathrm{cart}^{1,1}(B^n \times \mathcal{Y})$, see Theorem 2.12 and Section 3. Moreover, assuming in addition that the first homotopy group $\pi_1(\mathcal{Y})$ is commutative, in Section 4 and Section 5 we will prove in any dimension $n \geq 2$ that for every $T \in \mathrm{cart}^{1,1}(B^n \times \mathcal{Y})$ there exists a sequence of smooth maps $\{u_k\} \subset C^1(B^n, \mathcal{Y})$ such that $G_{u_k} \rightharpoonup T$

and $\mathcal{E}_{1,1}(u_k) \to \mathcal{E}_{1,1}(T)$ as $k \to \infty$, Theorem 2.13. Consequently, we show that a closure-compactness property holds in $\mathrm{cart}^{1,1}(B^n \times \mathcal{Y})$, Theorem 2.17. We stress that the commutativity hypothesis on $\pi_1(\mathcal{Y})$ cannot be removed, see Remark 5.2.

In Section 6, extending the classical notion of total variation of vector-valued maps, compare e.g. [1], we introduce in a natural way the *total variation* of functions $u \in BV(B^n, \mathcal{Y})$, given by

$$\mathcal{E}_{TV}(u) := \int_{B^n} |\nabla u(x)|\, dx + \left|D^C u\right|(B^n) + \int_{J_u} \mathcal{H}^1(l_x)\, d\mathcal{H}^{n-1}(x)\,,$$

where, for any $x \in J_u$, we let $\mathcal{H}^1(l_x)$ denote the length of a *geodesic arc* l_x in $\mathcal{Y}$ with initial and final points $u^-(x)$ and $u^+(x)$. Extending the density result of Bethuel [5], in Theorem 6.5 we will show that for every $u \in BV(B^n, \mathcal{Y})$ we can find a sequence of maps $\{u_k\} \subset R_1^\infty(B^n, \mathcal{Y})$ such that $u_k \rightharpoonup u$ as $k \to \infty$ weakly in the BV-sense and

$$\lim_{k\to\infty} \int_{B^n} |Du_k|\, dx = \mathcal{E}_{TV}(u)\,.$$

If $n = 1$, the class $R_1^\infty(B^n, \mathcal{Y})$ agrees with $C^1(B^n, \mathcal{Y})$. If $n \geq 2$, it is given by all the maps $u \in W^{1,1}(B^n, \mathcal{Y})$ which are smooth except on a singular set which is discrete, if $n = 2$, and is the finite union of smooth $(n-2)$-dimensional subsets of B^n with smooth boundary, if $n \geq 3$. Therefore, if $\pi_1(\mathcal{Y}) = 0$, we obtain that smooth maps in $C^1(B^n, \mathcal{Y})$ are dense in $BV(B^n, \mathcal{Y})$ in the strong sense above mentioned.

However, in Section 7 we will show that $\mathcal{E}_{TV}(u)$ does not agree with the *relaxed* of the total variation

$$\widetilde{\mathcal{E}_{TV}}(u) := \inf\left\{\liminf_{k\to\infty} \int_{B^n} |Du_k| dx \mid \{u_k\} \subset C^1(B^n, \mathcal{Y}),\ u_k \rightharpoonup u \text{ weakly in the } BV\text{-sense}\right\}$$

if $n \geq 2$, and we have $\widetilde{\mathcal{E}_{TV}}(u) < \infty$, Theorem 7.3, and that

$$\widetilde{\mathcal{E}_{TV}}(u) = \inf\{\mathcal{E}_{1,1}(T) \mid T \in \mathcal{T}_u\}\,,$$

Theorem 7.4, where $\mathcal{T}_u$ is the class of Cartesian currents T in $\mathrm{cart}^{1,1}(B^n \times \mathcal{Y})$ with underlying BV-function u_T equal to u, this way obtaining the representation formula

$$\widetilde{\mathcal{E}_{TV}}(u) = \int_{B^n} |\nabla u(x)|\, dx + \left|D^C u\right|(B^n) + \inf\left\{\int_{J_c(T)} \mathcal{L}_T(x) d\mathcal{H}^{n-1}(x) \mid T \in \mathcal{T}_u\right\}.$$

We finally specify the above relaxation results to $u \in W^{1,1}(B^n, \mathcal{Y})$ and/or $\mathcal{Y} = S^1$, recovering in particular previous results in [13, 8], and [19].

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1. Cartesian currents in dimension one

In this section we discuss some features of 1-dimensional *Cartesian currents* in $B^1 \times \mathcal{Y}$ and, in particular, we discuss a gap phenomenon and the relaxed area functional.

First let us introduce a few notation about BV-functions and Cartesian currents in the general context $B^n \times \mathcal{Y}$.

Vector valued *BV* -functions. Let $u : B^n \to \mathbb{R}^N$ be a function in $BV(B^n, \mathbb{R}^N)$, *i.e.*, $u = (u^1, \dots u^N)$ with all components $u^j \in BV(B^n)$. The *Jump set* of u is the countably $\mathcal{H}^{n-1}$-rectifiable set J_u in B^n given by the union of the complements of the Lebesgue sets of the u^j's. Let $\nu = \nu_u(x)$ be a unit vector in $\mathbb{R}^n$ orthogonal to J_u at $\mathcal{H}^{n-1}$-a.e. point $x \in J_u$. Let $u^\pm(x)$ denote the one-sided approximate limits of u on J_u, so that for $\mathcal{H}^{n-1}$-a.e. point $x \in J_u$

$$\lim_{\rho \to 0^+} \rho^{-n} \int_{B_\rho^\pm(x)} |u(x) - u^\pm(x)|\, dx = 0\,,$$

where $B_\rho^\pm(x) := \{y \in B_\rho(x) : \pm\langle y - x, \nu(x)\rangle \geq 0\}$. Note that a change of sign of ν induces a permutation of u^+ and u^- and that only for scalar functions there is a canonical choice of the sign of ν which ensures that $u^+(x) > u^-(x)$. The distributional derivative of u is the sum of a "gradient" measure, which is absolutely continuous with respect to the Lebesgue measure, of a "jump" measure, concentrated on a set that is σ-finite with respect to the $\mathcal{H}^{n-1}$-measure, and of a "Cantor-type" measure. More precisely,

$$Du = D^a u + D^J u + D^C u\,,$$

where

$$D^a u = \nabla u \cdot dx\,, \qquad D^J u = (u^+(x) - u^-(x)) \otimes \nu(x)\, \mathcal{H}^{n-1} \llcorner J_u\,,$$

$\nabla u := (\nabla_1 u, \dots, \nabla_n u)$ being the approximate gradient of u, compare e.g. [2] or [14, Vol. I]. We also recall that $\{u_k\}$ is said to converge to u *weakly in the BV-sense*, $u_k \rightharpoonup u$, if $u_k \to u$ strongly in $L^1(B^n, \mathbb{R}^N)$ and $Du_k \rightharpoonup Du$ weakly in the sense of (vector-valued) measures. We will finally denote

$$BV(B^n, \mathcal{Y}) := \{u \in BV(B^n, \mathbb{R}^N) \mid u(x) \in \mathcal{Y} \quad \text{for } \mathcal{L}^n\text{-a.e. } x \in B^n\}\,.$$

Cartesian currents. The class of Cartesian currents $\mathrm{cart}(B^n \times \mathbb{R}^N)$, compare [14, Vol. I], is defined as the class of integer multiplicity rectifiable currents T in $\mathcal{R}_n(B^n \times \mathbb{R}^N)$ which have no inner boundary, $\partial T \llcorner B^n \times \mathbb{R}^N = 0$, have finite mass, $\mathbf{M}(T) < \infty$, and are such that

$$\|T\|_1 < \infty\,, \quad \pi_\#(T) = [\![\, B^n \,]\!] \quad \text{and} \quad T^{\bar{0}0} \geq 0\,,$$

where

$$\|T\|_1 := \sup\{T(\varphi(x,y)|y|\,dx) \mid \varphi \in C^0_c(B^n \times \mathbb{R}^N) \text{ and } \|\varphi\| \le 1\}$$

and $T^{\bar{0}0}$ is the Radon measure in $B^n \times \mathbb{R}^N$ given by

$$T^{\bar{0}0}(\varphi(x,y)) = T(\varphi(x,y)\,dx) \qquad \forall\, \varphi \in C^0_c(B^n \times \mathbb{R}^N)\,.$$

Finally, here and in the sequel $\pi : \mathbb{R}^{n+N} \to \mathbb{R}^n$ and $\widehat{\pi} : \mathbb{R}^{n+N} \to \mathbb{R}^N$ denote the projections onto the first n and the last N coordinates, respectively.

It is shown in [14, Vol. I] that for every $T \in \text{cart}(B^n \times \mathbb{R}^N)$ there exists a function $u_T \in BV(B^n, \mathbb{R}^N)$ such that

$$T(\phi(x,y)\,dx) = \int_{B^n} \phi(x, u_T(x))\,dx \tag{1.1}$$

for all $\phi \in C^0(B^n \times \mathbb{R}^N)$ such that $|\phi(x,y)| \le C\,(1+|y|)$, and

$$(-1)^{n-i} T(\varphi(x)\widehat{dx^i} \wedge dy^j) = \langle D_i u^j_T, \varphi\rangle := -\int_{B^n} u^j_T(x) \cdot D_i\varphi(x)\,dx$$

for all $\varphi \in C^1_c(B^n)$, where

$$\widehat{dx^i} := dx^1 \wedge \cdots dx^{i-1} \wedge dx^{i-1} \wedge \cdots \wedge dx^n\,.$$

In particular, we have $\|T\|_1 = \|u_T\|_{L^1(B^n,\mathbb{R}^N)}$.

Definition 1.1. *If $n = 1$ we set*

$$\text{cart}(B^1 \times \mathcal{Y}) := \left\{T \in \text{cart}(B^1 \times \mathbb{R}^N) \mid \text{spt}\, T \subset \overline{B}^1 \times \mathcal{Y}\right\}\,.$$

Notice that the class $\text{cart}(B^1 \times \mathcal{Y})$ contains the weak limits of sequences of graphs of smooth maps $u_k : B^1 \to \mathcal{Y}$ with equibounded $W^{1,1}$-energies. Moreover, it is closed under weak convergence in $\mathcal{D}_1(B^1 \times \mathcal{Y})$ with equibounded masses. Finally, the BV-function u_T associated to currents T in $\text{cart}(B^1 \times \mathcal{Y})$ clearly belongs to $BV(B^1, \mathcal{Y})$.

Restriction over one point. Let $T \in \text{cart}(B^1 \times \mathcal{Y})$. Since T has finite mass, $\eta \mapsto T(\chi_{B_r(x)} \wedge \eta)$, where $x \in B^1$ and $0 < r < 1 - |x|$, defines a current in $\mathcal{D}_1(\mathcal{Y})$. The 1*-dimensional restriction of T over the point x*

$$\widehat{\pi}_\#\,(T \llcorner \{x\} \times \mathcal{Y}) \in \mathcal{D}_1(\mathcal{Y})$$

is the limit

$$\widehat{\pi}_\#(T \llcorner \{x\} \times \mathcal{Y})(\eta) := \lim_{r\to 0^+} T(\chi_{B_r(x)} \wedge \eta)\,, \qquad \eta \in \mathcal{D}^1(\mathcal{Y})\,.$$

Canonical decomposition. There is a canonical way to decompose a current $T \in \operatorname{cart}(B^1 \times \mathcal{Y})$. We first observe that the 1-dimensional restriction of T over any point x in the jump set J_{u_T} of u_T is given by

$$\widehat{\pi}_\#(T \llcorner \{x\} \times \mathcal{Y}) = \Gamma_x ,$$

Γ_x being a 1-dimensional integral chain on $\mathcal{Y}$ such that $\partial \Gamma_x = \delta_{u_T^+(x)} - \delta_{u_T^-(x)}$, where $u_T^+(x)$ and $u_T^-(x)$ here and in the sequel denote the right and left limits of u_T at x, respectively. Therefore, by applying Federer's decomposition theorem [9], we find an indecomposable 1-dimensional integral chain γ_x on $\mathcal{Y}$, satisfying $\partial \gamma_x = \delta_{u_T^+(x)} - \delta_{u_T^-(x)}$, and an integral 1-cycle C_x in $\mathcal{Y}$, satisfying $\partial C_x = 0$, such that

$$\Gamma_x = \gamma_x + C_x \qquad \text{and} \qquad \mathbf{M}(\Gamma_x) = \mathbf{M}(\gamma_x) + \mathbf{M}(C_x) . \tag{1.2}$$

Currents associated to graphs of BV-functions. Next we associate to any $T \in \operatorname{cart}(B^1 \times \mathcal{Y})$ a current $G_T \in \mathcal{D}_1(B^1 \times \mathcal{Y})$ carried by the graph of the function $u_T \in BV(B^1, \mathcal{Y})$ corresponding to T, and acting in a linear way on forms ω in $\mathcal{D}^1(B^1 \times \mathcal{Y})$ as follows. We first split $\omega = \omega^{(0)} + \omega^{(1)}$ according to the number of vertical differentials, so that

$$\omega^{(0)} = \phi(x, y)\, dx \qquad \text{and} \qquad \omega^{(1)} = \sum_{j=1}^{N} \phi^j(x, y)\, dy^j$$

for some $\phi, \phi^j \in C_0^\infty(B^1 \times \mathcal{Y})$. We then decompose G_T into its *absolutely continuous*, *Cantor*, and *Jump* parts

$$G_T := T^a + T^C + T^J$$

and define $T^C(\omega^{(0)}) = T^J(\omega^{(0)}) = 0$ and

$$\begin{aligned}
T^a(\omega^{(0)}) &:= \int_{B^1} \phi(x, u_T(x))\, dx \\
T^a(\omega^{(1)}) &:= \sum_{j=1}^{N} \int_{B^1} \phi^j(x, u_T(x)) \nabla u_T^j(x)\, dx \\
T^C(\omega^{(1)}) &:= \sum_{j=1}^{N} \Big\langle D^C u_T^j, \phi^j(\cdot, u_T(\cdot)) \Big\rangle \\
T^J(\omega^{(1)}) &:= \sum_{j=1}^{N} \int_{J_{u_T}} \left(\int_{\gamma_x} \phi^j(x, y)\, dy^j \right) \cdot \nu(x)\, d\mathcal{H}^0(x) .
\end{aligned}$$

Here, γ_x is the indecomposable 1-dimensional integral chain defined by means of the 1-dimensional restriction of T over the point $x \in J_{u_T}$, see (1.2).

Notice that the definition of G_T obviously depends on γ_x and hence, in conclusion, on the current $T \in \mathrm{cart}(B^1 \times \mathcal{Y})$. Moreover, we readily infer that the mass of G_T is given by

$$\mathbf{M}(G_T) = \mathbf{M}(T^a) + \mathbf{M}(T^C) + \mathbf{M}(T^J),$$

where

$$\begin{aligned}\mathbf{M}(T^a) &= \int_{B^1} \sqrt{1 + |\nabla u_T(x)|^2}\, dx,\\ \mathbf{M}(T^C) &= |D^C u_T|(B^1),\\ \mathbf{M}(T^J) &= \int_{J_{u_T}} \mathcal{H}^1(\gamma_x)\, d\mathcal{H}^0(x).\end{aligned}$$

A density result. We recall from [14] that if $u : B^1 \to \mathcal{Y}$ is smooth, or at least e.g. $u \in W^{1,1}(B^1, \mathcal{Y})$, the current G_u integration of 1-forms in $\mathcal{D}^1(B^1 \times \mathcal{Y})$ over the *rectifiable graph of* u is defined in a weak sense by $G_u := (Id \bowtie u)_\# [\![B^1]\!]$, *i.e.*, by letting $G_u(\omega) = (Id \bowtie u)^\#(\omega)$ for every $\omega \in \mathcal{D}^1(B^1 \times \mathcal{Y})$, where $(Id \bowtie u)(x) := (x, u(x))$. Moreover, the mass of G_u agrees with the *area* $\mathcal{A}(u)$ *of the graph* of u

$$\mathbf{M}(G_u) = \mathcal{A}(u) := \int_{B^1} \sqrt{1 + |Du(x)|^2}\, dx.$$

By a straightforward adaptation of the proof of Theorem 1.8 below, we readily obtain the following strong density result for the mass of G_T.

Proposition 1.2. *For every $T \in \mathrm{cart}(B^1 \times \mathcal{Y})$ there exists a sequence of smooth maps $\{u_k\} \subset C^1(B^1, \mathcal{Y})$ such that $u_k \rightharpoonup u_T$ weakly in the BV-sense, $G_{u_k} \rightharpoonup G_T$ weakly in $\mathcal{D}_1(B^1 \times \mathcal{Y})$ and $\mathbf{M}(G_{u_k}) \to \mathbf{M}(G_T)$ as $k \to \infty$.*

Vertical Homology. Let now $\mathcal{Z}^{1,1}(B^1 \times \mathcal{Y})$ denote the class of vertically closed forms

$$\mathcal{Z}^{1,1}(B^1 \times \mathcal{Y}) := \{\omega \in \mathcal{D}^1(B^1 \times \mathcal{Y}) \mid d_y \omega^{(1)} = 0\},$$

where $d = d_x + d_y$ denotes the splitting of the exterior differential d into a horizontal and a vertical differential. We say that $T_k \rightharpoonup T$ weakly in $\mathcal{Z}_{1,1}(B^1 \times \mathcal{Y})$ if $T_k(\omega) \to T(\omega)$ for every $\omega \in \mathcal{Z}^{1,1}(B^1 \times \mathcal{Y})$.

Homological vertical part. By Proposition 1.2, since by Stokes' theorem $\partial G_{u_k} \llcorner B^1 \times \mathcal{Y} = 0$, whereas $G_{u_k} \rightharpoonup G_T$, we obtain that

$$\partial G_T \llcorner B^1 \times \mathcal{Y} = 0.$$

Remark 1.3. In higher dimension $n \geq 2$ in general G_T has a non-zero boundary, *i.e.*, $\partial G_T \llcorner B^n \times \mathcal{Y} \neq 0$, see Remark 2.2.

Setting then
$$S_T := T - G_T ,$$
by (1.1) we infer that $S_T(\phi(x,y)\,dx) = 0$ and $S_T(d\phi) = 0$ for every $\phi \in C_0^\infty(B^1 \times \mathcal{Y})$. Therefore, by homological reasons, since
$$\inf\{\mathbf{M}(C) \mid C \in \mathcal{Z}_1(\mathcal{Y})\,,\ C \text{ is non trivial in } \mathcal{Y}\} > 0\,,$$
similarly to [14, Vol. II, Section 5.3.1] we infer that
$$S_T = \sum_{i=1}^{I} \delta_{x_i} \times C_i \qquad \text{on} \quad \mathcal{Z}^{1,1}(B^1 \times \mathcal{Y})\,,$$
where $\{x_i : i = 1, \dots, I\}$ is a finite disjoint set of points in B^1, possibly intersecting the Jump set J_{u_T}, and C_i is a non-trivial homological integral 1-cycle in $\mathcal{Y}$. Notice that the integral 1-homology group $H_1(\mathcal{Y})$ is finitely generated.

Remark 1.4. Setting
$$S_{T,\text{sing}} := T - G_T - \sum_{i=1}^{I} \delta_{x_i} \times C_i\ ,$$
it turns out that $S_{T,\text{sing}}$ is nonzero only possibly on forms ω with non-zero vertical component, $\omega^{(1)} \neq 0$, and such that $d_y\omega^{(1)} \neq 0$. Therefore, $S_{T,\text{sing}}$ is a *homologically trivial* integer multiplicity rectifiable current in $\mathcal{R}_1(B^1 \times \mathcal{Y})$.

Consequently, setting for $T \in \text{cart}(B^1 \times \mathcal{Y})$
$$T^H := \sum_{i=1}^{I} \delta_{x_i} \times C_i\ , \tag{1.3}$$
T decomposes into the absolutely continuous, Cantor, Jump, Homological, and Singular parts,
$$T = T^a + T^C + T^J + T^H + S_{T,\text{sing}}\,.$$

Gap phenomenon. However, a *gap phenomenon* occurs in $\text{cart}(B^1 \times \mathcal{Y})$. More precisely, if we set
$$\widetilde{\mathbf{M}}(T) := \inf\left\{ \liminf_{k\to\infty} \mathbf{M}(G_{u_k}) \,|\, \{u_k\} \subset C^1(B^1, \mathcal{Y}),\ G_{u_k} \rightharpoonup T \text{ weakly in } \mathcal{D}_1(B^1 \times \mathcal{Y}) \right\},$$
we see that there exist Cartesian currents $T \in \text{cart}(B^1 \times \mathcal{Y})$ for which
$$\mathbf{M}(T) < \widetilde{\mathbf{M}}(T)\,.$$
For example, as in [14, Vol. I, Section 4.2.5], if $T = G_u + \delta_0 \times C$, where $u \equiv P \in \mathcal{Y}$ is a constant map and $C \in \mathcal{Z}^1(\mathcal{Y})$ is a 1-cycle in $\mathcal{Y}$, it readily follows that for every

smooth sequence $\{u_k\} \subset C^1(B^1, \mathcal{Y})$ such that $G_{u_k} \rightharpoonup T$ weakly in $\mathcal{D}_1(B^1 \times \mathcal{Y})$ we have that

$$\liminf_{k\to\infty} \mathbf{M}(G_{u_k}) \geq \mathbf{M}(T) + 2d\,, \qquad d := \mathrm{dist}_{\mathcal{Y}}(P, \mathrm{spt}\, C)\,,$$

where $\mathrm{dist}_{\mathcal{Y}}$ denotes the geodesic distance in $\mathcal{Y}$.

Remark 1.5. This gap phenomenon is due to the structure of the area integrand $u \mapsto \sqrt{1+|Du|^2}$, and it is typical of integrands with linear growth of the gradient, e.g., the total variation integrand $u \mapsto |Du|$, since the images of smooth approximating sequences may have to "connect" the point P to the cycle C, this way paying a cost in term of the distance d. This does not happen e.g. for the Dirichlet integrand $u \mapsto \frac{1}{2}|Du|^2$ in dimension 2, compare [15]. In this case, in fact, the connection from one point P to any 2-cycle $C \in \mathcal{Z}_2(\mathcal{Y})$ can be obtained by means of "cylinders" of small 2-dimensional mapping area and, therefore, of small Dirichlet integral, on account of Morrey's ε-conformality theorem.

Homological theory. In order to study the currents which arise as weak limits of graphs of smooth maps $u_k : B^1 \to \mathcal{Y}$ with equibounded total variations, $\sup_k \|Du_k\|_{L^1} < \infty$, the previous facts lead us to consider vertical homology equivalence classes of currents in $\mathrm{cart}(B^1 \times \mathcal{Y})$. More precisely, we give the following

Definition 1.6. *We denote by* $\mathrm{cart}^{1,1}(B^1 \times \mathcal{Y})$ *the set of equivalence classes of currents in* $\mathrm{cart}(B^1 \times \mathcal{Y})$*, where*

$$T \sim \widetilde{T} \iff T(\omega) = \widetilde{T}(\omega) \quad \forall\, \omega \in \mathcal{Z}^{1,1}(B^1 \times \mathcal{Y})\,.$$

If $T \sim \widetilde{T}$, then the underlying BV-functions coincide, *i.e.*, $u_T = u_{\widetilde{T}}$. Therefore, we have $T^a = \widetilde{T}^a$ and $T^C = \widetilde{T}^C$, whereas in general $T^J \neq \widetilde{T}^J$. However, we have that

$$T^J + T^H = \widetilde{T}^J + \widetilde{T}^H \qquad \text{on} \quad \mathcal{Z}^{1,1}(B^1 \times \mathcal{Y})\,.$$

Jump-concentration points. For future use, we let

$$J_c(T) := J_{u_T} \cup \{x_i : i = 1, \dots, I\} \tag{1.4}$$

denote the set of points of *jump and concentration*, where the x_i's are given by (1.3). We infer that $J_c(T)$ is an at most countable set which does not depend on the representative T, *i.e.*, $J_c(T) = J_c(\widetilde{T})$ if $T \sim \widetilde{T}$. By extending the notion of 1-dimensional restriction $\widehat{\pi}_\#(T \llcorner \{x\} \times \mathcal{Y})$ to equivalence classes, we infer that $\widehat{\pi}_\#(T \llcorner \{x\} \times \mathcal{Y}) = 0$ if $x \notin J_c(T)$. As to jump-concentration points, letting

$$\mathcal{Z}^1(\mathcal{Y}) := \{\eta \in \mathcal{D}^1(\mathcal{Y}) \mid d_y \eta = 0\}\,,$$

if $x \in J_{u_T}$, with $x \neq x_i$, we infer that

$$\widehat{\pi}_\#(T \llcorner \{x\} \times \mathcal{Y}) = \gamma_x \qquad \text{on} \quad \mathcal{Z}^1(\mathcal{Y})\,,$$

where γ_x is the indecomposable 1-dimensional integral chain defined by (1.2), and if $x = x_i$, see (1.4),

$$\widehat{\pi}_\#(T \llcorner \{x\} \times \mathcal{Y}) = \gamma_{x_i} + C_i \qquad \text{on} \quad \mathcal{Z}^1(\mathcal{Y}),$$

where $C_i \in \mathcal{Z}_1(\mathcal{Y})$ is the non-trivial 1-cycle defined by (1.3), and $\gamma_{x_i} = 0$ if $x_i \notin J_{u_T}$.

Vertical minimal connection. For every Cartesian current $T \in \text{cart}^{1,1}(B^1 \times \mathcal{Y})$ and every point $x \in J_c(T)$ we will denote by

$$\begin{aligned}\Gamma_T(x) := \{\gamma \in \text{Lip}([0,1],\mathcal{Y}) \mid \gamma(0) = u_T^-(x),\ \gamma(1) = u_T^+(x), \\ \gamma_\#[\![(0,1)]\!](\eta) = \widehat{\pi}_\#(T \llcorner \{x\} \times \mathcal{Y})(\eta)\ \forall\, \eta \in \mathcal{Z}^1(\mathcal{Y})\}\end{aligned} \tag{1.5}$$

the family of all smooth curves γ in $\mathcal{Y}$, with end points $u_T^{\pm}(x)$, such that their image current $\gamma_\#[\![\,(0,1)\,]\!]$ agrees with the 1-dimensional restriction $\widehat{\pi}_\#(T \llcorner \{x\} \times \mathcal{Y})$ on closed 1-forms in $\mathcal{Z}^1(\mathcal{Y})$. Moreover, we denote by

$$\mathcal{L}_T(x) := \inf\{\mathcal{L}(\gamma) \mid \gamma \in \Gamma_T(x)\}, \qquad x \in J_c(T), \tag{1.6}$$

the *minimal length* of curves γ connecting the "vertical part" of T over x to the graph of u_T. For future use, we remark that the infimum in (1.6) is attained, *i.e.*,

$$\forall\, x \in J_c(T), \quad \exists\, \gamma \in \Gamma_T(x) \quad : \quad \mathcal{L}(\gamma) = \mathcal{L}_T(x). \tag{1.7}$$

Relaxed area functional. We finally introduce the functional

$$\mathcal{A}(T, B) := \int_B \sqrt{1 + |\nabla u_T(x)|^2}\, dx + |D^C u_T|(B) + \int_{J_c(T)\cap B} \mathcal{L}_T(x)\, d\mathcal{H}^0(x)$$

for every Borel set $B \subset B^1$, and we let

$$\mathcal{A}(T) := \mathcal{A}(T, B^1).$$

Notice that for every $T \in \text{cart}^{1,1}(B^1 \times \mathcal{Y})$ we have

$$\min\{\mathbf{M}(\widetilde{T}) : \widetilde{T} \sim T\} \le \mathcal{A}(T). \tag{1.8}$$

Main results. We first prove the following lower semicontinuity property.

Theorem 1.7. *Let $T \in \text{cart}^{1,1}(B^1 \times \mathcal{Y})$. For every sequence of smooth maps $\{u_k\} \subset C^1(B^1, \mathcal{Y})$ such that $G_{u_k} \rightharpoonup T$ weakly in $\mathcal{Z}_{1,1}(B^1 \times \mathcal{Y})$, we have*

$$\liminf_{k\to\infty} \mathbf{M}(G_{u_k}) \ge \mathcal{A}(T).$$

Then we prove the following density result.

Theorem 1.8. *Let $T \in \mathrm{cart}^{1,1}(B^1 \times \mathcal{Y})$. There exists a sequence of smooth maps $\{u_k\} \subset C^1(B^1, \mathcal{Y})$ such that $G_{u_k} \rightharpoonup T$ weakly in $\mathcal{Z}_{1,1}(B^1 \times \mathcal{Y})$ and $\mathbf{M}(G_{u_k}) \to \mathcal{A}(T)$ as $k \to \infty$.*

As a consequence, if we denote, in the same spirit as *Lebesgue's relaxed area*,

$$\widetilde{\mathcal{A}}(T) := \inf\{\liminf_{k\to\infty} \mathcal{A}(u_k) \,|\, \{u_k\} \subset C^1(B^1, \mathcal{Y}) \,,\; G_{u_k} \rightharpoonup T \text{ weakly in } \mathcal{Z}_{1,1}(B^1 \times \mathcal{Y})\} \,,$$

by Theorems 1.7 and 1.8 we readily conclude that

$$\mathcal{A}(T) = \widetilde{\mathcal{A}}(T) \qquad \forall\, T \in \mathrm{cart}^{1,1}(B^1 \times \mathcal{Y}) \,.$$

Properties. From Theorems 1.7 and 1.8, (1.8) and the closure of the class $\mathrm{cart}(B^1 \times \mathcal{Y})$ we infer:

(i) the functional $T \mapsto \mathcal{A}(T)$ is lower semicontinuous in $\mathrm{cart}^{1,1}(B^1 \times \mathcal{Y})$ with respect to the weak convergence in $\mathcal{Z}_{1,1}(B^1 \times \mathcal{Y})$;
(ii) the class $\mathrm{cart}^{1,1}(B^1 \times \mathcal{Y})$ is closed and compact under weak convergence in $\mathcal{Z}_{1,1}(B^1 \times \mathcal{Y})$ with equibounded $\mathcal{A}$-energies.

We finally notice that similar properties hold if one considers the total variation integrand $u \mapsto |Du|$ instead of the area integrand $u \mapsto \sqrt{1 + |Du|^2}$. In particular, setting

$$\mathcal{E}_{1,1}(T) := \int_{B^1} |\nabla u_T(x)|\, dx + |D^C u_T|(B^1) + \int_{J_c(T)} \mathcal{L}_T(x)\, d\mathcal{H}^0(x) \,,$$

for every $T \in \mathrm{cart}^{1,1}(B^1 \times \mathcal{Y})$ we have

$$\mathcal{E}_{1,1}(T) = \inf\Big\{ \liminf_{k\to\infty} \int_{B^1} |Du_k|\, dx \mid \{u_k\} \subset C^1(B^1, \mathcal{Y}) \,, \\ G_{u_k} \rightharpoonup T \quad \text{weakly in } \mathcal{Z}_{1,1}(B^1 \times \mathcal{Y}) \Big\} \,.$$

Remark 1.9. For future use, we denote

$$\mathcal{Y}_\varepsilon := \{y \in \mathbb{R}^N \mid \mathrm{dist}(y, \mathcal{Y}) \le \varepsilon\}$$

the ε-neighborhood of $\mathcal{Y}$ and we observe that, since $\mathcal{Y}$ is smooth, there exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon \le \varepsilon_0$ the nearest point projection Π_ε of $\mathcal{Y}_\varepsilon$ onto $\mathcal{Y}$ is a well defined Lipschitz map with Lipschitz constant $L_\varepsilon \to 1^+$ as $\varepsilon \to 0^+$. Note that for $0 < \varepsilon \le \varepsilon_0$ the set $\mathcal{Y}_\varepsilon$ is equivalent to $\mathcal{Y}$ in the sense of the algebraic topology. In particular, we have

$$\pi_1(\mathcal{Y}_\varepsilon) = \pi_1(\mathcal{Y}) \,.$$

Proof of Theorem 1.7. Let $\{x_i\}_{i>I} \subset B^1$ be the at most countable set of discontinuity points in $J_{u_T} \setminus \{x_i : i = 1, \dots, I\}$, see (1.4). By the properties of $\mathcal{Y}$ we have

$$\mathcal{L}_T(x_i) \le C \cdot |u_T^+(x_i) - u_T^-(x_i)| \qquad \forall\, i > I\,,$$

where $C = C(\mathcal{Y}) > 0$ is an absolute constant, see (1.6). Therefore, since

$$|D^J u_T|(B^1) = \sum_{i=1}^{\infty} |u_T^+(x_i) - u_T^-(x_i)| < \infty\,,$$

for every $\varepsilon > 0$ we find $l(\varepsilon) > I$ such that

$$\sum_{i=l(\varepsilon)+1}^{\infty} \mathcal{L}_T(x_i) < \varepsilon\,. \tag{1.9}$$

After rearranging in an increasing way the set $\{x_i : i \le l(\varepsilon)\}$, and setting $x_0 = -1$, $x_{l(\varepsilon)+1} = 1$, we let

$$2\delta = 2\delta(\varepsilon) := \min\{|x_i - x_{i+1}| : i = 0, \dots, l(\varepsilon)\} > 0\,.$$

For $i \in \{1, \dots, l(\varepsilon)\}$, due to the weak convergence $u_k \rightharpoonup u_T$ in the BV-sense, possibly passing to a subsequence, we find the existence of sequences of points $a_k^i \in]x_i - \delta/k, x_i[$ and $b_k^i \in]x_i, x_i + \delta/k[$ such that

$$\mathrm{dist}_{\mathcal{Y}}\big(u_k(a_k^i), u_T^-(x_i)\big) < \frac{1}{k} \qquad \text{and} \qquad \mathrm{dist}_{\mathcal{Y}}\big(u_k(b_k^i), u_T^+(x_i)\big) < \frac{1}{k} \tag{1.10}$$

for every k, where $\mathrm{dist}_{\mathcal{Y}}$ denotes the geodesic distance in $\mathcal{Y}$.

Let $\gamma_k^i : [0,1] \to \mathcal{Y}$ be the Lipschitz reparametrization with constant velocity of the smooth curve $u_{k|[a_k^i, b_k^i]}$. From the weak convergence $G_{u_k} \rightharpoonup T$ we infer that

$$\gamma^i_{k\#}[\![\, (0,1)\,]\!](\eta) \to \widehat{\pi}_\#(T \llcorner \{x\} \times \mathcal{Y})(\eta) \qquad \forall\, \eta \in \mathcal{Z}^1(\mathcal{Y}) \tag{1.11}$$

as $k \to \infty$, where $\widehat{\pi}_\#(T \llcorner \{x\} \times \mathcal{Y})$ is the previously defined restriction of T over x. Moreover, by connecting the end points $u_k(a_k^i)$ and $u_k(b_k^i)$ with $u_T^-(x_i)$ and $u_T^+(x_i)$, respectively, due to (1.10) we find a sequence of Lipschitz arcs $\widetilde{\gamma}_k^i : [0,1] \to \mathcal{Y}$, with end points $\widetilde{\gamma}_k^i(0) = u_T^-(x_i)$ and $\widetilde{\gamma}_k^i(1) = u_T^+(x_i)$, such that $\big(\widetilde{\gamma}^i_{k\#}[\![\, (0,1)\,]\!] - \gamma^i_{k\#}[\![\, (0,1)\,]\!]\big)(\eta) \to 0$ for every $\eta \in \mathcal{Z}^1(\mathcal{Y})$ as $k \to \infty$ and

$$\mathcal{L}(\widetilde{\gamma}_k^i) \le \mathcal{L}(\gamma_k^i) + \frac{2}{k} \qquad \forall\, k\,.$$

By the construction we also infer that $\{\widetilde{\gamma}_k^i\}_k$ is a sequence of equibounded and equicontinuous maps. Therefore, by Ascoli's theorem, possibly passing to a subsequence, we find that $\widetilde{\gamma}_k^i$ converges uniformly to a Lipschitz arc $\widetilde{\gamma}^i : [0,1] \to \mathcal{Y}$, with end points $u_T^{\mp}(x_i)$, satisfying by (1.11)

$$\widetilde{\gamma}^i_{\#}[\![\, (0,1)\,]\!](\eta) = \widehat{\pi}_\#(T \llcorner \{x\} \times \mathcal{Y})(\eta) \qquad \forall\, \eta \in \mathcal{Z}^1(\mathcal{Y})\,.$$

We then obtain that $\widetilde{\gamma}^i \in \Gamma_T(x_i)$, according to the definition (1.5). Moreover, by the lower semicontinuity of the length functional with respect to the uniform convergence, we have

$$\mathcal{L}(\widetilde{\gamma}^i) \leq \liminf_{k\to\infty} \mathcal{L}(\widetilde{\gamma}^i_k) \,.$$

By (1.6) and by the above estimates we conclude that

$$\mathcal{L}_T(x_i) \leq \liminf_{k\to\infty} \mathcal{L}(\gamma^i_k) \qquad \forall\, i = 1, \ldots, l(\varepsilon)\,. \tag{1.12}$$

Now, since by the weak BV-convergence of $u_k \rightharpoonup u_T$ we have

$$\int_{B^1} \sqrt{1 + |\nabla u_T(x)|^2}\, dx + |D^C u_T|(B^1) \leq \liminf_{k\to\infty} \mathcal{A}(u_k)\,,$$

by the previous argument, taking into account (1.9) and (1.12), we readily infer that

$$\mathcal{A}(T) - \varepsilon \leq \liminf_{k\to\infty} \mathcal{A}(u_k)$$

and hence the assertion, by letting $\varepsilon \searrow 0$. □

Proof of Theorem 1.8. Let $\{x_i\}_{i>I}$, $l(\varepsilon)$ and $\delta = \delta(\varepsilon)$ be defined as in the proof of Theorem 1.7, so that (1.9) holds true. Let $\gamma^i \in \Gamma_T(x_i)$ be such that $\mathcal{L}(\gamma^i) \leq \mathcal{L}_T(x_i) + \varepsilon \cdot 2^{-i}$, see (1.5) and (1.6). For fixed $\delta \in (0, \delta(\varepsilon))$, and for every $i = 1, \ldots, l(\varepsilon)$, we first define $u^\varepsilon_\delta : [x_i - \delta, x_i + \delta] \to \mathcal{Y}$ by reparametrising with the same orientation the arc γ_i, *i.e.*,

$$u^\varepsilon_\delta(x) := \gamma^i\left(\frac{1}{2} + \frac{1}{2\delta}(x - x_i)\right).$$

Setting $I_i :=]x_i + \delta, x_{i+1} - \delta[$ if $i = 1, \ldots, l(\varepsilon) - 1$, and $I_1 :=]-1, x_1 - \delta[$, $I_{l(\varepsilon)} :=]x_{l(\varepsilon)} + \delta, 1[$, we then extend u^ε_δ to the whole of B^1 by letting $u^\varepsilon_\delta(x) := u_T(\Psi_i(x))$ if $x \in I_i$ for some $i = 0, \ldots, l(\varepsilon)$, where Ψ_i is the bijective and increasing affine map between the intervals I_i and $]x_i, x_{i+1}[$. We then apply a mollification procedure to the function u^ε_δ, defining this way a smooth map $v^\varepsilon_\delta : B^1 \to \mathbb{R}^N$ such that

$$\|v^\varepsilon_\delta - u^\varepsilon_\delta\|_{L^1(B^1)} \leq \delta \qquad \text{and} \qquad \int_{B^1} |Dv^\varepsilon_\delta|\, dx \leq |Du^\varepsilon_\delta|(B^1) + \delta\,.$$

Since u_T is continuous outside the Jump set J_{u_T} and (1.9) holds true, for every $\sigma > 0$ we find $\eta = \eta(\sigma, \delta, \varepsilon) > 0$ such that, in the a.e. sense,

$$\forall\, x, y \in B^1\,, \quad |x - y| < \eta \quad \Longrightarrow \quad |u^\varepsilon_\delta(x) - u^\varepsilon_\delta(y)| < \sigma + \varepsilon\,.$$

As a consequence, we may and do define v^ε_δ in such a way that in particular

$$\operatorname{dist}(v^\varepsilon_\delta(x), \mathcal{Y}) < \varepsilon \qquad \forall\, x \in B^1\,.$$

Setting now $w^\varepsilon_\delta := \Pi_\varepsilon \circ v^\varepsilon_\delta : B^1 \to \mathcal{Y}$, compare Remark 1.9, taking first δ small with respect to ε, and letting then $\varepsilon \to 0$, by a diagonal procedure we find a smooth approximating sequence. □

2. Cartesian currents, BV-energy and weak limits

In this section we deal with the weak limits of graphs of smooth maps $u_k : B^n \to \mathcal{Y}$ with equibounded $W^{1,1}$-energies. We first state a few preliminary results.

Homological facts. Since $H_1(\mathcal{Y})$ has no torsion, there are generators $[\gamma_1], \dots, [\gamma_{\overline{s}}]$, *i.e.* integral 1-cycles in $\mathcal{Z}_1(\mathcal{Y})$, such that

$$H_1(\mathcal{Y}) = \left\{ \sum_{s=1}^{\overline{s}} n_s\, [\gamma_s] \mid n_s \in \mathbb{Z} \right\} ,$$

see e.g. [14], Vol. I, Section 5.4.1. By de Rham's theorem the first real homology group is in duality with the first cohomology group $H^1_{dR}(\mathcal{Y})$, the duality being given by the natural pairing

$$\langle [\gamma], [\omega] \rangle := \gamma(\omega) = \int_\gamma \omega\,, \qquad [\gamma] \in H_1(\mathcal{Y}; \mathbb{R})\,, \quad [\omega] \in H^1_{dR}(\mathcal{Y})\,.$$

We will then denote by $[\omega^1], \dots, [\omega^{\overline{s}}]$ a dual basis in $H^1_{dR}(\mathcal{Y})$ so that $\gamma_s(\omega^r) = \delta_{sr}$, where δ_{sr} denotes the Kronecker symbols.

$\mathcal{D}_{n,1}$-currents. For $p = 1, \dots, n$, every differential p-form $\omega \in \mathcal{D}^p(B^n \times \mathcal{Y})$ splits as a sum $\omega = \sum_{j=0}^{\overline{p}} \omega^{(j)}$, where $\overline{p} := \min(p, M)$, $M = \dim(\mathcal{Y})$, and the $\omega^{(j)}$'s are the p-forms that contain exactly j differentials in the vertical $\mathcal{Y}$ variables. We denote by $\mathcal{D}^{p,1}(B^n \times \mathcal{Y})$ the subspace of $\mathcal{D}^p(B^n \times \mathcal{Y})$ of p-forms of the type $\omega = \omega^{(0)} + \omega^{(1)}$, and by $\mathcal{D}_{p,1}(B^n \times \mathcal{Y})$ the dual space of $\mathcal{D}^{p,1}(B^n \times \mathcal{Y})$. Every $(p, 1)$-current $T \in \mathcal{D}_{p,1}(B^n \times \mathcal{Y})$ splits as $T = T_{(0)} + T_{(1)}$, where $T_{(j)}(\omega) := T(\omega^{(j)})$. For example, if $u \in W^{1,1}(B^n, \mathcal{Y})$, then G_u is an $(n, 1)$-current in $\mathcal{D}_{n,1}(B^n \times \mathcal{Y})$ defined in an approximate sense by

$$G_u := (Id \bowtie u)_\# [\![B^n]\!]\,, \tag{2.1}$$

where $(Id \bowtie u)(x) := (x, u(x))$, compare [14], see also [4].

Weak $\mathcal{D}_{n,1}$-convergence. If $\{T_k\} \subset \mathcal{D}_{n,1}(B^n \times \mathcal{Y})$, we say that $\{T_k\}$ converges weakly in $\mathcal{D}_{n,1}(B^n \times \mathcal{Y})$, $T_k \rightharpoonup T$, if $T_k(\omega) \to T(\omega)$ for every $\omega \in \mathcal{D}^{n,1}(B^n \times \mathcal{Y})$. Trivially, the class $\mathcal{D}_{n,1}(B^n \times \mathcal{Y})$ is closed under weak convergence.

$\mathcal{E}_{1,1}$-norm. For $\omega \in \mathcal{D}^{n,1}(B^n \times \mathcal{Y})$ and $T \in \mathcal{D}_{n,1}(B^n \times \mathcal{Y})$ we set

$$\|\omega\|_{\mathcal{E}_{1,1}} := \max \left\{ \sup_{x,y} \frac{|\omega^{(0)}(x, y)|}{1 + |y|}\,, \int_{B^n} \sup_y |\omega^{(1)}(x, y)|\, dx \right\} ,$$

$$\|T\|_{\mathcal{E}_{1,1}} := \sup \left\{ T(\omega) \mid \omega \in \mathcal{D}^{n,1}(B^n \times \mathcal{Y})\,,\ \|\omega\|_{\mathcal{E}_{1,1}} \le 1 \right\} .$$

It is not difficult to show that $\|T\|_{\mathcal{E}_{1,1}}$ is a norm on $\{T \in \mathcal{D}_{n,1}(B^n \times \mathcal{Y}) : \|T\|_{\mathcal{E}_{1,1}} < \infty\}$. Moreover, $\|\cdot\|_{\mathcal{E}_{1,1}}$ is weakly lower semicontinuous in $\mathcal{D}_{n,1}$, so that $\{T \in \mathcal{D}_{n,1}(B^n \times \mathcal{Y}) : \|T\|_{\mathcal{E}_{1,1}} < \infty\}$ is closed under weak $\mathcal{D}_{n,1}$-convergence with equibounded $\mathcal{E}_{1,1}$-norms. Finally, if $\sup_k \|T_k\|_{\mathcal{E}_{1,1}} < \infty$ there is a subsequence that weakly converges to some $T \in \mathcal{D}_{n,1}(B^n \times \mathcal{Y})$ with $\|T\|_{\mathcal{E}_{1,1}} < \infty$.

Boundaries. The exterior differential d splits into a horizontal and a vertical differential $d = d_x + d_y$. Of course $\partial_x T(\omega) := T(d_x\omega)$ defines a boundary operator $\partial_x : \mathcal{D}_{n,1}(B^n \times \mathcal{Y}) \to \mathcal{D}_{n-1,1}(B^n \times \mathcal{Y})$. Now, for any $\omega \in \mathcal{D}^{n-1,1}(B^n \times \mathcal{Y})$, $d_y\omega$ belongs to $\mathcal{D}^{n,1}(B^n \times \mathcal{Y})$ if and only if $d_y\omega^{(1)} = 0$. Then $\partial_y T$ makes sense only as an element of the dual space of $\mathcal{Z}^{n-1,1}(B^n \times \mathcal{Y})$, where

$$\mathcal{Z}^{p,1}(B^n \times \mathcal{Y}) := \{\omega \in \mathcal{D}^{p,1}(B^n \times \mathcal{Y}) \mid d_y\omega^{(1)} = 0\}.$$

Graphs of *BV*-maps. We introduce a class of $\mathcal{D}_{n,1}$-currents associated to the graphs of BV-functions. To this aim, we observe that any form $\omega = \omega^{(1)} \in \mathcal{D}^{n,1}(B^n \times \mathcal{Y})$ can be written as

$$\omega^{(1)} = \sum_{i=1}^{n}\sum_{j=1}^{N}(-1)^{n-i}\phi_i^j(x,y)\,\widehat{dx^i} \wedge dy^j \tag{2.2}$$

for some $\phi_i^j \in C_0^\infty(B^n \times \mathcal{Y})$, and we will set $\phi^j := (\phi_1^j, \dots, \phi_n^j)$.

Definition 2.1. *We say that a current $G \in \mathcal{D}_{n,1}(B^n \times \mathcal{Y})$ is in* BV-graph$(B^n \times \mathcal{Y})$ *if it decomposes into its absolutely continuous, Cantor, and Jump parts*

$$G := G^a + G^C + G^J,$$

where $G^C_{(0)} = G^J_{(0)} = 0$, and its action on forms in $\mathcal{D}^{n,1}(B^n \times \mathcal{Y})$ is given for any $\phi \in C_c^\infty(B^n \times \mathcal{Y})$ by

$$G(\phi(x,y)\,dx) = G^a(\phi(x,y)\,dx) := \int_{B^n} \phi(x,u(x))\,dx$$

for some function $u = u(G) \in BV(B^n, \mathcal{Y})$ and, on forms $\omega = \omega^{(1)}$ satisfying (2.2)*, by*

$$\begin{aligned}
G^a(\omega^{(1)}) &:= \sum_{j=1}^{N}\int_{B^n} \langle \nabla u^j, \phi^j(x,u(x))\rangle\,dx\\
G^C(\omega^{(1)}) &:= \sum_{j=1}^{N}\int_{B^n} \phi^j(x,u(x))\,dD^C u^j\\
G^J(\omega^{(1)}) &:= \sum_{j=1}^{N}\sum_{i=1}^{n}\int_{J_u}\left(\int_{\gamma_x}\phi_i^j(x,y)\,dy^j\right)\nu_i\,d\mathcal{H}^{n-1}(x),
\end{aligned}$$

where γ_x is a 1*-dimensional integral chain in $\mathcal{Y}$ satisfying $\partial\gamma_x = \delta_{u^+(x)} - \delta_{u^-(x)}$ and $\nu = (\nu_1, \dots, \nu_n)$ is the unit normal to J_u at x, for $\mathcal{H}^{n-1}$-a.e. $x \in J_u$.*

Remark 2.2. If $n \geq 2$ in general the current G has a non-zero boundary in $B^n \times \mathcal{Y}$, even if $u \in W^{1,1}(B^n, \mathcal{Y})$, *i.e.*, if $G = G^a$. Take for example $n = 2$, $\mathcal{Y} = S^1 \subset \mathbb{R}^2$, and $u(x) = x/|x|$, so that $G = G_u := (\mathrm{Id} \bowtie u)_\# [\![B^2]\!]$ and hence

$$\partial G \llcorner B^2 \times S^1 = -\delta_0 \times [\![S^1]\!] ,$$

where δ_0 is the unit Dirac mass at the origin. However, as we shall see in Remark 6.10 below, the boundary ∂G is null on every $(n-1)$-form $\widetilde{\omega}$ in $B^n \times \mathcal{Y}$ which has no "vertical" differentials.

Weak limits of smooth graphs. Let $\{u_k\} \subset C^1(B^n, \mathcal{Y})$ be a sequence of smooth maps with equibounded $W^{1,1}$-energies, $\sup_k \|Du_k\|_{L^1} < \infty$. The currents G_{u_k} carried by the graphs of the u_k's are well defined currents in $\mathcal{D}_{n,1}(B^n \times \mathcal{Y})$ with equibounded $\mathcal{E}_{1,1}$-norms. Therefore, possibly passing to a subsequence, we infer that $G_{u_k} \rightharpoonup T$ weakly in $\mathcal{D}_{n,1}(B^n \times \mathcal{Y})$ to some current $T \in \mathcal{D}_{n,1}(B^n \times \mathcal{Y})$, and $u_k \rightharpoonup u_T$ weakly in the BV-sense to some function $u_T \in BV(B^n, \mathcal{Y})$. Therefore, we clearly have that

$$T(\phi(x, y)\, dx) = \int_{B^n} \phi(x, u_T(x))\, dx \qquad \forall\, \phi \in C_c^\infty(B^n \times \mathcal{Y}) . \tag{2.3}$$

Moreover, by lower semicontinuity we have $\|T\|_{\mathcal{E}_{1,1}} < \infty$ whereas, since the G_{u_k}'s have no boundary in $B^n \times \mathcal{Y}$, by the weak convergence we also infer

$$\partial T = 0 \quad \text{on} \quad \mathcal{Z}^{n-1,1}(B^n \times \mathcal{Y}) . \tag{2.4}$$

Currents associated to graphs of BV-functions. Arguing as in Section 1, we associate to the weak limit current T a current $G_T \in BV$-graph$(B^n \times \mathcal{Y})$, see Definition 2.1, where the function $u = u(G_T) \in BV(B^n, \mathcal{Y})$ is given by u_T and the γ_x's in the definition of the jump part G_T^J are the indecomposable 1-dimensional integral chains defined as in the previous section, but for $\mathcal{H}^{n-1}$-a.e. $x \in J_{u_T}$, since $\|T\|_{\mathcal{E}_{1,1}} < \infty$, compare (1.2) and Definition 2.8 below. In general $\partial G_T \llcorner B^n \times \mathcal{Y} \neq 0$. However, setting

$$S_T := T - G_T ,$$

we clearly have $S_T(\phi(x, y)\, dx) = 0$ for every $\phi \in C_c^\infty(B^n \times \mathcal{Y})$. Moreover, we also have:

Proposition 2.3. $S_T(\omega) = 0$ *for every form* $\omega = \omega^{(1)}$ *such that* $\omega = d_y \widetilde{\omega}$ *for some* $\widetilde{\omega} \in \mathcal{D}^{n-1,0}(B^n \times \mathcal{Y})$.

Proof. Write $\widetilde{\omega} := \omega_\varphi \wedge \eta$ for some $\eta \in C_0^\infty(\mathcal{Y})$ and $\varphi = (\varphi^1, \dots, \varphi^n) \in C_0^\infty(B^n, \mathbb{R}^n)$, where

$$\omega_\varphi := \sum_{i=1}^n (-1)^{i-1} \varphi^i(x)\, \widehat{dx^i} . \tag{2.5}$$

Since

$$d(\omega_\varphi \wedge \eta) = \mathrm{div}\varphi(x)\eta(y)\,dx + (-1)^{n-1}\omega_\varphi \wedge d_y\eta$$

and $T(d(\omega_\varphi \wedge \eta)) = \partial T(\omega_\varphi \wedge \eta) = 0$, we have

$$(-1)^n T(\mathrm{div}\varphi(x)\eta(y)\,dx) = T(\omega_\varphi \wedge d_y\eta)\,,$$

so that

$$S_T(\omega_\varphi \wedge d_y\eta) = (-1)^n T(\mathrm{div}\varphi(x)\eta(y)\,dx) - G_T(\omega_\varphi \wedge d_y\eta)\,.$$

Moreover, since $T_{(0)} = G_{T(0)}$, by (2.3) we have

$$T(\mathrm{div}\varphi(x)\eta(y)\,dx) = \int_{B^n} \mathrm{div}\varphi(x)\eta(u_T(x))\,dx = -\langle D(\eta \circ u_T), \varphi\rangle$$

whereas, taking $\phi_i^j = \varphi^i D_{y_j}\eta$ in (2.2), by the definition of G_T, since $\partial\gamma_x = \delta_{u_T^+(x)} - \delta_{u_T^-(x)}$ we infer

$$\begin{aligned}(-1)^{n-1}G_T(\omega_\varphi \wedge d_y\eta) \;=\; & \sum_{j=1}^N \int_{B^n} \frac{\partial\eta}{\partial y^j}(u_T(x))\langle\nabla u_T^j(x), \varphi(x)\rangle\,dx \\ & + \sum_{j=1}^N \int_{B^n} \frac{\partial\eta}{\partial y^j}(u_T(x))\,\varphi(x)\,dD^C u_T^j \\ & + \int_{J_{u_T}} \big(\eta(u_T^+(x)) - \eta(u_T^-(x))\big)\langle\varphi(x), \nu(x)\rangle\,d\mathcal{H}^{n-1}\,.\end{aligned}$$

Finally, by the chain rule for the derivative $D(\eta \circ u_T)$ we obtain

$$(-1)^{n-1}G_T(\omega_\varphi \wedge d_y\eta) = \langle D(\eta \circ u_T), \varphi\rangle$$

and hence that $S_T(\omega_\varphi \wedge d_y\eta) = 0$. □

In conclusion, similarly to [14, Vol. II, Section 5.4.3], we infer that the weak limit current T is given by

$$T = G_T + S_T\,, \qquad S_T = \sum_{s=1}^{\overline{s}} \mathbb{L}_s(T) \times \gamma_s \qquad \text{on} \quad \mathcal{Z}^{n,1}(B^n \times \mathcal{Y})\,, \tag{2.6}$$

where $\mathbb{L}_s(T) \in \mathcal{D}_{n-1}(B^n)$ is defined by

$$\mathbb{L}_s(T) = (-1)^{n-1}\pi^\#(S_T \llcorner \widehat{\pi}^\#\omega^s)\,, \qquad s = 1, \ldots, \overline{s}\,, \tag{2.7}$$

so that

$$\mathbb{L}_s(T)(\phi) = S_T(\pi^\#\phi \wedge \widehat{\pi}^\#\omega^s) \qquad \forall\,\phi \in \mathcal{D}^{n-1}(B^n)\,.$$

Notice that by (2.4) we have

$$\partial\, \mathbb{L}_s(T) \llcorner B^n = (-1)^{n-1} \pi_\#((\partial G_T) \llcorner \widehat{\pi}^\# \omega^s) \qquad \forall\, s = 1, \dots, \overline{s}\,.$$

Finally, setting

$$S_{T,\mathrm{sing}} := T - G_T - \sum_{s=1}^{\overline{s}} \mathbb{L}_s(T) \times \gamma_s\,, \tag{2.8}$$

see Remark 1.4, it turns out that $S_{T,\mathrm{sing}}$ is nonzero only possibly on forms ω with non-zero vertical component, $\omega^{(1)} \neq 0$, and such that $d_y \omega^{(1)} \neq 0$.

Parametric polyconvex lower semicontinuous extension of the total variation. Following [14], Vol. II, Section 1.2, we recall that the *parametric polyconvex lower semicontinuous extension* $\|\cdot\|_{TV}$ of the total variation integrand of mappings from B^n to $\mathbb{R}^N$ has the form

$$\|\xi\|_{TV} := |\xi_{(1)}| \quad \forall \xi \in \Lambda_n \mathbb{R}^{n+N} \quad \text{such that} \quad \xi^{\overline{0}0} \geq 0, \tag{2.9}$$

where $\xi^{\overline{0}0}$ denotes the coefficient of the first component of any n-vector $\xi \in \Lambda_n \mathbb{R}^{n+N}$ and $|\xi_{(1)}|$ is the euclidean norm of the component $\xi_{(1)}$ of ξ in $\Lambda_{n-1}\mathbb{R}^n \otimes \Lambda_1 \mathbb{R}^N$. We have

Proposition 2.4. *The parametric polyconvex lower semicontinuous extension $F(x,u,\xi) : B^n \times \mathbb{R}^N \times \Lambda_n \mathbb{R}^{n+N} \to \overline{\mathbb{R}}^+$ of the total variation integrand of mappings from B^n into any smooth manifold $\mathcal{Y} \subset \mathbb{R}^N$ is given by*

$$F(x,u,\xi) := \begin{cases} \|\xi\|_{TV} & \textit{if} \quad u \in \mathcal{Y},\ \xi \in \Lambda_n(\mathbb{R}^n \times T_u \mathcal{Y}) \\ +\infty & \textit{otherwise}\,, \end{cases} \tag{2.10}$$

where $\|\xi\|_{TV}$ is given by (2.9) *and $T_u\mathcal{Y}$ is the tangent space to $\mathcal{Y}$ at u.*

Parametric total variation. If $T \in \mathcal{D}_{n,1}(B^n \times \mathcal{Y})$ is such that $\|T\|_{\mathcal{E}_{1,1}} < \infty$, we denote by

$$T = \|T\|_{\mathcal{E}_{1,1}} \llcorner \overrightarrow{T}$$

the Radon-Nikodym decomposition of T with respect to the $\mathcal{E}_{1,1}$-norm, T being identified with the $\mathbb{R}^{1+Nn}$-valued linear functional

$$T := \left(T^{\overline{0}0}, (T^{\overline{i}j})_{\mathbb{R}^{Nn}}\right), \qquad i = 1, \dots n\,, \quad j = 1, \dots N\,,$$

where

$$T^{\overline{0}0}(\phi) := T(\phi\, dx)\,, \qquad T^{\overline{i}j}(\phi) := T(\phi\, \widehat{dx^i} \wedge dy^j)\,, \qquad \phi \in C_0^\infty(B^n \times \mathcal{Y})\,.$$

Definition 2.5. *The* parametric variational integral *associated to the total variation integral is defined for every Borel set $B \subset B^n$ by*

$$\mathcal{F}_{1,1}(T, B \times \mathcal{Y}) := \int_{B \times \mathcal{Y}} F\big(\pi(z), \widehat{\pi}(z), \overrightarrow{T}(z)\big)\, d\|T\|_{\mathcal{E}_{1,1}}(z)$$

where $F(x,u,\xi)$ is given by (2.10)*, and we let $\mathcal{F}_{1,1}(T) := \mathcal{F}_{1,1}(T, B^n \times \mathcal{Y})$.*

Gap phenomenon. If $T \in \mathcal{D}_{n,1}(B^n \times \mathcal{Y})$ is the weak limit of a sequence $\{G_{u_k}\}$ of graphs of smooth maps $\{u_k\} \subset C^1(B^n, \mathcal{Y})$ with equibounded $W^{1,1}$-energies, since $\mathcal{F}_{1,1}(G_{u_k}) = \|Du_k\|_{L^1}$, by the *lower semicontinuity* of $\mathcal{F}_{1,1}$ with respect to the weak convergence in $\mathcal{D}_{n,1}$ we infer that $\mathcal{F}_{1,1}(T) < \infty$. Moreover, if T decomposes as in (2.6) on the whole of $\mathcal{D}^{n,1}(B^n \times \mathcal{Y})$, *i.e.*, the singular part $S_{T,\text{sing}}$ defined in (2.8) vanishes, and if the $\mathbb{L}_s(T)$'s are integer multiplicity rectifiable currents, an explicit formula can be obtained. However, similarly to the case of dimension $n = 1$, a gap phenomenon occurs. More precisely, in general for every smooth sequence $\{u_k\} \subset C^1(B^n, \mathcal{Y})$ such that $G_{u_k} \rightharpoonup T$ weakly in $\mathcal{D}_{n,1}(B^n \times \mathcal{Y})$ we have that

$$\liminf_{k\to\infty} \mathcal{F}_{1,1}(G_{u_k}) \geq \mathcal{F}_{1,1}(T) + C$$

for some absolute constant $C > 0$, see Remark 1.5.

Vertical homology classes. As in Definition 1.6, we are therefore led to consider *vertical homology equivalence classes* of currents satisfying the same structure properties as weak limits of graphs of smooth maps $u_k : B^n \to \mathcal{Y}$ with equibounded total variation, $\sup_k \|Du_k\|_{L^1} < \infty$. More precisely, we say that

$$T \sim \widetilde{T} \iff T(\omega) = \widetilde{T}(\omega) \quad \forall\, \omega \in \mathcal{Z}^{n,1}(B^n \times \mathcal{Y})\,. \tag{2.11}$$

Moreover, we will say that $T_k \rightharpoonup T$ weakly in $\mathcal{Z}_{n,1}(B^n \times \mathcal{Y})$ if $T_k(\omega) \to T(\omega)$ for every $\omega \in \mathcal{Z}^{n,1}(B^n \times \mathcal{Y})$.

Definition 2.6. *We denote by* $\mathcal{E}_{1,1}$-graph$(B^n \times \mathcal{Y})$ *the set of equivalence classes, in the sense of* (2.11)*, of currents* T *in* $\mathcal{D}_{n,1}(B^n \times \mathcal{Y})$ *which have no interior boundary,*

$$\partial T = 0 \quad on \quad \mathcal{Z}^{n-1,1}(B^n \times \mathcal{Y})\,,$$

finite $\mathcal{E}_{1,1}$*-norm,* i.e.

$$\|T\|_{\mathcal{E}_{1,1}} := \sup\left\{ T(\omega) \mid \omega \in \mathcal{Z}^{n,1}(B^n \times \mathcal{Y})\,,\ \|\omega\|_{\mathcal{E}_{1,1}} \leq 1 \right\} < \infty\,,$$

and decompose as

$$T = G_T + S_T\,, \qquad S_T = \sum_{s=1}^{\bar{s}} \mathbb{L}_s(T) \times \gamma_s \qquad on \quad \mathcal{Z}^{n,1}(B^n \times \mathcal{Y})\,,$$

where $G_T \in BV$-graph$(B^n \times \mathcal{Y})$*, see Definition* 2.1*, and* $\mathbb{L}_s(T)$ *is an integer multiplicity rectifiable current in* $\mathcal{R}_{n-1}(B^n)$ *for every* s*.*

Remark 2.7. If $\widetilde{T} \sim T$, in general $G_{\widetilde{T}} \neq G_T$. However, the corresponding BV-functions coincide, *i.e.*, $u(G_T) = u(G_{\widetilde{T}})$, see Definition 2.1. This yields that we may refer to the underlying functions $u_T \in BV(B^n, \mathcal{Y})$ associated to currents T in $\mathcal{E}_{1,1}$-graph$(B^n \times \mathcal{Y})$.

Jump-concentration set. Moreover, if $\mathcal{L}(T)$ denotes the $(n-1)$-rectifiable set given by the union of the sets of positive multiplicity of the $\mathbb{L}_s(T)$'s, we infer that the union

$$J_c(T) := J_{u_T} \cup \mathcal{L}(T) \tag{2.12}$$

does not depend on the choice of the representative in T. As in dimension one, the countably $\mathcal{H}^{n-1}$-rectifiable set $J_c(T)$ is said to be the set of points of *jump-concentration* of T.

Restriction over points of jump-concentration. Let $T \in \mathcal{E}_{1,1}$-graph$(B^n \times \mathcal{Y})$ and let $\nu_T : J_c(T) \to S^{n-1}$ denote an extension to $J_c(T)$ of the unit normal ν_{u_T} to the Jump set J_{u_T}. For any $k = 1, \dots, n-1$, let P be an oriented k-dimensional subspace in $\mathbb{R}^n$ and $P_\lambda := P + \sum_{i=1}^{n-k} \lambda_i \nu_i$ the family of oriented k-planes parallel to P, where $\lambda := (\lambda_1, \dots, \lambda_{n-k}) \in \mathbb{R}^{n-k}$, span$(\nu_1, \dots, \nu_{n-k})$ being the orthogonal space to P. Since T has finite $\mathcal{E}_{1,1}$-norm, similarly to the case of normal currents, for $\mathcal{L}^{n-k}$-a.e. λ such that $P_\lambda \cap B^n \neq \emptyset$, the *slice* $T \llcorner \pi^{-1}(P_\lambda)$ of T over $\pi^{-1}(P_\lambda)$ is a well defined k-dimensional current in $\mathcal{E}_{1,1}$-graph$((B^n \cap P_\lambda) \times \mathcal{Y})$ with finite $\mathcal{E}_{1,1}$-norm. Moreover, for any such λ we have

$$J_c(T \llcorner \pi^{-1}(P_\lambda)) = J_c(T) \cap P_\lambda \qquad \text{in the } \mathcal{H}^{k-1}\text{-a.e. sense},$$

whereas the BV-function associated to $T \llcorner \pi^{-1}(P_\lambda)$ is equal to the restriction $u_{T|P_\lambda}$ of u_T to P_λ. Therefore, in the particular case $k = 1$, as in Section 1 the 1-dimensional restriction

$$\widehat{\pi}_\#\big((T \llcorner \pi^{-1}(P_\lambda)) \llcorner \{x\} \times \mathcal{Y}\big) \in \mathcal{D}_1(\mathcal{Y}) \tag{2.13}$$

of the 1-dimensional current $T \llcorner \pi^{-1}(P_\lambda)$ over any point $x \in J_c(T) \cap P_\lambda$ such that $\nu_T(x)$ does not belong to P is well defined. In this case, from the slicing properties of BV-functions, if $x \in (J_c(T) \setminus J_{u_T}) \cap P_\lambda$ we have $u_{T|P_\lambda}(x) = u_T(x)$. Moreover, if $x \in J_{u_T} \cap P_\lambda$, the one-sided approximate limits of u_T are equal to the one-sided limits of the restriction $u_{T|P_\lambda}$, *i.e.*

$$u^+_{T|P_\lambda}(x) = u^+_T(x) \quad \text{and} \quad u^-_{T|P_\lambda}(x) = u^-_T(x),$$

provided that $\langle \nu, \nu_{u_T}(x) \rangle > 0$, where ν is an orienting unit vector to P, compare Theorem 3.2. We finally infer that for $\mathcal{H}^{n-1}$-a.e. point $x \in J_c(T)$ the 1-dimensional restriction (2.13), up to the orientation, does not depend on the choice of the oriented 1-space P and on $\lambda \in \mathbb{R}^{n-1}$, provided that $x \in P_\lambda$ and $\nu_T(x)$ does not belong to P. As a consequence we may and do give the following:

Definition 2.8. *For $\mathcal{H}^{n-1}$-a.e. point $x \in J_c(T)$, the 1-dimensional restriction $\widehat{\pi}_\#(T \llcorner \{x\} \times \mathcal{Y})$ is well-defined by (2.13) for any oriented 1-space P and $\lambda \in \mathbb{R}^{n-1}$ such that $x \in P_\lambda$ and $\langle \nu, \nu_T(x) \rangle > 0$, where ν is the orienting unit vector to P.*

***BV*-energy.** The gap phenomenon and the properties previously described lead us to define the *BV-energy* of a current $T \in \mathcal{E}_{1,1}$-graph$(B^n \times \mathcal{Y})$ as follows.

Definition 2.9. *For $\mathcal{H}^{n-1}$-a.e. point $x \in J_c(T)$ we define $\Gamma_T(x)$ and $\mathcal{L}_T(x)$ by* (1.5) *and* (1.6)*, respectively, where this time $\widehat{\pi}_\#(T \llcorner \{x\} \times \mathcal{Y})$ is the* 1*-dimensional restriction given by Definition* 2.8.

Definition 2.10. *The BV*-energy *of a current $T \in \mathcal{E}_{1,1}$*-graph$(B^n \times \mathcal{Y})$ *is defined for every Borel set $B \subset B^n$ by*

$$\mathcal{E}_{1,1}(T, B \times \mathcal{Y}) := \int_B |\nabla u_T(x)|\, dx + \left| D^C u_T \right| (B) + \int_{J_c(T)\cap B} \mathcal{L}_T(x)\, d\mathcal{H}^{n-1}(x)\,.$$

We also let

$$\mathcal{E}_{1,1}(T) := \mathcal{E}_{1,1}(T, B^n \times \mathcal{Y})\,.$$

Of course, if $T = G_u$ is the current integration of n-forms in $\mathcal{D}^{n,1}(B^n \times \mathcal{Y})$ over the graph of a smooth $W^{1,1}$-function $u : B^n \to \mathcal{Y}$, then

$$\mathcal{E}_{1,1}(u) = \mathcal{E}_{1,1}(G_u) = \|Du\|_{L^1}\,.$$

Definition 2.11. *We denote by* $\mathrm{cart}^{1,1}(B^n \times \mathcal{Y})$ *the class of currents T in $\mathcal{E}_{1,1}$-graph$(B^n \times \mathcal{Y})$ such that $\mathcal{E}_{1,1}(T) < \infty$.*

Lower semicontinuity. Using the lower semicontinuity result in dimension $n = 1$, see Theorem 1.7, and applying arguments as for instance in [7], in Section 3 we will prove in any dimension:

Theorem 2.12. *Let $n \geq 2$ and $T \in$* $\mathrm{cart}^{1,1}(B^n \times \mathcal{Y})$*. For every sequence of smooth maps $\{u_k\} \subset C^1(B^n, \mathcal{Y})$ such that $G_{u_k} \rightharpoonup T$ weakly in $\mathcal{Z}_{n,1}(B^n \times \mathcal{Y})$, we have*

$$\liminf_{k\to\infty} \mathcal{E}_{1,1}(u_k) \geq \mathcal{E}_{1,1}(T)\,.$$

A strong density result. In all the results stated below, we shall always *assume that the first homotopy group $\pi_1(\mathcal{Y})$ is commutative*. We shall prove in any dimension $n \geq 2$

Theorem 2.13. *Let $T \in$* $\mathrm{cart}^{1,1}(B^n \times \mathcal{Y})$*. There exists a sequence of smooth maps $\{u_k\} \subset C^1(B^n, \mathcal{Y})$ such that $G_{u_k} \rightharpoonup T$ weakly in $\mathcal{Z}_{n,1}(B^n \times \mathcal{Y})$ and $\mathcal{E}_{1,1}(u_k) \to \mathcal{E}_{1,1}(T)$ as $k \to \infty$.*

More precisely, in Section 4 we will prove:

Theorem 2.14. *Let $T \in$* $\mathrm{cart}^{1,1}(B^n \times \mathcal{Y})$*. We can find a sequence of currents $\{T_k\} \subset$* $\mathrm{cart}^{1,1}(B^n \times \mathcal{Y})$ *such that*

$$T_k \rightharpoonup T \quad \text{weakly in } \mathcal{Z}_{n,1}(B^n \times \mathcal{Y})\,, \qquad \mathcal{E}_{1,1}(T_k) \to \mathcal{E}_{1,1}(T)$$

and for all k the corresponding function $u_k := u_{T_k}$ in $BV(B^n, \mathcal{Y})$ has no Cantor part, i.e, $|D^C u_k| = 0$ for every k. Moreover, u_k weakly converges to u_T in the BV-sense and

$$\lim_{k\to\infty} |Du_k|(B^n) = |Du_T|(B^n)\,.$$

In Section 5 we will then prove

Theorem 2.15. *Let* $T \in \mathrm{cart}^{1,1}(B^n \times \mathcal{Y})$ *be such that the corresponding* BV*-function* $u_T \in BV(B^n, \mathcal{Y})$ *has no Cantor part, i.e,* $|D^C u_T| = 0$*. There exists a sequence of smooth maps* $\{u_k\} \subset C^1(B^n, \mathcal{Y})$ *such that* $G_{u_k} \rightharpoonup T$ *weakly in* $\mathcal{Z}_{n,1}(B^n \times \mathcal{Y})$ *and the energy* $\mathcal{E}_{1,1}(u_k) \to \mathcal{E}_{1,1}(T)$ *as* $k \to \infty$*.*

By a diagonal argument we then clearly obtain Theorem 2.13.

Relaxed total variation functional. As a consequence, setting

$$\widetilde{\mathcal{E}_{1,1}}(T) := \inf\left\{ \liminf_{k\to\infty} \int_{B^n} |Du_k|\, dx \ :\ \{u_k\} \subset C^1(B^n, \mathcal{Y}),\right.$$
$$\left. G_{u_k} \rightharpoonup T \quad \text{weakly in } \mathcal{Z}_{n,1}(B^n \times \mathcal{Y}) \right\},$$

by Theorems 2.12 and 2.13 we conclude that

$$\mathcal{E}_{1,1}(T) = \widetilde{\mathcal{E}_{1,1}}(T) \qquad \forall\, T \in \mathrm{cart}^{1,1}(B^n \times \mathcal{Y}).$$

Properties. By Theorems 2.12 and 2.13 we readily infer the following lower semicontinuity result.

Proposition 2.16. *Let* $\{T_k\} \subset \mathrm{cart}^{1,1}(B^n \times \mathcal{Y})$ *converge weakly in* $\mathcal{Z}_{n,1}(B^n \times \mathcal{Y})$*,* $T_k \rightharpoonup T$*, to some* $T \in \mathrm{cart}^{1,1}(B^n \times \mathcal{Y})$*. Then*

$$\mathcal{E}_{1,1}(T) \le \liminf_{k\to\infty} \mathcal{E}_{1,1}(T_k).$$

As a consequence of Theorem 2.13, in the final part of this section we prove that the class of Cartesian currents $\mathrm{cart}^{1,1}(B^n \times \mathcal{Y})$ is closed under weak convergence with equibounded energies.

Theorem 2.17. *Let* $\{T_k\} \subset \mathrm{cart}^{1,1}(B^n \times \mathcal{Y})$ *converge weakly in* $\mathcal{Z}_{n,1}(B^n \times \mathcal{Y})$*,* $T_k \rightharpoonup T$*, to some* $T \in \mathcal{D}_{n,1}(B^n \times \mathcal{Y})$*, and* $\sup_k \mathcal{E}_{1,1}(T_k) < \infty$*. Then* $T \in \mathrm{cart}^{1,1}(B^n \times \mathcal{Y})$*.*

By the relative compactness of $\mathcal{E}_{1,1}$-bounded sets in $\mathcal{D}_{n,1}(B^n \times \mathcal{Y})$, we then readily infer the following compactness property.

Proposition 2.18. *Let* $\{T_k\} \subset \mathrm{cart}^{1,1}(B^n \times \mathcal{Y})$ *be such that* $\sup_k \mathcal{E}_{1,1}(T_k) < \infty$*. Then, possibly passing to a subsequence,* $T_k \rightharpoonup T$ *weakly in* $\mathcal{Z}_{n,1}(B^n \times \mathcal{Y})$ *to some* $T \in \mathrm{cart}^{1,1}(B^n \times \mathcal{Y})$*.*

Proof of Theorem 2.17. By Theorem 2.13, and by a diagonal procedure, we may and will assume that $T_k = G_{u_k}$ for some smooth sequence $\{u_k\} \subset C^1(B^n, \mathcal{Y})$. As a consequence, by the first part of this section we infer that T satisfies (2.4) and

(2.6). It then remains to show that the $\mathbb{L}_s(T)$'s in (2.6) are integer multiplicity rectifiable current in $\mathcal{R}_{n-1}(B^n)$. In this case, in fact, since $\|T\|_{\mathcal{E}_{1,1}} < \infty$, we obtain that $T \in \mathcal{E}_{1,1}$-graph$(B^n \times \mathcal{Y})$, see Definition 2.6, and hence, by lower semicontinuity, Theorem 2.12, and the condition $\sup_k \mathcal{E}_{1,1}(G_{u_k}) < \infty$, we conclude that $\mathcal{E}_{1,1}(T) < \infty$, which yields $T \in \text{cart}^{1,1}(B^n \times \mathcal{Y})$, according to Definition 2.11. To prove that the $\mathbb{L}_s(T)$'s are integer multiplicity rectifiable currents we make use of the following slicing argument.

As before, let P be an oriented 1-space in $\mathbb{R}^n$ and $\{P_\lambda\}_{\lambda\in\mathbb{R}^{n-1}}$ the family of oriented straight lines parallel to P. For $\mathcal{H}^{n-1}$-a.e. λ the *slice* $T \llcorner \pi^{-1}(P_\lambda)$ of T over $\pi^{-1}(P_\lambda)$ is well defined on $\mathcal{Z}^{1,1}((B^n \cap P_\lambda) \times \mathcal{Y})$ and $G_{u_k} \llcorner \pi^{-1}(P_\lambda)$ belongs to $\text{cart}^{1,1}((B^n \cap P_\lambda) \times \mathcal{Y})$ for every k. Moreover, since $G_{u_k} \rightharpoonup T$ weakly in $\mathcal{Z}_{n,1}$, for $\mathcal{H}^{n-1}$-a.e. λ, passing to a subsequence we have $G_{u_k} \llcorner \pi^{-1}(P_\lambda) \rightharpoonup T \llcorner \pi^{-1}(P_\lambda)$ weakly in $\mathcal{Z}_{1,1}((B^n \cap P_\lambda) \times \mathcal{Y})$, with $\sup_k \mathbf{M}(G_{u_k} \llcorner \pi^{-1}(P_\lambda)) < \infty$, so that by the closure-compactness of $\text{cart}^{1,1}$ on 1-dimensional domains, we infer that $T \llcorner \pi^{-1}(P_\lambda) \in \text{cart}^{1,1}((B^n \cap P_\lambda) \times \mathcal{Y})$.

Therefore, the 0-dimensional slices $\mathbb{L}_s(T) \llcorner \pi^{-1}(P_\lambda)$ are rectifiable in $\mathcal{R}_0(B^n \cap P_\lambda)$, as $T \llcorner \pi^{-1}(P_\lambda)$ belongs to $\text{cart}^{1,1}((B^n \cap P_\lambda) \times \mathcal{Y})$ and $\mathbb{L}_s(T) \llcorner \pi^{-1}(P_\lambda) = \mathbb{L}_s(T \llcorner \pi^{-1}(P_\lambda))$. Since the $\mathbb{L}_s(T)$'s are flat chains, see Lemma 2.19 below, arguing as in [12], by White's rectifiability criterion [23], see also [3], we infer that $\mathbb{L}_s(T)$ is an integer multiplicity rectifiable current in $\mathcal{R}_{n-1}(B^n)$ for every s, as required. $\square$

Lemma 2.19. *The* $\mathbb{L}_s(T)$*'s are flat chains in* B^n.

Proof. By Theorem 2.13, we may and will assume that T is the weak limit of G_{u_k} for some smooth sequence $\{u_k\} \subset C^1(B^n, \mathcal{Y})$ such that $\sup_k \|u_k\|_{W^{1,1}} < \infty$. The proof follows the same lines as the proof of [17, Theorem 2.15]. Since $u_k \in BV(B^n, \mathcal{Y})$ is smooth, for all k and s we infer that $\mathbb{L}_s(G_{u_k}) := \pi_\#(G_{u_k} \llcorner \widehat{\pi}^\# \omega^s)$ is a flat chain with equibounded *flat norms*. Recall that the flat norm $\mathbf{F}\big(\mathbb{L}_s(G_{u_k})\big)$ of $\mathbb{L}_s(G_{u_k})$ is given by

$$\mathbf{F}\big(\mathbb{L}_s(G_{u_k})\big) := \sup\{\mathbb{L}_s(G_{u_k})(\phi) \mid \phi \in \mathcal{D}^{n-1}(B^n)\,,\ \mathbf{F}(\phi) \le 1\}\,,$$

where

$$\mathbf{F}(\phi) := \max\left\{\sup_{x\in B^n} \|\phi(x)\|\,,\ \sup_{x\in B^n} \|d\phi(x)\|\right\}.$$

Next, since $u_k \rightharpoonup u_T$ weakly in the BV-sense, we deduce that $\{\mathbb{L}_s(G_{u_k})(\phi)\}_k$ is a Cauchy sequence for every ϕ such that $\mathbf{F}(\phi) \le 1$. If $\mathcal{F}^{n-1}(B^n)$ denotes a countable dense subset of smooth forms ϕ in $\mathcal{D}^{n-1}(B^n)$ satisfying $\mathbf{F}(\phi) \le 1$, by a diagonal argument we infer that

$$\sup\left\{\big(\mathbb{L}_s(G_{u_k}) - \mathbb{L}_s(G_{u_h})\big)(\phi) \mid \phi \in \mathcal{F}^{n-1}(B^n)\right\}$$

is small for k, h large. This yields that $\{\mathbb{L}_s(G_{u_k})\}_k$ is a Cauchy sequence with respect to the flat norm, *i.e.*, that

$$\mathbf{F}\big(\mathbb{L}_s(G_{u_k}) - \mathbb{L}_s(G_{u_h})\big) := \sup\left\{\big(\mathbb{L}_s(G_{u_k}) - \mathbb{L}_s(G_{u_h})\big)(\phi) \mid \phi \in \mathcal{D}^{n-1}(B^n)\,,\ \mathbf{F}(\phi) \le 1\right\}$$

is small for k, h large and therefore, due to weak convergence of G_{u_k} to T, that $R_s := \pi_\#(T \llcorner \widehat{\pi}^\# \omega^s)$ is a flat chain. Similarly, by using a trivial extension of Theorem 6.7 below, we infer that $D_s := \pi_\#(G_T \llcorner \widehat{\pi}^\# \omega^s)$ is a flat chain and hence, since $(-1)^{n-1}\,\mathbb{L}_s(T) = R_s - D_s$, compare (2.6) and (2.7), we conclude that $\mathbb{L}_s(T)$ is a flat chain, too. $\square$

3. Lower semicontinuity

In this section we prove Theorem 2.12, by recovering it from the one dimensional case. To this aim, we recall the following properties from BV-functions theory, compare [2, Section 3.11].

One-dimensional restrictions of BV-functions. Let $\Omega \subset \mathbb{R}^n$ be an open set. Given $\nu \in S^{n-1}$ we denote by π_ν the hyperplane in $\mathbb{R}^n$ orthogonal to ν and by Ω_ν the orthogonal projection of Ω on π_ν. For any $y \in \Omega_\nu$ we let

$$\Omega_y^\nu := \{t \in \mathbb{R} \mid y + t\nu \in \Omega\}$$

denote the (non-empty) section of Ω corresponding to y. Accordingly, for any function $u : B \subset \Omega \to \mathbb{R}^N$ and any $y \in B_\nu$ the function $u_y^\nu : B_y^\nu \to \mathbb{R}^N$ is defined by

$$u_y^\nu(t) := u(y + t\nu)\,.$$

Proposition 3.1. *Let $u \in L^1(\Omega, \mathbb{R}^N)$. Then $u \in BV(\Omega, \mathbb{R}^N)$ if and only if there exist n linearly independent unit vectors ν_i such that $u_y^{\nu_i} \in BV(\Omega_y^{\nu_i}, \mathbb{R}^N)$ for $\mathcal{L}^{n-1}$-a.e. $y \in \Omega_{\nu_i}$ and*

$$\int_{\Omega_{\nu_i}} |Du_y^{\nu_i}|(\Omega_y^{\nu_i})\, d\mathcal{L}^{n-1}(y) < \infty \qquad \forall\, i = 1, \dots, n\,.$$

Theorem 3.2. *If $u \in BV(\Omega, \mathbb{R}^N)$ and $\nu \in S^{n-1}$, then*

$$\begin{aligned} \langle Du, \nu\rangle &= \mathcal{L}^{n-1} \llcorner \Omega_\nu \otimes Du_y^\nu\,, \quad & \langle D^a u, \nu\rangle &= \mathcal{L}^{n-1} \llcorner \Omega_\nu \otimes D^a u_y^\nu\,,\\ \langle D^J u, \nu\rangle &= \mathcal{L}^{n-1} \llcorner \Omega_\nu \otimes D^J u_y^\nu\,, & \langle D^C u, \nu\rangle &= \mathcal{L}^{n-1} \llcorner \Omega_\nu \otimes D^C u_y^\nu\,. \end{aligned}$$

In addition, for $\mathcal{L}^{n-1}$-a.e. $y \in \Omega_\nu$ the precise representative *u^* has classical directional derivatives along ν $\mathcal{L}^1$-a.e. in Ω_y^ν, the function $(u^*)_y^\nu$ is a good representative in the equivalence class of u_y^ν, its Jump set is $(J_u)_y^\nu$ and*

$$\frac{\partial u^*}{\partial \nu}(y + t\nu) = \langle \nabla u(y + t\nu), \nu\rangle \qquad \textit{for } \mathcal{L}^1\textit{-a.e. } t \in \Omega_y^\nu\,.$$

Finally, $\sigma(t) := \langle \nu, \nu_u(y + t\nu)\rangle \neq 0$ for $\mathcal{L}^{n-1}$-a.e. $y \in \Omega_\nu$ and $\mathcal{L}^1$-a.e. $t \in \Omega_y^\nu$, and

$$\begin{cases} \lim\limits_{s\downarrow t} u^*(y + s\nu) = u^+(y + t\nu), \ \lim\limits_{s\uparrow t} u^*(y + s\nu) = u^-(y + t\nu) & \textit{if} \quad \sigma(t) > 0\\ \lim\limits_{s\downarrow t} u^*(y + s\nu) = u^-(y + t\nu), \ \lim\limits_{s\uparrow t} u^*(y + s\nu) = u^+(y + t\nu) & \textit{if} \quad \sigma(t) < 0. \end{cases}$$

One-dimensional restrictions of Cartesian currents. If $T \in \mathrm{cart}^{1,1}(B^n, \mathcal{Y})$, taking $\Omega = B^n$, for any $\nu \in S^{n-1}$ the 1-dimensional slice

$$T^\nu_y := T \llcorner (B^n)^\nu_y \times \mathcal{Y}$$

defines a Cartesian current $T^\nu_y \in \mathrm{cart}^{1,1}((B^n)^\nu_y \times \mathcal{Y})$ for $\mathcal{L}^{n-1}$-a.e. $y \in (B^n)_\nu$. Also, by Theorem 3.2 and by Definition 2.10, we infer that the BV-energy of T^ν_y is given for $\mathcal{L}^{n-1}$-a.e. $y \in (B^n)_\nu$ by

$$\begin{aligned}\mathcal{E}_{1,1}(T^\nu_y, A^\nu_y \times \mathcal{Y}) = \int_{A^\nu_y} |\langle \nabla u_T(y+t\nu), \nu\rangle| \, dt + |D^C(u_T)^\nu_y|(A^\nu_y) \\ + \sum_{t\in (J_c(T)\cap A)^\nu_y} \mathcal{L}_T(y+t\nu)\end{aligned} \tag{3.1}$$

for any open set $A \subset B^n$.

Proof of Theorem 2.12. We follow [2, Theorem 5.4], [7]. Since $\{u_k\} \subset C^1(B^n, \mathcal{Y})$ is such that $G_{u_k} \rightharpoonup T$ weakly in $\mathcal{Z}_{n,1}(B^n \times \mathcal{Y})$, for $\mathcal{L}^{n-1}$-a.e. $y \in (B^n)_\nu$ we infer that

$$(G_{u_k})^\nu_y \rightharpoonup T^\nu_y \qquad \text{weakly in} \quad \mathcal{Z}_{1,1}((B^n)^\nu_y \times \mathcal{Y})\,,$$

where

$$(G_{u_k})^\nu_y = G_{(u_k)^\nu_y}\,, \qquad (u_k)^\nu_y(t) := u_k(y+t\nu) \in C^1((B^n)^\nu_y, \mathcal{Y})\,.$$

Therefore, arguing as in the proof of Theorem 1.7, we readily infer that

$$\mathcal{E}_{1,1}(T^\nu_y, A^\nu_y \times \mathcal{Y}) \le \liminf_{k\to\infty} \mathcal{E}_{1,1}((u_k)^\nu_y, A^\nu_y) \tag{3.2}$$

for any open set $A \subset B^n$, where

$$\mathcal{E}_{1,1}((u_k)^\nu_y, A^\nu_y) = \mathcal{E}_{1,1}(G_{(u_k)^\nu_y}, A^\nu_y \times \mathcal{Y}) = \int_{A^\nu_y} |\langle \nabla u_k(y+t\nu), \nu\rangle| \, dt\,.$$

We now denote by ν_T an extension to the countably $\mathcal{H}^{n-1}$-rectifiable set $J_c(T)$ of the outward unit normal to the Jump set J_{u_T}. By the coarea formula, for any $\nu \in S^{n-1}$ and any open set $A \subset B^n$, we have

$$\int_{J_c(T)\cap A} |\langle \nu_T(x), \nu\rangle| \, f(x) \, d\mathcal{H}^{n-1}(x) = \int_{\pi_\nu} \sum_{t\in (J_c(T)\cap A)^\nu_y} f(y+t\nu) \, d\mathcal{L}^{n-1}(y)$$

for any Borel function $f : J_c(T) \cap A \to [0, +\infty]$. Moreover, Theorem 3.2 gives

$$\begin{aligned}\int_A |\langle \nabla u_T, \nu\rangle| \, dx \;&=\; \int_{\pi_\nu} \left(\int_{A^\nu_y} |\nabla (u_T)^\nu_y(t)| \, dt \right) \mathcal{L}^{n-1}(y) \\ \left| \left\langle D^C u_T, \nu \right\rangle \right| (A) \;&=\; \int_{\pi_\nu} \left| D^C (u_T)^\nu_y \right| (A^\nu_y) \, d\mathcal{L}^{n-1}(y)\,.\end{aligned}$$

Therefore, setting for every open set $A \subset B^n$ and $\nu \in S^{n-1}$

$$\begin{aligned}\mathcal{E}_{1,1}(T, A \times \mathcal{Y}, \nu) := \int_A |\langle \nabla u_T, \nu\rangle| \, dx + |\langle D^C u_T, \nu\rangle|(A) \\ + \int_{J_c(T)\cap A} |\langle \nu_T(x), \nu\rangle| \, \mathcal{L}_T(x) \, d\mathcal{H}^{n-1}(x) \,,\end{aligned}$$

by (3.1) we obtain the identity

$$\mathcal{E}_{1,1}(T, A \times \mathcal{Y}, \nu) = \int_{\pi_\nu} \mathcal{E}_{1,1}(T^\nu_y, A^\nu_y \times \mathcal{Y}) \, d\mathcal{L}^{n-1}(y) \,. \tag{3.3}$$

Similarly, for every k we obtain

$$\mathcal{E}_{1,1}(u_k, A, \nu) := \int_A |\langle \nabla u_k, \nu\rangle| \, dx = \int_{\pi_\nu} \mathcal{E}_{1,1}((u_k)^\nu_y, A^\nu_y) \, d\mathcal{L}^{n-1}(y) \,. \tag{3.4}$$

We also notice that

$$\mathcal{E}_{1,1}(T, A \times \mathcal{Y}, \nu) \le \mathcal{E}_{1,1}(T, A \times \mathcal{Y}) \qquad \text{and} \qquad \mathcal{E}_{1,1}(u_k, A, \nu) \le \mathcal{E}_{1,1}(u_k, A) \,.$$

Since

$$\lim_{k\to\infty} \int_{\pi_\nu} \left(\int_{A^\nu_y} |(u_k)^\nu_y - (u_T)^\nu_y| \, dt \right) d\mathcal{L}^{n-1}(y) = \lim_{k\to\infty} \int_A |u_k - u_T| \, dx = 0 \,,$$

we can find a sequence $\{k(h)\}$ such that

$$\liminf_{k\to\infty} \mathcal{E}_{1,1}(u_k, A, \nu) = \lim_{h\to\infty} \mathcal{E}_{1,1}(u_{k(h)}, A, \nu)$$

and $(G_{u_{k(h)}})^\nu_y$ converges to T^ν_y weakly in $\mathcal{Z}_{1,1}(A^\nu_y \times \mathcal{Y})$ as $h \to \infty$ for $\mathcal{L}^{n-1}$-a.e. $y \in \pi_\nu$. The lower semicontinuity property in dimension one, see (3.2), implies then

$$\liminf_{h\to\infty} \mathcal{E}_{1,1}((u_{k(h)})^\nu_y, A^\nu_y) \ge \mathcal{E}_{1,1}(T^\nu_y, A^\nu_y \times \mathcal{Y})$$

for $\mathcal{L}^{n-1}$-a.e. $y \in \pi_\nu$. Integrating both sides on π_ν, using Fatou's lemma and (3.3), (3.4), we get

$$\liminf_{k\to\infty} \mathcal{E}_{1,1}(u_k, A, \nu) = \lim_{h\to\infty} \mathcal{E}_{1,1}(u_{k(h)}, A, \nu) \ge \mathcal{E}_{1,1}(T, A \times \mathcal{Y}, \nu) \,.$$

Let $\lambda := \mathcal{L}^n + \mathcal{L}_T(\cdot)\, \mathcal{H}^{n-1} \llcorner J_c(T) + |D^C u_T|$ and let $\{\nu_i\} \subset S^{n-1}$ be a countable dense sequence. Choosing an $\mathcal{L}^n$-negligible set $E \subset B^n \setminus J_c(T)$ on which $|D^C u_T|$ is concentrated, we can define

$$\varphi_i(x) := \begin{cases} |\langle \nabla u_T(x), \nu_i\rangle| & \text{if} \quad x \in B^n \setminus (E \cup J_c(T)) \\ |\langle \nu_T(x), \nu_i\rangle| \, \mathcal{L}_T(x) & \text{if} \quad x \in J_c(T) \\ \dfrac{|\langle D^C u_T, \nu_i\rangle|}{|D^C u_T|}(x) & \text{if} \quad x \in E \end{cases}$$

and obtain from (3.3) that

$$\liminf_{k\to\infty} \mathcal{E}_{1,1}(u_k, A) \geq \liminf_{k\to\infty} \mathcal{E}_{1,1}(u_k, A, \nu_i) \geq \mathcal{E}_{1,1}(T, A \times \mathcal{Y}, \nu_i) = \int_A \varphi_i \, d\lambda$$

for any $i \in \mathbb{N}$ and any open set $A \subset B^n$. By the superadditivity of the $\liminf$ operator, we obtain that

$$\liminf_{k\to\infty} \mathcal{E}_{1,1}(u_k, B^n) \geq \sum_i \int_{A_i} \varphi_i \, d\lambda$$

for any finite family of pairwise disjoint open sets $A_i \subset B^n$. We now recall that by [2, Lemma 2.35]

$$\int_{B^n} \sup_{i\in\mathbb{N}} \varphi_i \, d\lambda = \sup\left\{ \sum_{i\in I} \int_{A_i} \varphi_i \, d\lambda \right\},$$

where the supremum is taken over all finite sets $I \subset \mathbb{N}$ and all families $\{A_i\}_{i\in I}$ of pairwise disjoint open sets with compact closure in B^n. We then conclude that

$$\begin{aligned} \liminf_{k\to\infty} \mathcal{E}_{1,1}(u_k, B^n) &\geq \int_{B^n} \sup_{i\in\mathbb{N}} \varphi_i \, d\lambda \\ &= \int_{B^n} |\nabla u_T(x)| dx + |D^C u_T|(B^n) + \int_{J_c(T)} \mathcal{L}_T(x) \, d\mathcal{H}^{n-1}(x) \\ &= \mathcal{E}_{1,1}(T, B^n \times \mathcal{Y}). \end{aligned}$$ □

4. The density theorem: part I

In this section we prove Theorem 2.14. To this aim we first recall that every $T \in \operatorname{cart}^{1,1}(B^n \times \mathcal{Y})$ decomposes as

$$T = G_T + S_T, \qquad S_T = \sum_{s=1}^{\bar{s}} \mathbb{L}_s(T) \times \gamma_s \qquad \text{on} \quad \mathcal{Z}^{n,1}(B^n \times \mathcal{Y}),$$

see Definition 2.11. Let $u = u_T \in BV(B^n, \mathcal{Y})$ be the BV-function associated to T, according to Remark 2.7. For every Borel set $B \subset B^n$ we have

$$\mathcal{E}_{1,1}(T, B \times \mathcal{Y}) = \int_B |\nabla u(x)| \, dx + |D^C u|(B) + \int_{J_c(T)\cap B} \mathcal{L}_T(x) \, d\mathcal{H}^{n-1}(x),$$

where $J_c(T)$, $\Gamma_T(x)$, and $\mathcal{L}_T(x)$ are given by (2.12), (1.5), and (1.6), respectively, compare Definition 2.10.

Slicing properties. Similarly to the case of normal currents, for every point $x_0 \in B^n$ and for a.e. radius $r \in (0, r_0)$, where $2r_0 := \operatorname{dist}(x_0, \partial B^n)$, the slice

$$\langle T, d_{x_0}, r\rangle = \langle G_T, d_{x_0}, r\rangle + \langle S_T, d_{x_0}, r\rangle,$$

where $d_{x_0}(x, y) := |x - x_0|$, is a well-defined Cartesian current in $\mathrm{cart}^{1,1}(\partial B_r(x_0) \times \mathcal{Y})$. More precisely, let $u_{(r,x_0)} := u_{|\partial B_r(x_0)}$ be the restriction of u to $\partial B_r(x_0)$, which is a function in $BV(\partial B_r(x_0), \mathcal{Y})$ with jump set satisfying $J_{u_{(r,x_0)}} = J_u \cap \partial B_r(x_0)$ in the $\mathcal{H}^{n-1}$-a.e. sense. The slice $\langle G_T, d_{x_0}, r\rangle$ is an $(n-1)$-dimensional current in BV-graph$(\partial B_r(x_0) \times \mathcal{Y})$ such that its action on forms in $\mathcal{D}^{n-1,1}(\partial B_r(x_0) \times \mathcal{Y})$, according to a straightforward extension of Definition 2.1, depends on the restriction $u_{(r,x_0)}$ and on the 1-dimensional integral chains γ_x in $\mathcal{Y}$ associated to the current $G_T \in BV$-graph$(B^n \times \mathcal{Y})$, so that in particular $\partial \gamma_x = \delta_{u^+_{(r,x_0)}(x)} - \delta_{u^-_{(r,x_0)}(x)}$ for $\mathcal{H}^{n-1}$-a.e. $x \in J_{u_{(r,x_0)}}$. Also,

$$\langle S_T, d_{x_0}, r\rangle = \sum_{s=1}^{\bar{s}} \langle \mathbb{L}_s(T), \delta_{x_0}, r\rangle \times \gamma_s \qquad \text{on} \quad \mathcal{Z}^{n-1,1}(\partial B_r(x_0) \times \mathcal{Y}),$$

where $\delta_{x_0}(x) := |x - x_0|$. Finally, letting

$$J_c(\langle T, d_{x_0}, r\rangle) := J_{u_{(r,x_0)}} \cup \mathcal{L}(\langle T, d_{x_0}, r\rangle),$$

where $\mathcal{L}(\langle T, d_{x_0}, r\rangle)$ denotes the $(n-2)$-rectifiable set given by the union of the sets of positive multiplicity of the $\langle \mathbb{L}_s(T), \delta_{x_0}, r\rangle$'s, we have, in the $\mathcal{H}^{n-1}$-a.e. sense,

$$J_c(\langle T, d_{x_0}, r\rangle) = J_c(T) \cap \partial B_r(x_0),$$

where $J_c(T)$ is given by (2.12). In this case we say that r is a *good radius* for T at x_0. Moreover, by the argument preceding Definition 2.8, we also infer that for any good radius

$$\mathcal{L}_{\langle T, d_{x_0}, r\rangle}(x) = \mathcal{L}_T(x) \qquad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in J_c(\langle T, d_{x_0}, r\rangle).$$

As a consequence, according to Definition 2.10, we infer that the BV-energy of $\langle T, d_{x_0}, r\rangle$ is given by

$$\begin{aligned}\mathcal{E}_{1,1}(\langle T, d_{x_0}, r\rangle, \partial B_r(x_0) \times \mathcal{Y}) = {} & \int_{\partial B_r(x_0)} |\nabla_\tau u_{(r,x_0)}| \, d\mathcal{H}^{n-1} + |D^C_\tau u|(\partial B_r(x_0)) \\ & + \int_{J_c(T) \cap \partial B_r(x_0)} \mathcal{L}_T(x) \, d\mathcal{H}^{n-2}(x),\end{aligned} \tag{4.1}$$

where D_τ and ∇_τ denote the distributional derivative and the approximate gradient with respect to an orthonormal frame τ tangential to $\partial B_r(x_0)$, respectively.

Proof of Theorem 2.14. We make use of an inductive argument on the dimension n. More precisely, we will assume that Theorem 2.13 holds true in dimension $n-1$, and we use Theorem 1.7 in the case $n=2$. Therefore, taking into account the slicing properties previously outlined, we may and will assume that for every $x_0 \in B^n$ and for a.e. radius $r \in (0, r(x_0))$, where $r(x_0) > 0$ is suitably chosen, by the inductive hypothesis we find a sequence of smooth functions $\{v_k\} \subset C^1(\partial B_r(x_0), \mathcal{Y})$ such that

$$G_{v_k} \rightharpoonup \langle T, d_{x_0}, r\rangle \qquad \text{weakly in} \quad \mathcal{Z}_{n-1,1}(\partial B_r(x_0) \times \mathcal{Y})$$

and

$$\int_{\partial B_r(x_0)} |D_\tau v_k| \, d\mathcal{H}^{n-1} \to \mathcal{E}_{1,1}(\langle T, d_{x_0}, r\rangle, \partial B_r(x_0) \times \mathcal{Y}) \,. \tag{4.2}$$

In particular, we have that $v_k \rightharpoonup u_{(r,x_0)}$ weakly in the BV-sense. We divide the proof of Theorem 2.14 in six steps.

Step 1: *Definition of the fine cover* $\mathcal{F}_m$. We define for every $m \in \mathbb{N}$ a suitable *fine cover* $\mathcal{F}_m$ of $B^n \setminus J_c(T)$ consisting of closed balls of radius smaller than $1/m$. To this aim, let μ_d and μ_{Jc} be the mutually singular Radon measures on B^n given for every Borel set $B \subset B^n$ by

$$\begin{aligned} \mu_d(B) &:= \int_B |\nabla u_T(x)| \, dx + |D^C u_T|(B) \,, \\ \mu_{Jc}(B) &:= \int_{J_c(T)\cap B} \mathcal{L}_T(x) \, d\mathcal{H}^{n-1}(x) \,. \end{aligned} \tag{4.3}$$

Definition 2.10 yields that the BV-energy of T decomposes into the "diffuse" and "jump-concentration" part, *i.e.*, setting

$$\mu_T := \mu_d + \mu_{Jc} \,,$$

for every Borel set $B \subset B^n$ we have

$$\mathcal{E}_{1,1}(T, B \times \mathcal{Y}) = \mu_T(B) = \mu_d(B) + \mu_{Jc}(B) \,.$$

By the decomposition of the derivative Du_T, compare [2, Proposition 3.92], we infer that for any point x_0 in $B^n \setminus J_c(T)$ we have

$$\liminf_{r\to 0} \frac{\mu_T(B_r(x_0))}{r^{n-1}} = \liminf_{r\to 0} \frac{|Du|(B_r(x_0))}{r^{n-1}} = 0 \,.$$

Moreover, since $\mu_{Jc} = \mu_{Jc} \llcorner J_c(T)$, where $J_c(T)$ is a countably $\mathcal{H}^{n-1}$-rectifiable set, and $\mu_T(J_c(T)) < \infty$, for every $m \in \mathbb{N}$ we find a closed subset $J_m \subset J_c(T)$ such that

$$J_m \subset J_{m+1} \qquad \text{and} \qquad \mu_T(J_c(T) \setminus J_m) = \mu_{Jc}(J_c(T) \setminus J_m) < \frac{1}{m} \qquad \forall\, m \,.$$

This yields in particular that

$$|D^J u_T|(J_{u_T} \setminus J_m) < \frac{1}{m} \,.$$

Setting now

$$\Omega := B^n \setminus J_c(T)\,,$$

J_m being closed, for every $x_0 \in \Omega$ there exists a positive radius $r = r(x_0, m)$, smaller than the distance of x_0 to the boundary ∂B^n, such that for every $0 < r < r(x_0, m)$

$$\overline{B}_r(x_0) \cap J_m = \emptyset\,.$$

Finally, by (4.1), if $x_0 \in \Omega$, for every $0 < r < r(x_0, m)$ we find a good radius $\rho \in (r/2, r)$ such that

$$\mathcal{E}_{1,1}(\langle T, d_{x_0}, \rho\rangle, \partial B_\rho(x_0) \times \mathcal{Y}) \le \frac{2}{r}\, \mathcal{E}_{1,1}(T, \overline{B}_r(x_0) \times \mathcal{Y})\,.$$

We then denote by $\mathcal{F}_m$ the union of all the closed balls centered at points $x_0 \in \Omega$ and with good radii $0 < r < \min\{r(x_0, m)/2, 1/m\}$ such that

$$\mathcal{E}_{1,1}(\langle T, d_{x_0}, r\rangle, \partial B_r(x_0) \times \mathcal{Y}) \le \frac{2}{r}\, \mathcal{E}_{1,1}(T, \overline{B}_{2r}(x_0) \times \mathcal{Y}) \tag{4.4}$$

and

$$\frac{1}{(2r)^{n-1}}\, \mathcal{E}_{1,1}(T, \overline{B}_{2r}(x_0) \times \mathcal{Y}) \le \frac{1}{m}\,. \tag{4.5}$$

The above construction yields that $\mathcal{F}_m$ is a *fine cover* of Ω such that

$$\bigcup \mathcal{F}_m \subset B^n \setminus J_m\,.$$

Step 2: *Covering argument*. We apply the following extension of the classical Vitali-Besicovitch covering theorem, see e.g. [2, Theorem 2.19], with respect to the positive Radon measure

$$\mu := \mathcal{L}^n + \mu_T = \mathcal{L}^n + \mu_d + \mu_{Jc}\,,$$

where $\mathcal{L}^n$ is the Lebesgue measure and μ_d, μ_{Jc} are given by (4.3). In the sequel, for any closed ball B we will denote by $\widetilde{B}$ the closed ball centered as B and with radius twice the radius of B, *i.e.*,

$$\widetilde{B} := \overline{B}_{2r}(x_0) \qquad \text{if} \qquad B = \overline{B}_r(x_0)\,.$$

Theorem 4.1 (Vitali-Besicovitch). *Let $\Omega \subset \mathbb{R}^n$ be a bounded Borel set, and let $\mathcal{F}$ be a fine cover of Ω made of closed balls. For every positive Radon measure μ in $\mathbb{R}^n$ there is a disjoint countable family $\mathcal{F}'$ of $\mathcal{F}$ such that*

$$\mu\Big(\Omega \setminus \bigcup \mathcal{F}'\Big) = 0\,.$$

Moreover, we have

$$\sum_{B \in \mathcal{F}'} \mu(\widetilde{B}) \le C \cdot \mu(\Omega)\,,$$

where $C = C(n) > 0$ is an absolute constant, only depending on the dimension n.

Proof. Following the notation in [2, Theorem 2.19], setting $A_0 := \Omega$, for every $h \in \mathbb{N}^+$, at the h^{th} step we may and do apply the Besicovitch theorem [2, Theorem 2.17] by selecting the fine cover of A_{h-1} given by all the closed balls B of $\mathcal{F}$ such that the corresponding balls $\widetilde{B}$ are contained in A_{h-1}. Besicovitch's theorem yields the existence of a countable family made of closed balls B which do not intersect more than ξ times and such that their doubles $\widetilde{B}$ do not intersect more that η times, where $\xi = \xi(n)$ and $\eta = \eta(n)$ are absolute constants. Therefore, the disjoint family $\mathcal{G}_h$ satisfies

$$\sum_{B\in\mathcal{G}_h} \mu(\widetilde{B}) \le \eta \cdot \mu(A_{h-1})$$

whereas, letting $A_h := A_{h-1} \setminus \bigcup \mathcal{G}_h$, we have

$$\mu(A_h) \le \delta\, \mu(A_{h-1}), \qquad \delta := 1 - \frac{1}{2\xi} < 1.$$

Therefore, since $\mu(A_h) \le \delta^h \cdot \mu(A_0)$ for every h, we obtain

$$\sum_{B\in\mathcal{G}_h} \mu(\widetilde{B}) \le \eta \cdot \delta^{h-1} \cdot \mu(\Omega)$$

and finally

$$\sum_{B\in\mathcal{F}'} \mu(\widetilde{B}) = \sum_{h=1}^{\infty} \sum_{B\in\mathcal{G}_h} \mu(\widetilde{B}) \le \sum_{h=1}^{\infty} \eta \cdot \delta^{h-1} \cdot \mu(\Omega)$$

which yields the assertion, by taking $C := \eta/(1-\delta)$. □

By Theorem 4.1 we obtain for every m a suitable denumerable disjoint family $\mathcal{F}'_m$ of closed balls contained in $B^n \setminus J_m$ and with radii smaller than $1/m$. We finally label

$$\mathcal{F}'_m = \{B_j\}_{j=1}^{\infty}, \qquad \Omega_m := \bigcup_{j=1}^{\infty} B_j$$

and notice that

$$\mu_{Jc}(\Omega_m) \le \mu_{Jc}(B^n \setminus J_m) < \frac{1}{m} \qquad \text{and} \qquad \mu_d(B^n \setminus \Omega_m) = 0\,. \tag{4.6}$$

Step 3: *Smoothing of the boundary data.* If $B_j = \overline{B}_r(x_0) \in \mathcal{F}'_m$, arguing as in Gagliardo's theorem [11, Theorem 1.II], that states the existence of a $W^{1,1}$-extension of any L^1-function, we are able to modify the boundary datum $\langle T, d_{x_0}, r\rangle$ to a smooth $W^{1,1}$-map with values into $\mathcal{Y}$. This can be done by paying an arbitrary small amount of energy.

More precisely, due to the inductive hypothesis, see (4.2), we find a sequence of smooth maps $\{v_h^{(j)}\} \subset W^{1,1}(\partial B_j, \mathcal{Y})$ such that $\|v_h^{(j)} - u_{|\partial B_j}\|_{L^1(\partial B_j)} \to 0$,

$$G_{v_h^{(j)}} \rightharpoonup \langle T, d_{x_0}, r \rangle \qquad \text{weakly in} \quad \mathcal{Z}_{n-1,1}(\partial B_j \times \mathcal{Y}) \tag{4.7}$$

as $h \to \infty$ and

$$\int_{\partial B_j} |D_\tau v_h^{(j)}| \, d\mathcal{H}^{n-1} \le \mathcal{E}_{1,1}(\langle T, d_{x_0}, r \rangle, \partial B_j \times \mathcal{Y}) \cdot (1 + 2^{-h}) \tag{4.8}$$

for every h. Taking k sufficiently large, we now define a map $W_k^{(j)} \in W^{1,1}(A^r_{\rho_k}, \mathbb{R}^N)$, where $0 < \rho_k < r$ and A^r_ρ denotes the annulus

$$A^r_\rho := \overline{B}_r(x_0) \setminus B_\rho(x_0) \,, \qquad 0 < \rho < r \,,$$

in such a way that $W^{(j)}_{k|\partial B_r(x_0)} = u_{|\partial B_r(x_0)}$ in the sense of traces,

$$W_k^{(j)}\left(x_0 + \rho_k \, \frac{x - x_0}{|x - x_0|} \right) = v_k^{(j)}\left(x_0 + r \, \frac{x - x_0}{|x - x_0|} \right)$$

and the energy $\int_{A^r_{\rho_k}} |DW_k^{(j)}| \, dx$ is arbitrarily small, if $\rho_k \nearrow r$ sufficiently rapidly.

The function $W_k^{(j)}$ is obtained by parametrizing in a sequence of annuli of the type $A^{\rho_{h+1}}_{\rho_h}$, for a suitable sequence $\{\rho_h\}_{h \ge k}$ of radii $\rho_h \nearrow r$, the affine homotopies

$$t_h \, v_h^{(j)} + (1 - t_h) \, v_{h+1}^{(j)} \,, \qquad t_h = t_h(\rho) \in [0, 1] \,, \qquad \rho := |x - x_0| \,,$$

where $t_h(\rho)$ is the affine map such that $t_h(\rho_h) = 1$ and $t_h(\rho_{h+1}) = 0$. Therefore, if we show that for every $t \in [0, 1]$ and $h \ge k$ the L^∞-distance of $t \, v_h^{(j)} + (1-t) \, v_{h+1}^{(j)}$ from $\mathcal{Y}$ is small, we find that

$$\operatorname{dist}(W_k^{(j)}(x), \mathcal{Y}) < \varepsilon_0 \qquad \text{for } \mathcal{L}^n\text{-a.e. } x \in A^r_{\rho_k} \tag{4.9}$$

and hence we may and do define $w_k^{(j)} := \Pi_{\varepsilon_0} \circ W_k^{(j)}$ on $A^r_{\rho_k}$, where Π_{ε_0} is the Lipschitz projection on $\mathcal{Y}$ given by Remark 1.9.

To prove (4.9), due to the L^1-convergence and to (4.8), by applying Poincaré inequality we find an absolute constant $c_n > 0$ such that, if k is sufficiently large, for $\mathcal{H}^{n-1}$-a.e. $x \in \partial B_r(x_0)$ and every $h \ge k$ we have

$$\begin{aligned}
&\int_{\partial B_r(x_0)} |v_h^{(j)}(x) - u(y)| \, d\mathcal{H}^{n-1}(y) \\
&\quad \le \int_{\partial B_r(x_0)} |v_h^{(j)}(x) - v_h^{(j)}(y)| \, d\mathcal{H}^{n-1}(y) + \|v_h^{(j)} - u\|_{L^1(\partial B_r(x_0))} \\
&\quad \le c_n \, r \int_{\partial B_r(x_0)} |D_\tau v_h^{(j)}| \, d\mathcal{H}^{n-1} + \|v_h^{(j)} - u\|_{L^1(\partial B_r(x_0))} \\
&\quad \le 2 \, c_n \, r \cdot \mathcal{E}_{1,1}(\langle T, d_{x_0}, r \rangle, \partial B_j \times \mathcal{Y}) \,.
\end{aligned}$$

As a consequence, by (4.4) and (4.5) we obtain

$$\int_{\partial B_r(x_0)} |v_h^{(j)}(x) - u(y)|\, d\mathcal{H}^{n-1}(y) \leq 2^{n+1} \cdot c_n \cdot \frac{r^{n-1}}{m}$$

and hence, by convexity, for any $t \in [0,1]$ we have

$$\begin{aligned}
&\int_{\partial B_r(x_0)} |t\, v_h^{(j)}(x) + (1-t)\, v_{h+1}^{(j)}(x) - u(y)|\, d\mathcal{H}^{n-1}(y) \\
&\leq \int_{\partial B_r(x_0)} |v_h^{(j)}(x) - u(y)|\, d\mathcal{H}^{n-1}(y) + \int_{\partial B_r(x_0)} |v_{h+1}^{(j)}(x) - u(y)|\, d\mathcal{H}^{n-1}(y) \\
&\qquad\qquad < \mathcal{H}^{n-1}(\partial B_r(x_0)) \cdot \varepsilon_0
\end{aligned}$$

provided that $m \in \mathbb{N}$ is large enough so that $2^{n+2} \cdot c_n \cdot 1/m < \varepsilon_0 \cdot n \cdot \omega_n$, where ω_n is the measure of the unit n-ball. Therefore, arguing as in Schoen-Uhlenbeck density theorem [21], we obtain

$$\operatorname{dist}(t\, v_h^{(j)}(x) + (1-t)\, v_{h+1}^{(j)}(x), \mathcal{Y}) < \varepsilon_0 \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in \partial B_r(x_0)\,, \tag{4.10}$$

which yields (4.9), as required.

We remark that due to the strong convergence (4.7) (4.8), the sequence $\{w_k^{(j)}\}_k$ this way obtained also satisfies the boundary condition

$$\langle G_{w_k^{(j)}}, d_{x_0}, r\rangle = \langle T, d_{x_0}, r\rangle\,. \tag{4.11}$$

Finally, for future use, we extend $w_k^{(j)}$ to the whole ball B_j by the map $\widetilde{w}_k^{(j)} : \overline{B}_{\rho_k}(x_0) \to \mathcal{Y}$ given by

$$\widetilde{w}_k^{(j)}(x) := \begin{cases} w_k^{(j)} \circ \psi_{(r,\sigma)}(x) & \text{if} \quad x \in A_{r-2\sigma}^{r-\sigma} \\ u \circ \phi_{(r,\sigma)}(x) & \text{if} \quad x \in B_{r-2\sigma}(x_0)\,, \end{cases} \tag{4.12}$$

where $\sigma := r - \rho_k$, $\psi_{(r,\sigma)} : A_{r-2\sigma}^{r-\sigma} \to A_{r-\sigma}^{r}$ is the reflection map

$$\psi_{(r,\sigma)}(x) := \big(-|x-x_0| + 2\,(r-\sigma)\big)\,\frac{x-x_0}{|x-x_0|}$$

and $\phi_{(r,\sigma)} : B_{r-2\sigma}(x_0) \to B_r(x_0)$ is the homothetic map

$$\phi_{(r,\sigma)}(x) := x_0 + \frac{r}{r-2\sigma}\,(x-x_0)\,.$$

Notice that $\widetilde{w}_k^{(j)}$ is smooth on $A_{r-2\sigma}^{r-\sigma}$ and that, taking σ small, by the property above we may and do assume that

$$|D\widetilde{w}_k^{(j)}|(\overline{B}_{\rho_k}(x_0)) \leq 2|Du|(\overline{B}_r(x_0))\,. \tag{4.13}$$

Step 4: *Approximation on the balls of* $\mathcal{F}'_m$. Let $B_j = \overline{B}_r(x_0) \in \mathcal{F}'_m$. Making use of arguments from [5], we now define an approximating sequence on B_j.

We first fix some notation. For any $\rho > 0$, we let

$$Q^n_\rho := [-\rho, \rho]^n \subset \mathbb{R}^n$$

denote the n-dimensional cube of side 2ρ and Σ^i_ρ the i-dimensional skeleton of Q^n_ρ, so that $\bigcup \Sigma^{n-1}_\rho = \partial Q^n_\rho$. Let $\|x\| := \max\{|x_1|, \dots, |x_n|\}$, so that

$$Q^n_\rho = \{x \in \mathbb{R}^n \,:\, \|x\| \le \rho\}, \qquad \partial Q^n_\rho = \{x \in \mathbb{R}^n \,:\, \|x\| = \rho\}.$$

If $v : Q^n_\rho \to \mathbb{R}^N$ is any given BV-function, and F is any i-face of Σ^i_ρ, in the sequel we will denote

$$E_{1,1}(v, F) := |Dv_{|F}|(F)$$

where $Dv_{|F}$ is the distributional derivative of the restriction $v_{|F}$ of v to F, and we let

$$E_{1,1}(v, \Sigma^i_\rho) := \sum_{F \in \Sigma^i_\rho} E_{1,1}(v, F).$$

Recall that $\mathcal{Y} \subset \mathbb{R}^N$, and denote by

$$B_{\mathcal{Y}}(y, \varepsilon) := \overline{B}^N(y, \varepsilon) \cap \mathcal{Y}$$

the intersection of $\mathcal{Y}$ with the closed N-ball of radius ε centered at y. If $y \in \mathcal{Y}$ and $0 < \varepsilon < \varepsilon_0$, we let $\Psi_{(y,\varepsilon)} : \mathbb{R}^N \to \overline{B}_{\mathcal{Y}}(y, \varepsilon)$ be the retraction map given by $\Psi_{(y,\varepsilon)}(z) := \Pi_\varepsilon \circ \xi_{(y,\varepsilon)}$, where

$$\xi_{(y,\varepsilon)}(z) := \begin{cases} z & \text{if} \quad z \in \overline{B}^N(y, \varepsilon) \\ \varepsilon \, \dfrac{z - y}{|z - y|} & \text{if} \quad z \in \mathbb{R}^N \setminus \overline{B}^N(y, \varepsilon) \end{cases}$$

and $\Pi_\varepsilon : \mathcal{Y}_\varepsilon \to \mathcal{Y}$ is the projection map given by Remark 1.9. Of course, $\Psi_{(y,\varepsilon)}$ is a Lipschitz continuous function with $\operatorname{Lip} \Psi_{(y,\varepsilon)} = \operatorname{Lip} \Pi_\varepsilon \to 1^+$ as $\varepsilon \to 0^+$.

First, letting $\rho = \rho_k$ from Step 3, by means of a deformation and slicing argument, we may and do define a bilipschitz homeomorphism $\psi_j : \overline{B}_\rho(x_0) \to Q^n_\rho$ such that $\|D\psi_j\|_\infty \le K$, $\|D\psi_j^{-1}\|_\infty \le K$ for some absolute constant $K > 0$, only depending on n. Moreover, we may and do define ψ_j in such a way that

$$\psi_j(\overline{B}_R(x_0)) = Q^n_R \qquad \forall\, R \in (\rho/2, \rho). \tag{4.14}$$

Finally, for any given BV-function $\widetilde{v} : \overline{B}_\rho(x_0) \to \mathcal{Y}$, smooth on $\partial B_\rho(x_0)$, if $v_j : Q^n_\rho \to \mathcal{Y}$ is the corresponding map given by $v_j := \widetilde{v} \circ \psi_j^{-1}$, we also may and do define ψ_j in such a way that

$$E_{1,1}(v_j, \Sigma^i_\rho) \le C \cdot \frac{1}{\rho} \cdot E_{1,1}(v_j, \Sigma^{i+1}_\rho) \qquad \forall\, i = 1, \dots, n-2, \tag{4.15}$$

where $C > 0$ is an absolute constant, not depending on $\widetilde{v}$.

Taking $\widetilde{v} = \widetilde{v}_j := \widetilde{w}_k^{(j)}$ from (4.12), *i.e.*, letting

$$v_j := \widetilde{w}_k^{(j)} \circ \psi_j^{-1} : Q_\rho^n \to \mathcal{Y}, \tag{4.16}$$

by (4.8) and (4.15) we readily infer that

$$E_{1,1}(v_j, \Sigma_\rho^i) \leq 2\, C\, K\, \rho^{i-n+1}\, \mathcal{E}_{1,1}(\langle T, d_{x_0}, r\rangle, \partial B_j \times \mathcal{Y}) \qquad \forall\, i = 1, \ldots, n-1$$

and hence, by (4.4), that

$$E_{1,1}(v_j, \Sigma_\rho^i) \leq \widetilde{C}\, \rho^{i-n}\, \mathcal{E}_{1,1}(T, \overline{B}_{2r}(x_0) \times \mathcal{Y}) \qquad \forall\, i = 1, \ldots, n-1\,. \tag{4.17}$$

On the other hand, since we may assume $\rho > r/2$, due to (4.5) and (4.13), by (4.17) we also obtain

$$\frac{1}{\rho^{i-1}} E_{1,1}(v_j, \Sigma_\rho^i) \leq \widetilde{C}\, \frac{1}{m} \qquad \forall\, i = 1, \ldots, n\,, \tag{4.18}$$

where in the above formulas $\widetilde{C} > 0$ is an absolute constant.

Remark 4.2. Let $\varepsilon_m := 1/\sqrt{m}$. By the Sobolev embedding theorem, if $m \in \mathbb{N}$ is sufficiently large, e.g., $m \geq 4\widetilde{C}^2$, the inequality (4.18), with $i = 1$, yields that the oscillation of v_j on the 1-skeleton Σ_ρ^1 is smaller than $\varepsilon_m/2$, if v_j is smooth. Therefore, the image $v_j(\Sigma_\rho^1)$ is contained in a small geodesic ball $B_{\mathcal{Y}}(y_j, \varepsilon_m/2)$ centered at some given point $y_j \in \mathcal{Y}$. Actually, since the total variation of 1-dimensional BV-functions estimates the oscillation, we infer that the above property holds for BV-function v_j, provided that in (4.18) we consider the total variation of the 1-dimensional restriction of v to Σ_ρ^1. We also notice that

$$\lim_{m \to +\infty} \varepsilon_m \cdot m = +\infty$$

whereas, on account of Remark 1.9,

$$\operatorname{Lip} \Psi_{(y_j, \varepsilon_m)} = \operatorname{Lip} \Pi_{\varepsilon_m} \to 1^+ \qquad \text{as} \quad m \to +\infty\,.$$

The case $n = 2$. In case of dimension $n = 2$, we define $w_j : Q_\rho^2 \to B_{\mathcal{Y}}(y_j, \varepsilon_m)$ by

$$w_j := \Psi_{(y_j, \varepsilon_m)} \circ v_j\,,$$

where v_j is given by (4.16), so that

$$|Dw_j|(Q_\rho^2) =: E_{1,1}(w_j, Q_\rho^2) \leq (\operatorname{Lip} \Pi_{\varepsilon_m}) \cdot E_{1,1}(v_j, Q_\rho^2)\,.$$

Remark 4.2 yields that w_j agrees with v_j on the boundary of Q_ρ^2. Moreover, letting $R := \rho - \sigma$, by (4.12), (4.14) and (4.16) we infer that w_j is smooth on $Q_\rho^2 \setminus Q_R^2$ and that

$$w_j(x) = \Psi_{(y_j, \varepsilon_m)} \circ (u \circ \phi_{(r,\sigma)}) \circ \psi_j^{-1}(x) \qquad \forall\, x \in Q_R^2\,.$$

Since the image of Q_R^2 by w_j is contained in the geodesic ball $B_{\mathcal{Y}}(y_j, \varepsilon_m)$, by means of a convolution argument we can approximate w_j on Q_R^2 by a smooth sequence $v_\varepsilon^{(j)} : Q_R^2 \to \overline{B}^N(y_j, \varepsilon_m)$ which converges in the L^1-sense to $w_{j|Q_R^2}$ and with total variation converging to the total variation $|Dw_j|(Q_R^2)$. We finally set $w_\varepsilon^{(j)} := \Pi_{\varepsilon_m} \circ v_\varepsilon^{(j)} : Q_R^2 \to \mathcal{Y}$, see Remark 1.9, so that clearly $w_\varepsilon^{(j)} \rightharpoonup w_j$ weakly in $BV(Q_R^2, \mathbb{R}^N)$, whereas

$$E_{1,1}(w_\varepsilon^{(j)}, Q_R^2) \le (\operatorname{Lip} \Pi_{\varepsilon_m}) \cdot E_{1,1}(v_\varepsilon^{(j)}, Q_R^2)\,,$$

so that

$$\limsup_{\varepsilon \to 0} E_{1,1}(w_\varepsilon^{(j)}, Q_R^2) \le (\operatorname{Lip} \Pi_{\varepsilon_m})^2 \cdot E_{1,1}(v_j, Q_R^2)\,. \tag{4.19}$$

Moreover, by suitably defining the convolution kernel, we may and do assume that the traces are equal, so that $w^{(j)}_{\varepsilon|\partial Q_R^2} = v^{(j)}_{\varepsilon|\partial Q_R^2} = w_{j|\partial Q_R^2}$. Most importantly, by the construction we may and do assume that the boundaries of the graphs agree on ∂Q_R^2, so that

$$\partial G_{w_\varepsilon^{(j)}} \llcorner \partial Q_R^2 \times \mathcal{Y} = \partial G_{v_\varepsilon^{(j)}} \llcorner \partial Q_R^2 \times \mathcal{Y} = \partial G_{w_j} \llcorner \partial Q_R^2 \times \mathcal{Y}\,. \tag{4.20}$$

Finally, letting $w_\varepsilon^{(j)} = w_j$ on $Q_\rho^2 \setminus Q_R^2$, we define $u_k^{(j)} : \overline{B}_r(x_0) \to \mathcal{Y}$ by

$$u_k^{(j)}(x) := \begin{cases} w_{\varepsilon_k}^{(j)} \circ \psi_j(x) & \text{if} \quad x \in \overline{B}_\rho(x_0) \\ w_k^{(j)}(x) & \text{if} \quad x \in \overline{B}_r(x_0) \setminus \overline{B}_\rho(x_0)\,, \end{cases}$$

where $\rho = \rho_k$ and $\varepsilon_k \searrow 0$ along a sequence.

The case $n \ge 3$. For $\delta := \rho(1-\eta)$, where $\eta := 1/q$ and $q \in \mathbb{N}^+$, we let $\Phi_{(\rho,\delta)} : Q_\rho^n \to Q_\delta^n$ be given by

$$\Phi_{(\rho,\delta)}(x) := (1-\eta)\, x\,.$$

Note that

$$E_{1,1}(v_j \circ \Phi^{-1}_{(\rho,\delta)}, \Sigma_\delta^i) = (1-\eta)^{i-1}\, E_{1,1}(v_j, \Sigma_\rho^i)\,, \tag{4.21}$$

so that (4.18) yields

$$\frac{1}{\delta^{i-1}} E_{1,1}(v_j \circ \Phi^{-1}_{(\rho,\delta)}, \Sigma_\delta^i) \le \widetilde{C}\, \frac{1}{m} \qquad \forall\, i = 1, \dots, n\,. \tag{4.22}$$

Define $w_j : Q_\delta^n \to B_{\mathcal{Y}}(y_j, \varepsilon_m)$ by

$$w_j := \Psi_{(y_j, \varepsilon_m)} \circ v_j \circ \Phi^{-1}_{(\rho,\delta)}\,, \tag{4.23}$$

where v_j is given by (4.16), so that

$$|Dw_j|(Q^n_\delta) =: E_{1,1}(w_j, Q^n_\delta) \le (\mathrm{Lip}\,\Pi_{\varepsilon_m}) \cdot E_{1,1}(v_j \circ \Phi^{-1}_{(\rho,\delta)}, Q^n_\delta)\,.$$

Remark 4.2 yields that w_j agrees with $v_j \circ \Phi^{-1}_{(\rho,\delta)}$ on the 1-skeleton Σ^1_δ of Q^n_δ. Moreover, letting $R := (\rho-\sigma)(1-\eta)$, by (4.12) and (4.14) we infer that w_j is smooth on $Q^n_\delta \setminus Q^n_R$ and that

$$w_j(x) = \Psi_{(y_j,\varepsilon_m)} \circ (u \circ \phi_{(r,\sigma)}) \circ \psi_j^{-1} \circ \Phi^{-1}_{(\rho,\delta)}(x) \qquad \forall\, x \in Q^n_R\,.$$

Now, since the image of Q^n_R by w_j is contained in the geodesic ball $B_{\mathcal{Y}}(y_j, \varepsilon_m)$, as in the case of dimension $n=2$, we approximate w_j by a smooth sequence $v^{(j)}_\varepsilon : Q^n_R \to \overline{B}^N(y_j, \varepsilon_m)$ which converges in the L^1-sense to $w_{j|Q^n_R}$, with total variation converging to the total variation $|Dw_j|(Q^n_R)$. Setting $w^{(j)}_\varepsilon := \Pi_{\varepsilon_m} \circ v^{(j)}_\varepsilon : Q^n_R \to \mathcal{Y}$, we have $w^{(j)}_\varepsilon \rightharpoonup w_j$ weakly in $BV(Q^n_R, \mathbb{R}^N)$, whereas

$$E_{1,1}(w^{(j)}_\varepsilon, Q^n_R) \le (\mathrm{Lip}\,\Pi_{\varepsilon_m}) \cdot E_{1,1}(v^{(j)}_\varepsilon, Q^n_R)\,,$$

so that again we have

$$\limsup_{\varepsilon\to 0} E_{1,1}(w^{(j)}_\varepsilon, Q^n_R) \le (\mathrm{Lip}\,\Pi_{\varepsilon_m})^2 \cdot E_{1,1}(v_j \circ \Phi^{-1}_{(\rho,\delta)}, Q^n_R)\,. \tag{4.24}$$

Moreover, we may and do assume that the traces of $w^{(j)}_\varepsilon$ and w_j on ∂Q^n_R are equal, $w^{(j)}_{\varepsilon|\partial Q^n_R} = w_{j|\partial Q^n_R}$, and that the boundaries of the graphs agree on ∂Q^n_R, *i.e.*,

$$\partial G_{w^{(j)}_\varepsilon} \llcorner \partial Q^n_R \times \mathcal{Y} = \partial G_{w_j} \llcorner \partial Q^n_R \times \mathcal{Y}\,. \tag{4.25}$$

Finally set $w^{(j)}_\varepsilon = w_j$ on $Q^n_\delta \setminus Q^n_R$.

In order to extend the approximating map to $Q^n_\rho \setminus Q^n_\delta$, we use an argument from [5]. If S_h is one of the $(n-1)$-faces of Σ^{n-1}_ρ, where $h = 1, \dots, 2n$, we may and do define a partition of S_h into $(q+1)^{n-1}$ small $(n-1)$-dimensional "cubes" $C_{l,h}$ in such a way that the following facts hold:

i) If $[C_{l,h}]_i$ denotes the i-dimensional skeleton of the boundary of $C_{l,h}$, the restriction of v_j to $[C_{l,h}]_i$ belongs to $W^{1,1}$, for every $i = 1, \dots, n-2$; in particular, v_j is continuous on the 1-skeleton $[C_{l,h}]_1$.

ii) If $n=3$, we have

$$\sum_{l=1}^{(q+1)^2} E_{1,1}(v_j, \partial C_{l,h}) \le K \left(E_{1,1}(v_j, \partial S_h) + \frac{q}{\rho} E_{1,1}(v_j, S_h) \right), \tag{4.26}$$

where $K > 0$ is an absolute constant.

iii) If $n \geq 4$, and $[S_h]_i$ denotes the i-dimensional skeleton of S_h, for every $i = 1, \dots, n-2$ we have

$$\sum_{l=1}^{(q+1)^{n-1}} E_{1,1}(v_j, [C_{l,h}]_i) \leq K \cdot \sum_{t=i}^{n-1} \left(\frac{q}{\rho}\right)^{t-i} \cdot E_{1,1}(v_j, [S_h]_t)\,, \tag{4.27}$$

where $K > 0$ is an absolute constant.

iv) All the $C_{l,h}$'s are bilipschitz homeomorphic to the $(n-1)$-cube $[-\rho/q, \rho/q]^{n-1}$ by linear maps $f_{l,h}$ such that $\|Df_{l,h}\|_\infty \leq K$, $\|Df_{l,h}^{-1}\|_\infty \leq K$.

Moreover, the inequality (4.18), with $i = 2, \dots, n-1$, yields that if $m \in \mathbb{N}$ is sufficiently large, and q satisfies

$$q < \frac{1}{5(n-2)\,\widetilde{C}} \cdot \frac{\varepsilon_m}{2} \cdot m\,,$$

we may and do define the partition of S_h in such a way that

$$E_{1,1}(v_j, [C_{l,h}]_1) \leq \frac{\varepsilon_m}{2} \qquad \forall\, l = 1, \dots, (q+1)^{n-1}\,, \quad \forall\, h = 1, \dots, 2n\,. \tag{4.28}$$

Therefore, in the sequel we will take

$$q := \text{integer part of } (\widehat{C} \cdot \varepsilon_m \cdot m) \tag{4.29}$$

for some fixed constant $\widehat{C} > 0$, say $\widehat{C} := 1/(12\,(n-2)\,\widetilde{C})$.

Remark 4.3. Again by Remark 4.2, since the image $v_j(\Sigma^1_\rho)$ is contained in $B_{\mathcal{Y}}(y_j, \varepsilon_m/2)$, the inequalities in (4.28) yield that the image of $[C_{l,h}]_1$ by v_j is contained in the geodesic ball $B_{\mathcal{Y}}(y_j, \varepsilon_m)$ for every l and h. By (4.23), this yields that the function w_j, and hence the $w_\varepsilon^{(j)}$'s, agrees with $v_j \circ \Phi^{-1}_{(\rho,\delta)}$ on the 1-skeleton $\widetilde{\Sigma}^1_\delta$ of ∂Q^n_δ given by

$$\widetilde{\Sigma}^1_\delta := \Phi_{(\rho,\delta)}\Bigg(\bigcup_{h=1}^{2n} \bigcup_{l=1}^{(q+1)^{n-1}} [C_{l,h}]_1\Bigg).$$

Finally, if $\pi_{(\rho,\delta)} : Q^n_\rho \setminus Q^n_\delta \to \partial Q^n_\rho$ is the projection map $\pi_{(\rho,\delta)}(x) := \rho\, x/\|x\|$, setting

$$\mathcal{M}_{(\rho,\delta)} := \pi^{-1}_{(\rho,\delta)} \circ \Phi_{(\rho,\delta)}\Bigg(\bigcup_{h=1}^{2n} \bigcup_{l=1}^{(q+1)^{n-1}} \partial C_{l,h}\Bigg)$$

it turns out that the $(n-1)$-skeleton

$$\mathcal{N}_{(\rho,\delta)} := \mathcal{M}_{(\rho,\delta)} \cup \partial Q^n_\rho \cup \partial Q^n_\delta$$

is the union of boundary of n-dimensional "cubes" $Q_{l,h}$, satisfying $C_{l,h} \subset \partial Q_{l,h}$ for every l and h, that partition $Q^n_\rho \setminus Q^n_\delta$. Moreover, each $Q_{l,h}$ is bilipschitz homeomorphic to the n-cube $[-\rho/q, \rho/q]^n$ by linear maps $\widetilde{f}_{l,h}$ such that $\|D\widetilde{f}_{l,h}\|_\infty \leq K$, $\|D\widetilde{f}^{-1}_{l,h}\|_\infty \leq K$, where $K > 0$ is an absolute constant.

We now extend the approximating map to the interior of $Q^n_\rho \setminus Q^n_\delta$, first considering the simpler case $n = 3$.

The case $n = 3$. We first set $w_j := v_j$ on ∂Q^3_ρ and

$$w_j := v_j \circ \pi_{(\rho,\delta)}(x) \qquad \text{on} \quad \mathcal{M}_{(\rho,\delta)} .$$

By Remark 4.3, the function w_j is smooth on the 2-skeleton $\mathcal{N}_{(\rho,\delta)}$. We then extend w_j to the whole of $Q^3_\rho \setminus Q^3_\delta$ by means of a radial extension on each cube $Q_{l,h}$, *i.e.*, by setting

$$w_j(x) := w_j\left(\widetilde{f}^{-1}_{l,h}\left(\frac{\rho}{q} \cdot \frac{\widetilde{f}_{l,h}(x)}{\|\widetilde{f}_{l,h}(x)\|}\right)\right), \qquad x \in Q_{l,h} , \qquad \forall\, l,\, h . \tag{4.30}$$

The function w_j this way constructed is smooth on the closure of $Q^3_\rho \setminus Q^3_\delta$, up to a discrete set of points. Moreover, denoting by $C > 0$ an absolute constant, possibly varying from line to line, but not depending on ρ or m, we have

$$E_{1,1}(w_j, Q_{l,h}) \leq C\, \frac{\rho}{q}\, E_{1,1}(w_j, \partial Q_{l,h}) ,$$

whereas

$$E_{1,1}(w_j, \partial Q_{l,h}) \leq C \left(E_{1,1}(v_j, C_{l,h}) + \frac{\rho}{q}\, E_{1,1}(v_j, \partial C_{l,h}) \right).$$

Therefore, by (4.26), and by summing on l and h, we estimate

$$E_{1,1}(w_j, Q^3_\rho \setminus Q^3_\delta) \leq C \left(\frac{\rho}{q}\, E_{1,1}(v_j, \Sigma^2_\rho) + \left(\frac{\rho}{q}\right)^2 E_{1,1}(v_j, \Sigma^1_\rho) \right).$$

Finally, by (4.29) and (4.17) we obtain, for $m > 1/\widehat{C}^2$,

$$E_{1,1}(w_j, Q^3_\rho \setminus Q^3_\delta) \leq C\, \frac{1}{\varepsilon_m \cdot m}\, \mathcal{E}_{1,1}(T, \overline{B}_{2r}(x_0) \times \mathcal{Y}) . \tag{4.31}$$

The case $n \geq 4$. According to Remark 4.3, we first set $w_j := v_j$ on ∂Q^n_ρ and

$$w_j := v_j \circ \pi_{(\rho,\delta)}(x) \qquad \text{on} \quad \pi^{-1}_{(\rho,\delta)}(\widetilde{\Sigma}^1_\delta) .$$

To extend w_j to the whole of $Q^n_\rho \setminus Q^n_\delta$, we argue by iteration on the dimension $i = 3 \ldots, n$. More precisely, if F is any i-dimensional face of $[Q_{l,h}]_i$ with disjoint interior from both ∂Q^n_ρ and ∂Q^n_δ, we extend w_j to the interior of F by means of a suitable radial extension of the boundary datum of w_j on ∂F similar to the one in (4.30), so that

$$E_{1,1}(w_j, F) \le C\, \frac{\rho}{q}\, E_{1,1}(w_j, \partial F)\,.$$

Therefore, by the construction, and for (4.27), we readily infer that

$$E_{1,1}(w_j, Q^n_\rho \setminus Q^n_\delta) \le C \sum_{i=1}^{n-1} \Big(\frac{\rho}{q}\Big)^{n-i} E_{1,1}(v_j, \Sigma^i_\rho)\,,$$

so that by (4.29) and (4.17) we obtain again, for $m > 1/\widehat{C}^2$,

$$E_{1,1}(w_j, Q^n_\rho \setminus Q^n_\delta) \le C\, \frac{1}{\varepsilon_m \cdot m}\, \mathcal{E}_{1,1}(T, \overline{B}_{2r}(x_0) \times \mathcal{Y})\,. \tag{4.32}$$

Remark 4.4. For future use, we notice that for any $n \ge 3$ the function w_j this way constructed is smooth on the closure of $Q^n_\rho \setminus Q^n_\delta$, up to a "smooth" closed $(n-3)$-dimensional set. This yields that the graph of w_j has no boundary in the interior of $Q^n_\rho \setminus Q^n_\delta$, *i.e.*,

$$\partial G_{w_j} = 0 \quad \text{on} \quad \mathcal{Z}^{n-1,1}(\mathrm{int}(Q^n_\rho \setminus Q^n_\delta) \times \mathcal{Y})\,.$$

We finally set for any $n \ge 3$

$$\widetilde{w}^{(j)}_\varepsilon(x) := \begin{cases} w^{(j)}_\varepsilon(x) & \text{if} \quad x \in Q^n_\delta \\ w_j(x) & \text{if} \quad x \in Q^n_\rho \setminus Q^n_\delta \end{cases}$$

and define $u^{(j)}_k : \overline{B}_r(x_0) \to \mathcal{Y}$ by

$$u^{(j)}_k(x) := \begin{cases} \widetilde{w}^{(j)}_{\varepsilon_k} \circ \psi_j(x) & \text{if} \quad x \in \overline{B}_\rho(x_0) \\ w^{(j)}_k(x) & \text{if} \quad x \in \overline{B}_r(x_0) \setminus \overline{B}_\rho(x_0)\,, \end{cases}$$

where $\rho = \rho_k$ and $\varepsilon_k \searrow 0$ along a sequence.

Step 5: *Approximating maps on the whole domain*. For any $n \ge 2$ we define now $u^{(m)}_k : B^n \to \mathcal{Y}$ by

$$u^{(m)}_k(x) := \begin{cases} u^{(j)}_k(x) & \text{if} \quad x \in B_j\,, \quad j \in \mathbb{N} \\ u_T(x) & \text{if} \quad x \in B^n \setminus \Omega_m\,, \end{cases} \qquad \Omega_m := \bigcup_{j=1}^{\infty} B_j\,. \tag{4.33}$$

By Step 4 we know that $u^{(j)}_k \in W^{1,1}(B_j, \mathcal{Y})$ for every j and k. Moreover, by (4.6), and since $u^{(j)}_k = u_T$ on ∂B_j for every j, we infer that $u^{(m)}_k$ is for every k a function in $BV(B^n, \mathcal{Y})$, with null Cantor part, $|D^C u^{(m)}_k| = 0$.

We now deal with the energy estimates of $u_k^{(m)}$, first considering the simpler case $n=2$.

The case $n=2$. By (4.19) and Step 3 we infer that

$$\limsup_{k\to\infty} E_{1,1}(u_k^{(m)},\Omega_m) \le (\mathrm{Lip}\,\Pi_{\varepsilon_m})^2 \cdot |Du_T|(\Omega_m)\,,$$

whereas by (4.6)

$$|Du_T|(\Omega_m) \le \mu_d(\Omega_m) + \frac{1}{m}\,.$$

By a diagonal argument, setting $u_m := u_{k_m}^{(m)}$ for a suitable sequence $k_m \to \infty$ as $m\to\infty$, we infer that

$$\lim_{m\to\infty} |Du_m|(B^2) = |Du_T|(B^2)\,.$$

The case $n \ge 3$. By (4.31) and (4.32) we infer that

$$\sum_{j=1}^{\infty} E_{1,1}(u_k^{(m)}, \psi_j^{-1}(Q_\rho^n \setminus Q_\delta^n)) \le C\,\frac{1}{\varepsilon_m\cdot m} \sum_{j=1}^{\infty} \mathcal{E}_{1,1}(T, \widetilde{B}_j \times \mathcal{Y})\,,$$

whereas by Theorem 4.1, on account of (4.3), we obtain

$$\sum_{j=1}^{\infty} \mathcal{E}_{1,1}(T, \widetilde{B}_j \times \mathcal{Y}) \le C\cdot \Big(\mathcal{E}_{1,1}(T, B^n\times\mathcal{Y}) + \mathcal{L}^n(B^n)\Big) < \infty\,,$$

and $1/(\varepsilon_m \cdot m) \to 0$ as $m\to\infty$, see Remark 4.2. On the other hand, by (4.24), and since $\eta \to 0$ as $m\to\infty$ in (4.21), as in the case $n=2$ we estimate the energy of $u_k^{(m)}$ on the sets $\psi_j^{-1}(Q_\delta^n)$. In particular, setting $u_m := u_{k_m}^{(m)}$ for suitable sequence $k_m\to\infty$ as $m\to\infty$, we infer that

$$\lim_{m\to\infty} \sum_{j=1}^{\infty} E_{1,1}(u_m, \psi_j^{-1}(Q_\delta^n)) = \mu_d(B^n)$$

and hence, by Step 3, that for any $n\ge 2$

$$\lim_{m\to\infty} |Du_m|(B^n) = |Du_T|(B^n)\,. \tag{4.34}$$

Moreover, in any dimension $n \ge 2$, since for every j the radius of the ball B_j in $\mathcal{F}'_m$ is smaller than $1/m$, and $u_k^{(m)} = u_T$ on ∂B_j, the above energy estimates and the Poincaré inequality yield that for m sufficiently large

$$\begin{aligned}\int_{B^n} |u_m - u_T|\,dx &= \sum_{j=1}^{\infty} \int_{B_j} |u_{k_m}^{(m)} - u_T|\,dx \le \sum_{j=1}^{\infty} C_n \cdot \frac{1}{m}\cdot |Du_T|(B_j)\\ &\le C_n\cdot \frac{1}{m}\cdot |Du_T|(B^n)\,,\end{aligned}$$

where $C_n > 0$ is an absolute constant. This proves the L^1-convergence of u_m to u_T as $m \to \infty$, and hence weakly in the BV-sense.

Finally, for future use, we observe that by the definition of u_m, on account of (4.6), the previous construction yields that the jump part of Du_m strictly converges to the jump part of Du_T. Therefore, denoting by

$$\widetilde{D}u_m := D^a u_m + D^C u_m \,, \qquad \widetilde{D}u_T := D^a u_T + D^C u_T \,,$$

the diffuse part of Du_m and Du_T, where we recall that the Cantor part $|D^C u_m|(B^n) = 0$ for every m, by (4.34) we have

$$\widetilde{D}u_m \rightharpoonup \widetilde{D}u_T \qquad \text{and} \qquad |\widetilde{D}u_m|(B^n) \to |\widetilde{D}u_T|(B^n)\,. \tag{4.35}$$

Step 6: *Approximating currents.* For every m and k let $T_k^{(m)} \in \mathcal{D}_{n,1}(B^n \times \mathcal{Y})$ be given by

$$T_k^{(m)} := \sum_{j=1}^{\infty} G_{u_k^{(j)}} \llcorner \operatorname{int}(B_j) \times \mathcal{Y} + T \llcorner (B^n \setminus \Omega_m) \times \mathcal{Y}\,,$$

where $u_k^{(j)} \in W^{1,1}(B_j, \mathcal{Y})$ is defined by (4.33). Since the boundary $\partial G_{u_k^{(j)}} \llcorner \operatorname{int}(B_j) \times \mathcal{Y} = 0$, whereas

$$\partial(G_{u_k^{(j)}} \llcorner \operatorname{int}(B_j) \times \mathcal{Y}) = \langle T, d_{x_0}, r\rangle\,,$$

we readily infer that $T_k^{(m)} \in \operatorname{cart}^{1,1}(B^n \times \mathcal{Y})$, with corresponding function in $BV(B^n, \mathcal{Y})$ given by $u_k^{(m)}$, see (4.33). Setting $T_m := T_{k_m}^{(m)}$, where the sequence $k_m \to \infty$ is defined as in Step 5, by (4.6) and (4.35) we readily infer that

$$\lim_{m\to\infty} \mathcal{E}_{1,1}(T_m, \Omega_m \times \mathcal{Y}) = |\widetilde{D}u_T|(B^n)\,, \tag{4.36}$$

which clearly yields that

$$\lim_{m\to\infty} \mathcal{E}_{1,1}(T_m, B^n \times \mathcal{Y}) = \mathcal{E}_{1,1}(T, B^n \times \mathcal{Y})\,.$$

It therefore remains to show that, possibly taking a subsequence,

$$T_m \rightharpoonup T \qquad \text{weakly in } \mathcal{Z}_{n,1}(B^n \times \mathcal{Y})\,. \tag{4.37}$$

By applying Theorem 2.15, the proof of which is independent of the one of Theorem 2.14, every T_m is the weak limit of a sequence of smooth graphs of maps $v_k^{(m)} \in C^1(B^n, \mathcal{Y})$, with energies converging to the energy of T_m. Therefore, since $\sup_m \mathcal{E}_{1,1}(T_m, B^n \times \mathcal{Y}) < \infty$, arguing as in the first part of Section 2, by a diagonal argument we may and do assume that, possibly passing to a subsequence, T_m

weakly converges in $\mathcal{Z}_{n,1}(B^n \times \mathcal{Y})$ to some current $\widetilde{T} \in \mathrm{cart}^{1,1}(B^n \times \mathcal{Y})$. Similarly, by the lower semicontinuity theorem for smooth graphs, Theorem 2.12, we infer that for any open set $A \subset B^n$ we have

$$\mathcal{E}_{1,1}(\widetilde{T}, A \times \mathcal{Y}) \leq \liminf_{m\to\infty} \mathcal{E}_{1,1}(T_m, A \times \mathcal{Y}) . \tag{4.38}$$

Moreover, since the sequence of functions $\{u_m\} \subset BV(B^n, \mathcal{Y})$ corresponding to the T_m's weakly converges in the BV-sense to $u_T \in BV(B^n, \mathcal{Y})$, we infer that u_T is the BV-function corresponding to $\widetilde{T}$.

We first show that $\widetilde{T}$ agrees with T on $\Omega \times \mathcal{Y}$, where

$$\Omega := B^n \setminus J_c(T) ,$$

$J_c(T)$ being the set of points of jump-concentration of T. Fix $m_0 \in \mathbb{N}$. Since

$$\Omega \subset \Omega_m \subset A_m , \qquad A_m := B^n \setminus J_m ,$$

and $\{J_m\}$ is an increasing sequence of closed sets, for any $m \geq m_0$ we infer that

$$A_{m_0} = \Omega_m \cup [(J_c(T) \setminus J_{m_0}) \setminus \Omega_m] ,$$

with disjoint union. Moreover, we recall that T_m is equal to T out of $\Omega_m \times \mathcal{Y}$. Therefore, since by (4.6)

$$\mathcal{E}_{1,1}(T, [(J_c(T) \setminus J_{m_0}) \setminus \Omega_m] \times \mathcal{Y}) \leq \frac{1}{m_0} ,$$

by (4.38) and (4.36) we obtain

$$\begin{aligned} \mathcal{E}_{1,1}(\widetilde{T}, A_{m_0} \times \mathcal{Y}) &\leq |\widetilde{D}u_T|(B^n) + \liminf_{m\to\infty} \mathcal{E}_{1,1}(T_m, [(J_c(T) \setminus J_{m_0}) \setminus \Omega_m] \times \mathcal{Y}) \\ &\leq |\widetilde{D}u_T|(B^n) + \liminf_{m\to\infty} \mathcal{E}_{1,1}(T, [(J_c(T) \setminus J_{m_0}) \setminus \Omega_m] \times \mathcal{Y}) \\ &\leq |\widetilde{D}u_T|(B^n) + 1/m_0 . \end{aligned}$$

By outer regularity, since $|\widetilde{D}u_T|(J_c(T)) = 0$ and $A_m \searrow \Omega$ as $m \to \infty$, we infer that

$$\mathcal{E}_{1,1}(\widetilde{T}, \Omega \times \mathcal{Y}) \leq |\widetilde{D}u_T|(\Omega) .$$

Therefore, decomposing the energy of $\widetilde{T}$ into its diffuse and jump-concentration part, see (4.3), we infer that the jump-concentration part is concentrated in the jump-concentration set of T, so that

$$J_c(\widetilde{T}) \subset J_c(T) \qquad \text{and} \qquad \widetilde{T} \llcorner \Omega \times \mathcal{Y} = T \llcorner \Omega \times \mathcal{Y} .$$

We now show that $\widetilde{T}$ agrees with T on $J_c(T) \times \mathcal{Y}$, which concludes the proof. As before, since T_m is equal to T out of $\Omega_m \times \mathcal{Y}$, and $\Omega_m \cap J_{m_0} = \emptyset$ if $m \geq m_0$, for every form $\omega \in \mathcal{Z}^{n,1}(B^n \times \mathcal{Y})$ we have

$$\begin{aligned} ((\widetilde{T} - T) \llcorner J_{m_0} \times \mathcal{Y})(\omega) &= ((\widetilde{T} - T_m) \llcorner J_{m_0} \times \mathcal{Y})(\omega) + ((T_m - T) \llcorner J_{m_0} \times \mathcal{Y})(\omega) \\ &= ((\widetilde{T} - T_m) \llcorner J_{m_0} \times \mathcal{Y})(\omega) \to 0 \end{aligned}$$

as $m \to \infty$, by the weak convergence of T_m to $\widetilde{T}$. This yields that

$$\widetilde{T} \llcorner J_{m_0} \times \mathcal{Y} = T \llcorner J_{m_0} \times \mathcal{Y}$$

and finally the assertion, by inner regularity, since $J_m \nearrow J_c(T)$ in the $\mathcal{H}^{n-1}$-sense as $m \to \infty$. □

5. The density theorem: part II

In this section we prove Theorem 2.15. Extending the notation from the previous section, see (4.3), in the sequel for every current $\widetilde{T} \in \mathrm{cart}^{1,1}(B^n \times \mathcal{Y})$ we will denote by $\mu_{Jc,\widetilde{T}}$ the Radon measure on B^n given for every Borel set $B \subset B^n$ by

$$\mu_{Jc,\widetilde{T}}(B) := \int_{J_c(\widetilde{T}) \cap B} \mathcal{L}_{\widetilde{T}}(x)\, d\mathcal{H}^{n-1}(x)\,, \tag{5.1}$$

that corresponds to the jump-concentration part of the BV-energy $\mathcal{E}_{1,1}(\widetilde{T}, B \times \mathcal{Y})$. We also recall that if $\widetilde{T} \in \mathrm{cart}^{1,1}(B^n \times \mathcal{Y})$ satisfies $|D^C u_{\widetilde{T}}| = 0$, for every Borel set $B \subset B^n$

$$\mathcal{E}_{1,1}(\widetilde{T}, B \times \mathcal{Y}) = \int_B |\nabla u_{\widetilde{T}}(x)|\, dx + \mu_{Jc,\widetilde{T}}(B)\,.$$

Moreover, for any $\widetilde{T}$ as above, in this section we will denote by $\mathbf{F}(\widetilde{T})$ the *flat norm* given by

$$\mathbf{F}(\widetilde{T}) := \sup\{\widetilde{T}(\phi) \mid \phi \in \mathcal{Z}^{n-1}(B^n \times \mathcal{Y})\,,\ \mathbf{F}(\phi) \le 1\}\,,$$

where

$$\mathbf{F}(\phi) := \max\left\{ \sup_{z \in B^n \times \mathcal{Y}} \|\phi(z)\|\,,\ \sup_{z \in B^n \times \mathcal{Y}} \|d\phi(z)\| \right\},$$

and we notice that the flat convergence $\mathbf{F}(T_k - T) \to 0$ yields the weak convergence $T_k \rightharpoonup T$ weakly in $\mathcal{Z}_{n,1}(\widetilde{B}^n \times \mathcal{Y})$, compare [22].

Proof of Theorem 2.15. It is based on the following:

Proposition 5.1. *Let $\widetilde{T} \in \mathrm{cart}^{1,1}(B^n \times \mathcal{Y})$ be such that $|D^C u_{\widetilde{T}}|(B^n) = 0$. Let $\varepsilon \in (0, 1/2)$ and $k \in \mathbb{N}$. We can find a current $\widehat{T} \in \mathrm{cart}^{1,1}(B^n \times \mathcal{Y})$ such that*

$$\begin{aligned} &\mathcal{E}_{1,1}(\widehat{T}, B^n \times \mathcal{Y}) \le \mathcal{E}_{1,1}(\widetilde{T}, B^n \times \mathcal{Y}) + \varepsilon^k\,, \qquad && \mathbf{F}(\widehat{T} - \widetilde{T}) \le \varepsilon^k\,, \\ &\mu_{Jc,\widehat{T}}(B^n) \le \frac{1}{2} \cdot \mu_{Jc,\widetilde{T}}(B^n) && \textit{and } |D^C u_{\widehat{T}}| = 0\,. \end{aligned} \tag{5.2}$$

In fact, for any $\varepsilon \in (0, 1/2)$ we apply iteratively Proposition 5.1 as follows. Letting $T_0^\varepsilon := T$, at the k^{th} step, in correspondence of $\widetilde{T} := T_{k-1}^\varepsilon$ we find $\widehat{T} := T_k^\varepsilon$ such that (5.2) holds true. By induction on $k \in \mathbb{N}$, we define $T^\varepsilon := T_\infty^\varepsilon \in \text{cart}^{1,1}(B^n \times \mathcal{Y})$ such that

$$\mathcal{E}_{1,1}(T^\varepsilon, B^n \times \mathcal{Y}) \leq \mathcal{E}_{1,1}(T, B^n \times \mathcal{Y}) + \sum_{k=1}^\infty \varepsilon^k \leq \mathcal{E}_{1,1}(T, B^n \times \mathcal{Y}) + 2\varepsilon$$

and $|D^C u_{T^\varepsilon}| = 0$. Moreover, since for every k

$$\mu_{Jc, T_k^\varepsilon}(B^n) \leq 2^{-k} \cdot \mu_{Jc,T}(B^n) \,,$$

letting $k \to \infty$ we obtain that $\mu_{Jc,T^\varepsilon}(B^n) = 0$. Finally, since

$$\mathbf{F}(T^\varepsilon - T) \leq \sum_{k=1}^\infty \mathbf{F}(T_k^\varepsilon - T_{k-1}^\varepsilon) \leq \sum_{k=1}^\infty \varepsilon^k \leq 2\varepsilon \,,$$

letting $T_k := T^{\varepsilon_k}$ for some sequence $\varepsilon_k \searrow 0$, and $u_k := u_{T_k}$, we infer that the sequence $\{T_k\} \subset \text{cart}^{1,1}(B^n \times \mathcal{Y})$ weakly converges to T with $\mathcal{E}_{1,1}(T_k) \to \mathcal{E}_{1,1}(T)$ as $k \to \infty$. Moreover, since $|D^C u_k|(B^n) = 0$ and $\mu_{Jc,T_k}(B^n) = 0$ for every k, we obtain that $u_k \in W^{1,1}(B^n, \mathcal{Y})$ and that T_k agrees with the current G_{u_k} given by the integration of forms in $\mathcal{Z}^{n,1}(B^n \times \mathcal{Y})$ over the rectifiable graph of u_k, see (2.1), so that $\mathcal{E}_{1,1}(T_k) = \mathcal{E}_{1,1}(u_k)$.

By means of Bethuel's density theorem [5], for every k we find a smooth sequence $\{u_h^{(k)}\}_h \subset C^1(B^n, \mathcal{Y})$ that strongly converges to u_k in the $W^{1,1}$-sense as $h \to \infty$. In fact, even if the first homotopy group $\pi_1(\mathcal{Y})$ is non-trivial, being commutative it is homeomorphic to the first homology group $H_1(\mathcal{Y})$. Therefore, the null-boundary condition

$$\partial G_{u_k} = 0 \quad \text{on} \quad \mathcal{Z}^{n-1,1}(B^n \times \mathcal{Y}) \tag{5.3}$$

allows to remove the $(n-2)$-dimensional singularities, compare [6] and e.g. [16]. Lower dimensional singularities are removed as in [5]. Since the strong convergence yields $G_{u_h^{(k)}} \rightharpoonup G_{u_k}$ with $\mathcal{E}_{1,1}(u_h^{(k)}) \to \mathcal{E}_{1,1}(u_k)$, the assertion follows by means of a diagonal argument. □

Remark 5.2. This is the exact point where the commutativity hypothesis on the first homotopy group $\pi_1(\mathcal{Y})$ is used, in addition to (5.3). If $\pi_1(\mathcal{Y})$ is non-abelian, even in dimension $n = 2$ we find functions $u \in W^{1,1}(B^2, \mathcal{Y})$, smooth outside the origin and satisfying (5.3), such that for every sequence of smooth maps $u_h : B^n \to \mathcal{Y}$ for which $G_{u_h} \rightharpoonup G_u$ weakly in $\mathcal{Z}_{n,1}(B^n \times \mathcal{Y})$ we have

$$\liminf_{h\to\infty} \int_{B^2} |Du_h| \, dx \geq C + \int_{B^2} |Du| \, dx$$

for some absolute constant $C > 0$, compare [17].

Proof of Proposition 5.1. We set $\widetilde{T} = T$, for simplicity, and divide the proof in four steps.

Step 1: *Blow-up argument.* We apply the argument by Federer [9, 4.2.19]. The rectifiable measure $\mu_{Jc,T}$ can be written as

$$\mu_{Jc,T} = \mathcal{L}_T\, \mathcal{H}^{n-1} \llcorner J_c(T)\,,$$

where the jump-concentration set $J_c(T)$ is countably $\mathcal{H}^{n-1}$-rectifiable and the density $\mathcal{L}_T(x)$ is a non-negative $\mathcal{H}^{n-1} \llcorner J_c(T)$-summable function on $J_c(T)$. Therefore, by [9, 3.2.29] there exists a countable family $\mathcal{G}$ of $(n-1)$-dimensional C^1-submanifolds $\mathcal{M}_j$ of B^n such that $\mu_{Jc,T}$-almost all of B^n is covered by $\mathcal{G}$. Moreover, since $\mu_{Jc,T}(B^n) < \infty$, we can find a positive number $\theta > 0$ so that the subset

$$J := \{x \in J_c(T) \mid \mathcal{L}_T(x) > \theta\}$$

satisfies the following properties:

$$\mathcal{H}^{n-1}(J) < \infty \qquad \text{and} \qquad \mu_{Jc}(B^n \setminus J) < \frac{1}{4} \cdot \mu_{Jc,T}(B^n)\,. \tag{5.4}$$

Let $\sigma > 0$ to be fixed. By [9, 2.10.19], by the Vitali-Besicovitch theorem, Theorem 3.2, and by the properties of the class $\operatorname{cart}^{1,1}(B^n \times \mathcal{Y})$ we can find a number $t_\sigma \in (0, 1)$, a countable disjoint family of closed balls B_j, contained in B^n and centered at points in J, and a bilipschitz homeomorphism ψ_σ from B^n onto itself satisfying the properties listed below, where $c > 0$ is an absolute constant, possibly varying from line to line, which is independent of σ and of the radii r_j of the balls B_j.

i) $\mu_{Jc,T}(B^n \setminus \bigcup_j B_j) = 0$.

ii) If $B_j := \overline{B}(p_j, r_j)$, for every j there is a manifold $\mathcal{M}_j$ of $\mathcal{G}$ such that $p_j \in \mathcal{M}_j$.

iii) Since $\mathcal{H}^{n-1}(J) < \infty$, then

$$\sum_{j=1}^{\infty} {r_j}^{n-1} \le c \cdot \mathcal{H}^{n-1}(J) < \infty\,. \tag{5.5}$$

iv) Letting $C_j := B(p_j, t_\sigma r_j) \cap \mathcal{M}_j$, we have

$$\mu_{Jc,T}(B(p_j, r_j) \setminus C_j) \le \sigma \cdot \mu_{Jc,T}(B(p_j, r_j)) \qquad \forall\, j\,. \tag{5.6}$$

v) If $p_j \notin J_{u_T}$, it is a Lebesgue point of u_T whereas, if $p_j \in J_{u_T}$, the one-sided approximate limits of u_T at p_j are well-defined.

vi) The 1-dimensional restriction $\widehat{\pi}_\#(T \llcorner \{p_j\} \times \mathcal{Y})$ is well-defined, compare Definition 2.8, and

$$\widehat{\pi}_\#(T \llcorner \{p_j\} \times \mathcal{Y}) = \Gamma_j$$

for some integral chain $\Gamma_j \in \mathcal{D}_1(\mathcal{Y})$.

vii) If $\eta_{p_j,\lambda} : \mathbb{R}^n \times \mathbb{R}^N \to \mathbb{R}^n \times \mathbb{R}^N$ denotes the "blow-up" map $\eta_{p_j,\lambda}(x,y) := \left(\dfrac{x-p_j}{\lambda}, y\right)$, the limit current

$$S_j(\omega) := \lim_{\lambda\to 0^+} \eta_{p_j,\lambda\#}T(\omega)\,, \qquad \omega \in \mathcal{Z}^{n,1}(B^n \times \mathcal{Y})$$

is well-defined, and the flat distance of T from S_j is small on $B_j \times \mathcal{Y}$, *i.e.*

$$\mathbf{F}(S_j \llcorner B_j \times \mathcal{Y} - T \llcorner B_j \times \mathcal{Y}) \le c\cdot\sigma\cdot {r_j}^{n-1}\,. \tag{5.7}$$

viii) Since $|Du_T|(B) \le \mu_T(B)$, we have

$$\frac{|Du_T|(B(p_j,r_j)\setminus C_j)}{\omega_{n-1}{r_j}^{n-1}} \le c\cdot\sigma\,, \tag{5.8}$$

where ω_{n-1} is the measure of the $(n-1)$-dimensional unit ball.

ix) Since $\mathcal{L}_T(p_j)$ is the $(n-1)$-dimensional density of $\mu_{Jc,T}$ at p_j, we have

$$|\mu_{Jc,T}(B_j) - \mathcal{L}_T(p_j)\cdot\omega_{n-1}\,{r_j}^{n-1}| \le \sigma\cdot\omega_{n-1}\,{r_j}^{n-1}\,. \tag{5.9}$$

x) $\mathrm{Lip}\,\psi_\sigma \le 2$ and $\mathrm{Lip}\,\psi_\sigma^{-1} \le 2$. Moreover, ψ_σ maps bijectively B_j onto B_j, with $\psi_{\sigma|\partial B_j} = Id_{|\partial B_j}$ and $\psi_\sigma(p_j) = p_j$ for all j, and ψ_σ is equal to the identity outside the union of the balls B_j.

xi) $\psi_\sigma(C_j) = B(p_j,\rho_j)\cap(p_j + \mathrm{Tan}(\mathcal{M}_j,p_j))$ for every j, where $\mathrm{Tan}(\mathcal{M}_j,p_j)$ is the $(n-1)$-dimensional tangent space to $\mathcal{M}_j$ at p_j and $\rho_j \in (r_j/2, r_j)$.

As a consequence, defining $T_j^\sigma \in \mathcal{D}_{n,1}(\mathrm{int}(B_j)\times\mathcal{Y})$ for any j by

$$T_j^\sigma := (\psi_\sigma \bowtie \mathrm{Id}_{\mathbb{R}^N})_\#(T \llcorner \mathrm{int}(B_j)\times\mathcal{Y})\,,$$

we infer that T_j^σ belongs to $\mathrm{cart}^{1,1}(\mathrm{int}(B_j)\times\mathcal{Y})$ and its corresponding function $u_j^\sigma := u_{T_j^\sigma} \in BV(\mathrm{int}(B_j),\mathcal{Y})$ is given by

$$u_j^\sigma := (u_T \circ \psi_\sigma^{-1})_{|\,\mathrm{int}(B_j)}\,.$$

Moreover, we clearly have

$$\mu_{Jc,T_j^\sigma} = \psi_{\sigma\#}(\mu_{Jc,T} \llcorner \mathrm{int}(B_j))\,.$$

Step 2: *Approximation on the balls* B_j. We now apply for every j a "dipole construction" to approximate almost all the Jump-concentration part of T_j^σ. Set

$$x = (\widetilde{x}, x_n) \in \mathbb{R}^{n-1}\times\mathbb{R}\,.$$

Without loss of generality we may and will assume that

$$B_j = \overline{B}^n_R\,, \qquad B(p_j,\rho_j) = B^n_r\,, \qquad 0 < r < R\,,$$

where $B^n_r := B^n(0,r)$, so that $R = r_j$ and $r = \rho_j$, and

$$B(p_j,\rho_j) \cap (p_j + \mathrm{Tan}(\mathcal{M}_j, p_j)) = D_r \times \{0\} \subset \mathbb{R}^{n-1} \times \mathbb{R}, \quad D_r := B^{n-1}(0_{\mathbb{R}^{n-1}}, r).$$

Let $y(\widetilde{x}) := (r - |\widetilde{x}|)$ denote the distance of $\widetilde{x}$ from the boundary of the $(n-1)$-disk D_r. For $\delta > 0$ small, let

$$\phi_\delta(x) := (\widetilde{x}, \varphi_\delta(y(\widetilde{x}))x_n)\,, \;\; x \in D_r \times [-1,1]\,, \quad \varphi_\delta(y) := \min\{y,\delta\}\,.$$

Let $\Omega_\delta := \phi_\delta(D_r \times [-1,1])$ be the "neighborhood" of $D_r \times \{0\}$ in B^n_R given by

$$\Omega_\delta = \{(\widetilde{x}, x_n) \mid \widetilde{x} \in D_r\,, \quad \rho \le \varphi_\delta(y(\widetilde{x}))\}\,,$$

where $\rho := |x_n|$, and let

$$\widetilde{\Omega}_\delta := \phi_\delta(D_r \times [-1/2, 1/2]) = \{(\widetilde{x}, x_n) \mid \widetilde{x} \in D_r\,,\; \rho \le \varphi_\delta(y(\widetilde{x}))/2\}\,.$$

Also, set

$$\Omega_{(r,\delta)} := \Omega_\delta \setminus (D_r \times \{0\})\,.$$

Let $v^\sigma_j : (\Omega_\delta \setminus \widetilde{\Omega}_\delta) \to \mathcal{Y}$ be given by $v^\sigma_j(x) := u^\sigma_j \circ \psi^\sigma_j(x)$, where $\psi^\sigma_j : \Omega_\delta \setminus \widetilde{\Omega}_\delta \to \Omega_{(r,\delta)}$ is the bijective map

$$\psi^\sigma_j(\widetilde{x}, x_n) := \left(\widetilde{x}, \left(2 - \frac{\varphi_\delta(y(\widetilde{x}))}{\rho}\right) x_n\right).$$

Since we have

$$|\nabla v^\sigma_j(x)| \le c\, |\nabla u^\sigma_j(\widetilde{x}, (2 - \varphi_\delta(y(\widetilde{x}))/\rho)\, x_n)| \cdot (1 + \varphi_\delta(y(\widetilde{x}))/\rho)\,,$$

and $\varphi_\delta(y(\widetilde{x}))/\rho \in]1/2, 1]$, we infer that $v^\sigma_j \in BV(\Omega_\delta \setminus \widetilde{\Omega}_\delta, \mathcal{Y})$, with

$$\int_{\Omega_\delta \setminus \widetilde{\Omega}_\delta} |\nabla v^\sigma_j|\, dx \le c \int_{\Omega_\delta} |\nabla u^\sigma_j|\, dx\,. \tag{5.10}$$

Moreover, the current

$$\overline{T}^\sigma_j := ((\psi^\sigma_j)^{-1} \bowtie \mathrm{Id}_{\mathbb{R}^N})_\#(T^\sigma_j \llcorner (\mathrm{int}(\Omega_{(r,\delta)}) \times \mathcal{Y}))$$

belongs to $\mathrm{cart}^{1,1}(\mathrm{int}(\Omega_\delta \setminus \widetilde{\Omega}_\delta) \times \mathcal{Y})$, its underlying BV-function is v^σ_j, and $\overline{T}^\sigma_j$ satisfies

$$\mu_{Jc,\overline{T}^\sigma_j}(\mathrm{int}(\Omega_\delta \setminus \widetilde{\Omega}_\delta)) \le \mu_{Jc,T^\sigma_j}(\mathrm{int}(\Omega_{(r,\delta)}))\,,$$

so that by (5.6) we have

$$\mu_{\overline{T}_j^\sigma}(\operatorname{int}(\Omega_\delta \setminus \widetilde{\Omega}_\delta)) \le c\,\sigma\,\mu_{T_j^\sigma}(B_r^n)\,. \tag{5.11}$$

We now define $w_j^\sigma : (\Omega_\delta \setminus \widetilde{\Omega}_\delta) \to \mathbb{R}^N$ by

$$w_j^\sigma(x) := \left(\frac{2\rho}{\varphi_\delta(y(\widetilde{x}))} - 1\right) \cdot v_j^\sigma(\widetilde{x}, x_n) + \left(2 - \frac{2\rho}{\varphi_\delta(y(\widetilde{x}))}\right) \cdot z_j^\pm\,,$$

where $\pm$ is the sign of x_n and $z_j^\pm$ are the one-sided approximate limits of u_j^σ at the point $0 \in J_{u_j^\sigma}$, so that

$$\lim_{\rho \to 0^+} \rho^{-n} \int_{B_\rho^\pm} |u_j^\sigma(x) - z_j^\pm|\, dx = 0\,,$$

if p_j belongs to the jump set of u_j^σ, and they agree with the Lebesgue value of u_j^σ at p_j, otherwise.

If $r - \delta \le |\widetilde{x}| \le r$ and $(r - |\widetilde{x}|)/2 < \rho < (r - |\widetilde{x}|)$, then

$$|\nabla w_j^\sigma|(x) \le \frac{c}{r - |\widetilde{x}|}\, |v_j^\sigma(x) - z_j^\pm| + c\,|\nabla v_j^\sigma(x)|\,,$$

whereas if $|\widetilde{x}| \le r - \delta$ and $\delta/2 < \rho < \delta$, we estimate

$$|\nabla w_j^\sigma|(x) \le \frac{c}{\delta}\, |v_j^\sigma(x) - z_j^\pm| + c\,|\nabla v_j^\sigma(x)|\,.$$

Moreover, by (5.8) and the Poincaré inequality we infer that the oscillation of u_j^σ on the upper and lower half-balls

$$B_r^\pm := \{x \in B_r^n \mid \pm x_n > 0\}$$

is smaller than $c\,\sigma$, so that

$$\|v_j^\sigma(x) - z_j^\pm\|_{\infty, \Omega_\delta \setminus \widetilde{\Omega}_\delta} \le c\,\sigma\,.$$

As a consequence, on account of (5.10) we obtain

$$\begin{aligned} \int_{\Omega_\delta \setminus \widetilde{\Omega}_\delta} |\nabla w_j^\sigma|\, dx &\le c\,\sigma\,\mathcal{L}^n(\Omega_\delta \setminus \widetilde{\Omega}_\delta) + c \int_{\Omega_\delta \setminus \widetilde{\Omega}_\delta} |\nabla v_j^\sigma|\, dx \\ &\le c\,\sigma\,\mathcal{L}^n(\Omega_\delta \setminus \widetilde{\Omega}_\delta) + c \int_{\Omega_\delta} |\nabla u_j^\sigma|\, dx \end{aligned} \tag{5.12}$$

which is small if δ and σ are small, by the absolute continuity. Also, since the oscillation of w_j^σ is smaller than $c\,\sigma$, by projecting w_j^σ into the manifold $\mathcal{Y}$, see

Remark 1.9, we may and will assume that w_j^σ is a function in $BV(\Omega_\delta \setminus \widetilde{\Omega}_\delta, \mathcal{Y})$. We finally observe that

$$w_j^\sigma(\widetilde{x}, \pm\varphi_\delta(y(\widetilde{x}))/2) = z_j^\pm \qquad \forall \widetilde{x} \in D_r .$$

Now, by means of the vertical part of the current $\overline{T}_j^\sigma$, we may and do define a current $\widetilde{T}_j^\sigma \in \mathrm{cart}^{1,1}(\mathrm{int}(\Omega_\delta \setminus \widetilde{\Omega}_\delta) \times \mathcal{Y})$, with underlying BV-function w_j^σ, such that

$$\mu_{Jc,\widetilde{T}_j^\sigma}(\mathrm{int}(\Omega_\delta \setminus \widetilde{\Omega}_\delta)) \le c\,\mu_{Jc,\overline{T}_j^\sigma}(\mathrm{int}(\Omega_\delta \setminus \widetilde{\Omega}_\delta))$$

and $\widetilde{T}_j^\sigma$ satisfies the boundary condition

$$\partial \widetilde{T}_j^\sigma = \partial T_j^\sigma \llcorner \partial\Omega_\delta \times \mathcal{Y} + [\![\, \partial\widetilde{\Omega}_\delta \cap B_r^+ \,]\!] \times \delta_{z_j^+} - [\![\, \partial\widetilde{\Omega}_\delta \cap B_r^- \,]\!] \times \delta_{z_j^-} .$$

In particular, by (5.11) and (5.12), taking δ small, we infer that $\widetilde{T}_j^\sigma$ satisfies the energy estimate

$$\begin{aligned}\mathcal{E}_{1,1}(\widetilde{T}_j^\sigma, \mathrm{int}(\Omega_\delta \setminus \widetilde{\Omega}_\delta) \times \mathcal{Y}) &= \int_{\Omega_\delta \setminus \widetilde{\Omega}_\delta} |\nabla w_j^\sigma|\,dx + \mu_{Jc,\widetilde{T}_j^\sigma}(\mathrm{int}(\Omega_\delta \setminus \widetilde{\Omega}_\delta)) \\ &\le c\,\sigma\, r^{n-1} + c\,\sigma\,\mu_{Jc,T_j^\sigma}(B_r^n) .\end{aligned}$$

Due to the property vi) above, setting

$$\widehat{T}_j^\sigma := \widetilde{T}_j^\sigma + T_j^\sigma \llcorner (B_R^n \setminus \Omega_\delta) \times \mathcal{Y} ,$$

we infer that $\widehat{T}_j^\sigma$ belongs to $\mathrm{cart}^{1,1}((B_R^n \setminus \tilde{\Omega}_\delta) \times \mathcal{Y})$, satisfies the boundary condition

$$\begin{aligned}\partial \widehat{T}_j^\sigma \;=\; & \partial T_j^\sigma \llcorner \partial B_R^n \times \mathcal{Y} - [\![\, \partial D_r \times \{0\} \,]\!] \times \Gamma_j \\ & + [\![\, \partial\widetilde{\Omega}_\delta \cap B_r^+ \,]\!] \times \delta_{z_j^+} - [\![\, \partial\widetilde{\Omega}_\delta \cap B_r^- \,]\!] \times \delta_{z_j^-}\end{aligned} \tag{5.13}$$

and the energy estimate

$$\begin{aligned}\mathcal{E}_{1,1}(\widehat{T}_j^\sigma, (B_R^n \setminus \widetilde{\Omega}_\delta) \times \mathcal{Y}) \;\le\; & \int_{B_R^n} |\nabla u_j^\sigma|\,dx \\ & + c\,\sigma\, r^{n-1} + c\,\sigma\,\mu_{Jc,T_j^\sigma}(B_r^n) .\end{aligned} \tag{5.14}$$

To extend $\widehat{T}_j^\sigma$ to a current in $\mathrm{cart}^{1,1}(\mathrm{int}(B_j) \times \mathcal{Y})$, we notice that $J_c(T_j^\sigma) = \psi_\sigma(J_c(T) \cap \mathrm{int}(B_j))$. Moreover, if $\gamma_j \in \Gamma_T(p_j)$ satisfies (1.7), of course γ_j belongs to $\Gamma_{T_j^\sigma}(p_j)$ and satisfies

$$\mathcal{L}(\gamma_j) = \mathcal{L}_{T_j^\sigma}(p_j) = \mathcal{L}_T(p_j)$$

and $\gamma_{j\#}[\![(0,1)]\!] = \Gamma_j$, see property vi). We define $v_j^\sigma : \widetilde{\Omega}_\delta \to \mathcal{Y}$ by setting

$$v_j^\sigma(x) := \gamma_j\Big(\frac{1}{2} + \frac{x_n}{\varphi_\delta(y(\widetilde{x}))}\Big), \qquad \widetilde{x} \in D_r, \quad \rho \le \varphi_\delta(y(\widetilde{x}))/2,$$

where the orientation of γ_j is chosen in such a way that $\gamma_j(0) = z_j^-$ and $\gamma_0(1) = z_j^+$, so that $\partial[\![\gamma_j]\!] = \delta_{z^+} - \delta_{z^-}$. Since

$$v_j^\sigma(x) := (v \circ \phi_\delta^{-1})(x), \qquad x \in \phi_\delta(D_r \times [-1/2, 1/2]),$$

where $v : D_r \times [-1/2, 1/2] \to \mathcal{Y}$ is given by $v(\widetilde{x}, t) := \widetilde{\gamma}_j(1/2 + t)$, we readily estimate

$$\begin{aligned} \int_{\widetilde{\Omega}_\delta} |Dv_j^\sigma|\, dx &\le \mathcal{L}(\gamma_j) \cdot (\mathcal{L}^{n-1}(D_{r-\delta}) + c\, \mathcal{L}^{n-1}(D_r \setminus D_{r-\delta})) \\ &\le \sigma\, r^{n-1} + \mathcal{L}^{n-1}(D_r) \cdot \mathcal{L}_{T_j^\sigma}(p_j) \end{aligned} \tag{5.15}$$

if $\delta > 0$ is small. Setting now

$$\widetilde{T}_j^{(\sigma)} := \widehat{T}_j^\sigma + G_{v_j^\sigma},$$

where $G_{v_j^\sigma}$ is the current integration over the graph of v_j^σ, the above construction and the boundary condition (5.13) yield that $\widetilde{T}_j^{(\sigma)}$ has no boundary in $\mathrm{int}(B_j) \times \mathcal{Y}$, so that $\widetilde{T}_j^{(\sigma)}$ belongs to $\mathrm{cart}^{1,1}(\mathrm{int}(B_j) \times \mathcal{Y})$. Moreover, by (5.14) and (5.15), on account of the property vi) above, we obtain that

$$\begin{aligned} \mathcal{E}_{1,1}(\widetilde{T}_j^{(\sigma)}, \mathrm{int}(B_j) \times \mathcal{Y}) \;\le\; & \mathcal{E}_{1,1}(T_j^\sigma, B_R^n \times \mathcal{Y}) \\ & + c\, \sigma\, r^{n-1} + c\, \sigma \mu_{Jc, T_j^\sigma}(B_r^n). \end{aligned} \tag{5.16}$$

We finally notice that $\widetilde{T}_j^{(\sigma)}$ agrees with T_j^σ outside $\Omega_\delta \times \mathcal{Y}$.

Step 3: *Flat distance.* We now show that for δ small enough

$$\mathbf{F}(\widetilde{T}_j^{(\sigma)} \llcorner B_R^n \times \mathcal{Y} - T_j^\sigma \llcorner B_R^n \times \mathcal{Y}) \le c \cdot \sigma \cdot R^{n-1}. \tag{5.17}$$

In fact, by the property vii) above the blow-up current

$$\widetilde{S}_j(\omega) := \lim_{\lambda \to 0^+} \eta_{0,\lambda\#} T_j^\sigma(\omega), \qquad \omega \in \mathcal{Z}^{n,1}(B_R^n \times \mathcal{Y})$$

is well-defined, and by property vi) it satisfies

$$\widetilde{S}_j = [\![B_R^+]\!] \times \delta_{z^+} + [\![B_R^-]\!] \times \delta_{z^-} + [\![D_r]\!] \times \Gamma_j,$$

where $\partial\Gamma_j = \delta_{z^+} - \delta_{z^-}$. On the other hand, (5.7) yields that

$$\mathbf{F}(\widetilde{S}_j \llcorner B^n_R \times \mathcal{Y} - T^\sigma_j \llcorner B^n_R \times \mathcal{Y}) \leq c \cdot \sigma \cdot R^{n-1} \,. \tag{5.18}$$

Also, by the definition of v^σ_j we infer that for $\delta > 0$ small

$$\mathbf{F}(\widetilde{S}_j \llcorner \widetilde{\Omega}_\delta \times \mathcal{Y} - G_{v^\sigma_j} \llcorner \widetilde{\Omega}_\delta \times \mathcal{Y}) \leq c \cdot \sigma \cdot r^{n-1} \,.$$

Moreover, the BV-energy of $\widetilde{T}^{(\sigma)}_j$ on $(\Omega_\delta \setminus \widetilde{\Omega}_\delta) \times \mathcal{Y}$ is small if δ is small, whereas $\widetilde{T}^{(\sigma)}_j$ agrees with T^σ_j outside $\Omega_\delta \times \mathcal{Y}$. By (5.18) we then obtain

$$\mathbf{F}(\widetilde{S}_j \llcorner (B^n_R \setminus \widetilde{\Omega}_\delta) \times \mathcal{Y} - \widetilde{T}^{(\sigma)}_j \llcorner (B^n_R \setminus \widetilde{\Omega}_\delta) \times \mathcal{Y}) \leq c \cdot \sigma \cdot R^{n-1}$$

and finally (5.17), as $r \in (R/2, R)$.

Step 4: *Approximation on the whole domain.* Setting now

$$T^{(\sigma)}_j := (\psi_\sigma^{-1} \bowtie \mathrm{Id}_{\mathbb{R}^N})_\#(\widetilde{T}^{(\sigma)}_j \llcorner \mathrm{int}(B_j) \times \mathcal{Y}) \,,$$

by (5.16), since $r = \rho_j \in (r_j/2, r_j)$, we infer that for every j

$$\mathcal{E}_{1,1}(T^{(\sigma)}_j, \mathrm{int}(B_j) \times \mathcal{Y}) \leq \int_{B_j} |\nabla u_T| \, dx + (1 + c\,\sigma)\, \mu_{J_c,T}(B_j) + c\,\sigma\, {r_j}^{n-1} \,, \tag{5.19}$$

whereas by (5.17), since $R = r_j$, we obtain that

$$\mathbf{F}(T^{(\sigma)}_j \llcorner \mathrm{int}(B_j) \times \mathcal{Y} - T \llcorner \mathrm{int}(B_j) \times \mathcal{Y}) \leq c \cdot \sigma \cdot {r_j}^{n-1} \,. \tag{5.20}$$

Let now $T^\sigma \in \mathrm{cart}^{1,1}(B^n \times \mathcal{Y})$ be given by

$$T^\sigma := \sum_{j=1}^\infty T^{(\sigma)}_j + T \llcorner \left(B^n \setminus \bigcup_{j=1}^\infty \mathrm{int}(B_j) \right) \times \mathcal{Y} \,.$$

By (5.19) and (5.5) we obtain that

$$\mathcal{E}_{1,1}(T^\sigma, B^n \times \mathcal{Y}) \leq \int_{B^n} |\nabla u_T| \, dx + (1 + c\,\sigma)\, \mu_{J_c,T}(B^n) + c\,\sigma\, \mathcal{H}^{n-1}(J) \,,$$

so that if $\sigma = \sigma(\varepsilon, k, J, \mu_{J_c,T}) > 0$ is small, we have

$$\mathcal{E}_{1,1}(T^\sigma, B^n \times \mathcal{Y}) \leq \mathcal{E}_{1,1}(T, B^n \times \mathcal{Y}) + \varepsilon^k \,.$$

Moreover, by (5.4) and (5.6), taking σ small, the above construction yields that

$$\begin{aligned}\mu_{Jc,T^\sigma}(B^n) &\le c\sum_{j=1}^{\infty}\mu_{Jc,T}(B_j\setminus C_j)+\mu_{Jc,T}(B^n\setminus J)\\ &\le c\,\sigma\,\mu_{Jc,T}(B^n)+\frac14\,\mu_{Jc,T}(B^n)<\frac12\cdot\mu_{Jc,T}(B^n)\,.\end{aligned}$$

Finally, by (5.20) we have

$$\begin{aligned}\mathbf{F}(T^\sigma-T) &\le \sum_{j=1}^{\infty}\mathbf{F}(T_j^{(\sigma)}\llcorner\,\mathrm{int}(B_j)\times\mathcal{Y}-T\llcorner\,\mathrm{int}(B_j)\times\mathcal{Y})\\ &\le c\cdot\sigma\sum_{j=1}^{\infty}{r_j}^{n-1}<\varepsilon^k\end{aligned}$$

if $\sigma=\sigma(\varepsilon,k)>0$ is small. Since Du_{T^σ} has no Cantor part, the proof is complete. □

6. The total variation of *BV*-functions

Extending the classical notion of total variation of vector-valued maps, to every map $u\in BV(B^n,\mathcal{Y})$ we associate in a natural way its *total variation*, essentially in the sense of Jordan, given for every Borel set $B\subset B^n$ by

$$\mathcal{E}_{TV}(u,B):=\int_B|\nabla u(x)|\,dx+|D^Cu|(B)+\int_{J_u\cap B}\mathcal{H}^1(l_x)\,d\mathcal{H}^{n-1}(x)\,. \tag{6.1}$$

Here, for any $x\in J_u$, we let $\mathcal{H}^1(l_x)$ denote the length of a *geodesic arc* l_x in $\mathcal{Y}$ with initial and final points $u^-(x)$ and $u^+(x)$. Moreover we set

$$\mathcal{E}_{TV}(u):=\mathcal{E}_{TV}(u,B^n)\,.$$

Note that if u is smooth, at least in $W^{1,1}(B^n,\mathcal{Y})$, then

$$\mathcal{E}_{TV}(u,B)=\mathcal{E}_{1,1}(u,B):=\int_B|Du|\,dx\,.$$

Moreover, clearly for every $u\in BV(B^n,\mathcal{Y})$ we have

$$|Du|(B)\le\mathcal{E}_{TV}(u,B)\,.$$

Lower semicontinuity. In a way similar to Theorems 1.7 and 2.12, it is not difficult to prove in any dimension n the following:

Proposition 6.1. *Let $u \in BV(B^n, \mathcal{Y})$. For every sequence of smooth maps $\{u_k\} \subset C^1(B^n, \mathcal{Y})$ such that $u_k \rightharpoonup u$ weakly in the BV-sense, we have*

$$\mathcal{E}_{TV}(u) \leq \liminf_{k\to\infty} \mathcal{E}_{TV}(u_k)\,.$$

The previous definition is motivated by the 1-dimensional case, $n = 1$. In fact, similarly to Theorem 1.8, we can prove the following:

Theorem 6.2. *For every $u \in BV(B^1, \mathcal{Y})$ there exists a sequence of smooth maps $\{u_k\} \subset C^\infty(B^1, \mathcal{Y})$ such that $u_k \rightharpoonup u$ weakly in the BV-sense and $\mathcal{E}_{TV}(u_k) \to \mathcal{E}_{TV}(u)$ as $k \to \infty$.*

Density results for Sobolev maps. If $n \geq 2$, we denote by $R_1^\infty(B^n, \mathcal{Y})$ the set of all the maps $u \in W^{1,1}(B^n, \mathcal{Y})$ which are smooth except on a singular set $\Sigma(u)$ of the type

$$\Sigma(u) = \bigcup_{i=1}^{r} \Sigma_i\,, \qquad r \in \mathbb{N}\,,$$

where Σ_i is a smooth $(n-2)$-dimensional subset of B^n with smooth boundary, if $n \geq 3$, and Σ_i is a point if $n = 2$. The following density results appear in [5].

Theorem 6.3. *The class $R_1^\infty(B^n, \mathcal{Y})$ is strongly dense in $W^{1,1}(B^n, \mathcal{Y})$.*

Theorem 6.4. *The class $C^1(B^n, \mathcal{Y})$ is dense in $R_1^\infty(B^n, \mathcal{Y})$ in the strong $W^{1,1}$-topology if and only if $\pi_1(\mathcal{Y}) = 0$.*

Using arguments from the proof of Theorem 2.13, it is not difficult to extend Theorem 6.3 to maps in $BV(B^n, \mathcal{Y})$, by proving:

Theorem 6.5. *For every $u \in BV(B^n, \mathcal{Y})$ there exists a sequence of maps $\{u_k\} \subset R_1^\infty(B^n, \mathcal{Y})$ such that $u_k \rightharpoonup u$ as $k \to \infty$ weakly in the BV-sense and*

$$\lim_{k\to\infty} \int_{B^n} |Du_k|\, dx = \mathcal{E}_{TV}(u, B^n)\,. \tag{6.2}$$

As a consequence, by using Theorem 6.4 we immediately obtain:

Corollary 6.6. *Suppose that $\pi_1(\mathcal{Y}) = 0$. For every $u \in BV(B^n, \mathcal{Y})$ there exists a sequence of smooth maps $\{u_k\} \subset C^1(B^n, \mathcal{Y})$ such that $u_k \rightharpoonup u$ as $k \to \infty$ weakly in the BV-sense and* (6.2) *holds true.*

Currents carried by BV-functions. Following Section 2, the structure of a function u in $BV(B^n, \mathcal{Y})$ suggests to associate to u a suitable current $G = T_u \in BV$-graph$(B^n \times \mathcal{Y})$, see Definition 2.1, where the function $u(T_u) \in BV(B^n, \mathcal{Y})$ is equal to u and the γ_x's in the definition of the jump part G_u^J agree for every $x \in J_u$ with an oriented geodesic arc l_x in $\mathcal{Y}$ with initial and final points respectively given by $u^-(x)$ and $u^+(x)$, so that $\partial[\![\, l_x \,]\!] = \delta_{u^+(x)} - \delta_{u^-(x)}$. We notice that the definition of T_u depends on the choice of the geodesics l_x. In particular, if $u \in W^{1,1}(B^n, \mathcal{Y})$, clearly $T_u = T_u^a$ and hence T_u agrees with the current G_u integration of forms in $\mathcal{D}^{n,1}(B^n \times \mathcal{Y})$ over the rectifiable graph of u, see (2.1). Now, Definition 2.5 yields that the parametric variational integral $\mathcal{F}_{1,1}$ associated to the total variation integral is such that for every Borel set $B \subset B^n$

$$\mathcal{F}_{1,1}(T_u, B \times \mathcal{Y}) = \mathcal{E}_{TV}(u, B) \qquad \forall\, u \in BV(B^n, \mathcal{Y})\,.$$

Moreover, arguing as in the proof of Theorem 2.13, we readily extend Theorems 6.2 and 6.5 by proving in any dimension $n \geq 2$

Theorem 6.7. *For every $u \in BV(B^n, \mathcal{Y})$ we find the existence of a sequence of maps $\{u_k\} \subset R_1^\infty(B^n, \mathcal{Y})$ such that $u_k \rightharpoonup u$ weakly in the BV-sense, $G_{u_k} \rightharpoonup T_u$ weakly in $\mathcal{Z}_{n,1}(B^n \times \mathcal{Y})$ and*

$$\lim_{k\to\infty} \int_{B^n} |Du_k|\, dx = \mathcal{E}_{TV}(u, B^n)\,.$$

Remark 6.8. If $n \geq 2$ in general the current T_u has a non zero boundary in $B^n \times \mathcal{Y}$, compare Remark 2.2. However, as shown by Proposition 6.9 below, ∂T_u is null on every $(n-1)$-form $\widetilde{\omega}$ in $B^n \times \mathcal{Y}$ which has no "vertical" differentials. To this purpose, following Proposition 2.3, any smooth $(n-1)$-form $\widetilde{\omega} \in \mathcal{D}^{n-1}(B^n \times \mathcal{Y})$ with no vertical differentials can be written as $\widetilde{\omega} := \omega_\varphi \wedge \eta$ for some $\eta \in C_0^\infty(\mathcal{Y})$ and $\varphi = (\varphi^1, \dots, \varphi^n) \in C_0^\infty(B^n, \mathbb{R}^n)$, where ω_φ is given by (2.5). Since $d_x\widetilde{\omega} = d\omega_\varphi \wedge \eta = \operatorname{div} \varphi(x)\, \eta(y)\, dx$, by Definition 2.1 we have

$$\begin{aligned} \partial_x T_u(\widetilde{\omega}) &:= T_u(d_x\widetilde{\omega}) = T_u(\operatorname{div} \varphi(x)\, \eta(y)\, dx) \\ &= \int_{B^n} \operatorname{div}\varphi(x) \cdot \eta(u(x))\, dx\,. \end{aligned}$$

We now show that $\partial_y T_u(\widetilde{\omega}) = -\partial_x T_u(\widetilde{\omega})$, which yields the assertion.

Proposition 6.9. *We have*

$$\begin{aligned} \partial_y T_u(\omega_\varphi \wedge \eta) &:= T_u(d_y(\omega_\varphi \wedge \eta)) \\ &= -\int_{B^n} \operatorname{div}\varphi(x) \cdot \eta(u(x))\, dx =: \langle D(\eta \circ u), \varphi \rangle\,. \end{aligned}$$

Proof. Since

$$\begin{aligned} d_y(\omega_\varphi \wedge \eta) &= (-1)^{n-1} \omega_\varphi \wedge d_y\eta \\ &= \sum_{j=1}^N \sum_{i=1}^n (-1)^{n-i} \varphi^i(x) \frac{\partial \eta}{\partial y^j}(y)\, \widehat{dx^i} \wedge dy^j \end{aligned}$$

taking $\phi_i^j = \varphi^i \, \eta_{,y_j}$ in (2.2), by the definition of T_u we infer

$$\begin{aligned}(-1)^{n-1} T_u(\omega_\varphi \wedge d_y \eta) \; = \; & \sum_{j=1}^N \int_{B^n} \frac{\partial \eta}{\partial y^j}(u(x)) \langle \nabla u^j(x), \varphi(x)\rangle \, dx \\ & + \sum_{j=1}^N \int_{B^n} \frac{\partial \eta}{\partial y^j}(u(x)) \, \varphi(x) \, dD^C u^j \\ & + \int_{J_u} \big(\eta(u^+(x)) - \eta(u^-(x)\big) \langle \varphi(x), \nu(x)\rangle \, d\mathcal{H}^{n-1} \, .\end{aligned}$$

Therefore, by the chain rule formula for the distributional derivative of $\eta \circ u$, compare [2], we obtain the assertion, as

$$T_u(d_y(\omega_\varphi \wedge \eta)) = (-1)^{n-1} T_u(\omega_\varphi \wedge d_y \eta)) = \langle D(\eta \circ u), \varphi \rangle \, . \qquad \square$$

Remark 6.10. If G is any current in BV-graph$(B^n \times \mathcal{Y})$ with corresponding function $u(G) \in BV(B^n, \mathcal{Y})$ equal to u, see Definition 2.1, arguing as in Proposition 6.9 we obtain again that

$$\partial_x G(\omega_\varphi \wedge \eta) = -\partial_y G(\omega_\varphi \wedge \eta) = \int_{B^n} \mathrm{div}\varphi(x) \cdot \eta(u(x)) \, dx \, .$$

Example 6.11. Of course, compare Section 2, every Cartesian current T in $\mathrm{cart}^{1,1}(B^n \times \mathcal{Y})$ may be decomposed as

$$T = T_u + S_T \qquad \text{on} \quad \mathcal{Z}^{n,1}(B^n \times \mathcal{Y}) \, , \tag{6.3}$$

where $u = u_T \in BV(B^n, \mathcal{Y})$ is the BV-function corresponding to T and $T_u \in BV$-graph$(B^n \times \mathcal{Y})$ is defined as above, by means of geodesic arcs connecting $u^-(x)$ and $u^+(x)$ at the points x in the jump set J_u. However, even in dimension $n = 1$ and in the particular case $\mathcal{Y} = S^1$, the unit sphere, in general it may happen that the BV-energy of T cannot be recovered by the sum of the BV-energies of its component T_u and S_T in (6.3). If $\mathcal{Y} = S^1$, in fact, we have $S_{T,\mathrm{sing}} = 0$, *i.e.*, the equivalence classes of elements in $\mathrm{cart}^{1,1}(B^n \times S^1)$ have a unique representative, and the energies $\mathcal{E}_{1,1}(T)$ and $\mathcal{F}_{1,1}(T)$ are equal, *i.e.*, no gap phenomenon occurs. Consider the current $T^\theta \in \mathrm{cart}^{1,1}(B^1 \times S^1)$ given by

$$T^\theta := [\![(-1,0)]\!] \times \delta_{P_0} + [\![(0,1)]\!] \times \delta_{P_\theta} + \delta_0 \times \gamma_\theta \, , \qquad \theta \in [0, 2\pi] \, ,$$

where $P_\theta = (\cos\theta, \sin\theta)$ and γ_θ is the simple arc in S^1 connecting the points P_0 and P_θ in the counterclockwise sense. If $\pi < \theta < 2\pi$ we clearly have

$$T_u = [\![(-1,0)]\!] \times \delta_{P_0} + [\![(0,1)]\!] \times \delta_{P_\theta} + \delta_0 \times \widetilde{\gamma}_\theta \, ,$$

where $\widetilde{\gamma}_\theta$ is the simple arc in S^1 connecting the points P_0 and P_θ in the clockwise sense, so that we may decompose T^θ as in (6.3) with $S_T = \delta_0 \times [\![S^1]\!]$. Since

$$\mathcal{F}_{1,1}(T_u) = \mathcal{H}^1(\widetilde{\gamma}_\theta) = 2\pi - \theta \,, \qquad \mathcal{F}_{1,1}(S_T) = 2\pi \,,$$

we infer that the sum of the energies $\mathcal{F}_{1,1}(T_u) + \mathcal{F}_{1,1}(S_T)$ is greater than the energy of T^θ, as clearly

$$\mathcal{E}_{1,1}(T^\theta) = \mathcal{F}_{1,1}(T^\theta) = \mathcal{H}^1(\gamma_\theta) = \theta \,.$$

7. The relaxed *BV*-energy of functions

In this section we analyze the lower semicontinuous envelope of the total variation, defined for every function $u \in BV(B^n, \mathcal{Y})$ by

$$\widetilde{\mathcal{E}_{TV}}(u) := \inf\left\{ \liminf_{k\to\infty} \int_{B^n} |Du_k|\, dx \;\mid\; \{u_k\} \subset C^1(B^n, \mathcal{Y}) \,, \right.$$
$$\left. u_k \rightharpoonup u \quad \text{weakly in the } BV\text{-sense} \right\} .$$

Remark 7.1. Of course one may equivalently require that $u_k \to u$ strongly in $L^1(B^n, \mathbb{R}^N)$.

We first recall the following facts.

Definition 7.2. *For every* $k = 2, \dots, n$ *and* $\Gamma \in \mathcal{D}_{n-k}(B^n)$*, we denote by*

$$m_{i,B^n}(\Gamma) := \inf\{\mathbf{M}(L) \mid L \in \mathcal{R}_{n-k+1}(B^n) \,, \quad (\partial L) \llcorner B^n = \Gamma\}$$

the integral mass *of* Γ *and by*

$$m_{r,B^n}(\Gamma) := \inf\{\mathbf{M}(D) \mid D \in \mathcal{D}_{n-k+1}(B^n) \,, (\partial D) \llcorner B^n = \Gamma\}$$

the real mass *of* Γ. *Moreover, in case* $m_{i,B^n}(\Gamma) < \infty$*, we say that an integer multiplicity rectifiable current* $L \in \mathcal{R}_{n-k+1}(B^n)$ *is an* integral minimal connection *of* Γ *if* $(\partial L) \llcorner B^n = \Gamma$ *and* $\mathbf{M}(L) = m_{i,B^n}(\Gamma)$.

We also recall that by Federer's theorem [10], and by Hardt-Pitts' result [18], respectively, in the cases $k = n$ and $k = 2$ we have that

$$m_{i,B^n}(\Gamma) = m_{r,B^n}(\Gamma) \,. \tag{7.1}$$

Vertical homology classes. Let $u \in W^{1,1}(B^n, \mathcal{Y})$ and let G_u be the current integration of forms in $\mathcal{D}^{n,1}(B^n \times \mathcal{Y})$ over the rectifiable graph of u, see (2.1). We have that $\partial G_u(\omega) = 0$ if $\omega \in \mathcal{D}^{n-1,1}(B^n \times \mathcal{Y})$ with $\omega^{(1)} = 0$ or $d_y\omega = 0$. Setting

$$\mathcal{B}^{p,1}(B^n \times \mathcal{Y}) := \{\omega \in \mathcal{D}^{p,1}(B^n \times \mathcal{Y}) \mid \exists\, \eta \in \mathcal{D}^{p-1,0}(B^n \times \mathcal{Y}) \;:\; \omega^{(1)} = d_y\eta\}$$

and

$$\mathcal{H}^{p,1}(B^n \times \mathcal{Y}) := \frac{\mathcal{Z}^{p,1}(B^n \times \mathcal{Y})}{\mathcal{B}^{p,1}(B^n \times \mathcal{Y})},$$

then $\partial G_u = 0$ on $\mathcal{B}^{n-1,1}(B^n \times \mathcal{Y})$ and $\partial_y \partial G_u = 0$, whence $\partial G_u(\omega)$ depends only on the cohomology class of $\omega \in \mathcal{Z}^{n-1,1}(B^n \times \mathcal{Y})$. As a consequence ∂G_u induces a functional $(\partial G_u)_\star$ on $\mathcal{H}^{n-1,1}(B^n \times \mathcal{Y})$ given by

$$(\partial G_u)_\star(\omega + \mathcal{B}^{n-1,1}) := \partial G_u(\omega + \mathcal{B}^{n-1,1}) = \partial G_u(\omega), \qquad \omega \in \mathcal{Z}^{n-1,1},$$

compare [14], Vol. II, Section 5.4.1. Therefore, since

$$\mathcal{H}^{p,1}(B^n \times \mathcal{Y}) \simeq \mathcal{D}^{p-1}(B^n) \otimes H^1_{dR}(\mathcal{Y}),$$

the homology map $(\partial G_u)_\star$ is uniquely represented as an element of $\mathcal{D}_{n-2}(B^n; H_1(\mathcal{Y}; \mathbb{R}))$. More explicitly, if $\phi \in \mathcal{D}^{n-2}(B^n)$, we have $[(\partial G_u)_\star(\phi)] \in H_1(\mathcal{Y}; \mathbb{R})$ and for $s = 1, \dots, \overline{s}$

$$\langle (\partial G_u)_\star(\phi), [\omega^s] \rangle = \partial G_u(\pi^\# \phi \wedge \widehat{\pi}^\# \omega^s),$$

$\langle , \rangle$ denoting the de Rham duality between $H_1(\mathcal{Y}; \mathbb{R})$ and $H^1_{dR}(\mathcal{Y})$: in general $(\partial G_u)_\star$ is non-trivial.

Singularities of Sobolev maps. Following [14], Vol. II, Section 5.4.2, we now set

$$\mathbb{P}(u) := (\partial G_u)_\star \in \mathcal{D}_{n-2}(B^n; H_1(\mathcal{Y}; \mathbb{R}))$$

and for each $\omega \in [\omega] \in H^1_{dR}(\mathcal{Y})$ we define the current $\mathbb{P}(u;\omega) := -\pi_\#((\partial G_u) \llcorner \widehat{\pi}^\# \omega) \in \mathcal{D}_{n-2}(B^n)$, so that

$$\mathbb{P}(u; \omega)(\phi) = -\partial G_u(\widehat{\pi}^\# \omega \wedge \pi^\# \phi) = G_u(\widehat{\pi}^\# \omega \wedge \pi^\# d\phi) = \int_{B^n} u^\# \omega \wedge d\phi$$

for every $\phi \in \mathcal{D}^{n-2}(B^n)$. We also define for every $\omega \in \mathcal{Z}^1(\mathcal{Y})$ the current $\mathbb{D}(u; \omega) := \pi_\#(G_u \llcorner \widehat{\pi}^\# \omega) \in \mathcal{D}_{n-1}(B^n)$, so that

$$\mathbb{D}(u; \omega)(\gamma) = G_u(\widehat{\pi}^\# \omega \wedge \pi^\# \gamma) = \int_{B^n} u^\# \omega \wedge \gamma \qquad \forall \gamma \in \mathcal{D}^{n-1}(B^n).$$

The following facts hold:

(i) for $s = 1, \dots, \overline{s}$

$$\mathbb{P}(u; \omega^s)(\phi) = \langle \mathbb{P}(u)(\phi), [\omega^s] \rangle,$$

i.e., $\mathbb{P}(u; \omega^s)$ does not depend on the representative in the cohomology class $[\omega^s]$;

(ii) $\partial\,\mathbb{P}(u) = 0$ and $\mathbb{P}(u) = \sum_{s=1}^{\overline{s}} \mathbb{P}(u;\omega^s)\otimes[\gamma_s]$, hence it does not depend on the choice of $\gamma_1,\dots,\gamma_{\overline{s}}$;

(iii) $\partial\,\mathbb{D}(u;\omega)(\phi) = \langle \mathbb{P}(u)(\phi),[\omega]\rangle$ and hence $\partial\,\mathbb{D}(u;\widetilde{\omega}^s)\llcorner B^n = \mathbb{P}(u;\widetilde{\omega}^s)$ for each representative $\widetilde{\omega}^s$ in $[\omega^s]$.

We can therefore set

$$\mathbb{D}_s(u) := \mathbb{D}(u;\omega^s), \quad \mathbb{P}_s(u) := \mathbb{P}(u;\omega^s) = \partial\,\mathbb{D}_s(u)\llcorner B^n, \quad s=1,\dots,\overline{s}\,. \tag{7.2}$$

Notice that if $T\in \operatorname{cart}^{1,1}(B^n\times\mathcal{Y})$ satisfies

$$T = G_u + S_T\,, \qquad S_T = \sum_{s=1}^{\overline{s}} \mathbb{L}_s(T)\times\gamma_s \qquad \text{on} \quad \mathcal{Z}^{n,1}(B^n\times\mathcal{Y})\,,$$

where $u = u_T \in W^{1,1}(B^n,\mathcal{Y})$ and $\mathbb{L}_s(T)\in\mathcal{R}_{n-1}(B^n)$, since

$$\begin{aligned}(-1)^{n-2}\partial G_u(\widehat{\pi}^\#\omega^s\wedge\pi^\#\phi) &= \partial G_u(\pi^\#\phi\wedge\widehat{\pi}^\#\omega^s) = -\partial S_T(\pi^\#\phi\wedge\widehat{\pi}^\#\omega^s)\\ &= -\partial\,\mathbb{L}_s(T)(\phi)\,,\end{aligned}$$

we infer that

$$\mathbb{P}_s(u) = (-1)^n\,\partial\,\mathbb{L}_s(T)\llcorner B^n \qquad \forall\, s=1,\dots,\overline{s}\,. \tag{7.3}$$

Finally, we clearly have $\mathbb{P}(u) = 0$ if u is smooth, say Lipschitz, or at least in $W^{1,2}(B^n,\mathcal{Y})$.

Results. In the sequel we shall *assume that the first homotopy group* $\pi_1(\mathcal{Y})$ *is commutative.* Moreover, we denote by

$$\mathcal{T}_u := \{T\in\operatorname{cart}^{1,1}(B^n,\mathcal{Y}) \mid u_T = u\} \tag{7.4}$$

the class of Cartesian currents T in $\operatorname{cart}^{1,1}(B^n\times\mathcal{Y})$ such that the underlying BV-function u_T is equal to u, compare Definition 2.11 and Remark 2.7. We first prove

Theorem 7.3. *For every* $u\in BV(B^n,\mathcal{Y})$ *we have* $\widetilde{\mathcal{E}_{TV}}(u)<\infty$.

From the results of the previous sections we then obtain the following representation result.

Theorem 7.4. *For any* $u\in BV(B^n,\mathcal{Y})$ *we have*

$$\begin{aligned}\widetilde{\mathcal{E}_{TV}}(u) &= \inf\{\mathcal{E}_{1,1}(T) \mid T\in\mathcal{T}_u\}\\ &= \int_{B^n}|\nabla u(x)|\,dx + |D^C u|(B^n)\\ &\quad + \inf\left\{\int_{J_c(T)}\mathcal{L}_T(x)\,d\mathcal{H}^{n-1}(x) \mid T\in\mathcal{T}_u\right\}\,,\end{aligned} \tag{7.5}$$

where $\mathcal{T}_u$, $J_c(T)$, *and* $\mathcal{L}_T(x)$ *are given by* (7.4), (2.12), *and Definition* 2.9, *respectively.*

Proof of Theorem 7.3. We observe that it suffices to show that the class $\mathcal{T}_u$ is non-empty, see (7.4). In this case, in fact, if $T \in \mathcal{T}_u$, by Theorem 2.13 we find a smooth sequence $\{u_k\} \subset C^1(B^n, \mathcal{Y})$ such that $G_{u_k} \rightharpoonup T$ weakly in $\mathcal{Z}_{n,1}(B^n \times \mathcal{Y})$ and $\|Du_k\|_{L^1} \to \mathcal{E}_{1,1}(T)$ as $k \to \infty$; this yields also that $u_k \rightharpoonup u_T$ weakly in the BV-sense, where $u_T = u$, whence $\widetilde{\mathcal{E}_{TV}}(u) < \infty$.

Now let us prove that $\mathcal{T}_u$ is non-empty. We first notice that, since $\mathcal{Y}$ is smooth and compact, there exists an absolute constant $C > 0$, depending on $\mathcal{Y}$, such that

$$\mathcal{E}_{TV}(u, B^n) < C\,|Du|(B^n) < \infty\,.$$

Let $\{u_k\}$ be the approximating sequence given by Theorem 6.7. Since $u_k \in R_1^\infty(B^n, \mathcal{Y})$, the real mass of the singularities is bounded by the L^1-norm of Du_k. More precisely, there exists an absolute constant $C > 0$ such that

$$m_{r,B^n}(\mathbb{P}_s(u_k)) \le C \int_{B^n} |Du_k|\,dx \qquad \forall s = 1, \dots, \overline{s}\,,$$

see Definition 7.2. In fact, we have

$$\begin{aligned} \mathbf{M}\big(\mathbb{D}_s(u_k)\big) &= \sup\left\{ \int_{B^n} \phi \wedge \big(u_k^{\#}\omega^s\big) \mid \phi \in \mathcal{D}^{n-1}(B^n)\,,\ \|\phi\| \le 1 \right\} \\ &\le C \int_{B^n} |Du_k|\,dx\,, \end{aligned}$$

see Proposition 7.6 below for the case $\mathcal{Y} = S^1$, so that the assertion follows from (7.2). Therefore, since by Hardt-Pitts' result (7.1) we have

$$m_{i,B^n}(\mathbb{P}_s(u_k)) = m_{r,B^n}(\mathbb{P}_s(u_k))\,,$$

we find for every s an integer multiplicity rectifiable current $\mathbb{L}_s^k \in \mathcal{R}_{n-1}(B^n)$ such that

$$\mathbb{P}_s(u_k) = (-1)^n\,(\partial\,\mathbb{L}_s^k) \llcorner B^n \qquad \text{and} \qquad \mathbf{M}(\mathbb{L}_s^k) \le C \int_{B^n} |Du_k|\,dx\,, \tag{7.6}$$

compare (7.3). As a consequence, letting

$$T_k := G_{u_k} + \sum_{s=1}^{\overline{s}} \mathbb{L}_s^k \times \gamma_s\,,$$

we readily find that $T_k \in \mathcal{D}_{n,1}(B^n \times \mathcal{Y})$ has no interior boundary

$$\partial T_k = 0 \quad \text{on} \quad \mathcal{Z}^{n-1,1}(B^n \times \mathcal{Y})$$

and finite BV-energy

$$\mathcal{E}_{1,1}(T_k) \le \int_{B^n} |Du_k|\,dx + C(\mathcal{Y}) \sum_{s=1}^{\overline{s}} \mathbf{M}(\mathbb{L}_s^k) \cdot \mathbf{M}(\gamma_s) < \infty$$

for some absolute constant $C(\mathcal{Y}) > 0$. In conclusion, by (7.6) we obtain a sequence $\{T_k\} \subset \mathrm{cart}^{1,1}(B^n \times \mathcal{Y})$ with equibounded energies

$$\sup_k \mathcal{E}_{1,1}(T_k) \leq \sup_k C \int_{B^n} |Du_k|\, dx \leq C\, \mathcal{E}_{TV}(u, B^n) < \infty\,,$$

where $C > 0$ is an absolute constant. Therefore, by compactness, Proposition 2.18, possibly passing to a subsequence we find that $T_k \rightharpoonup T$ weakly in $\mathcal{Z}_{n,1}(B^n \times \mathcal{Y})$ to some $T \in \mathrm{cart}^{1,1}(B^n \times \mathcal{Y})$ satisfying

$$\mathcal{E}_{1,1}(T) \leq \liminf_{k\to\infty} \mathcal{E}_{1,1}(T_k) < \infty$$

by lower semicontinuity, Proposition 2.16. In particular, since $u_k \rightharpoonup u$ weakly in the BV-sense, we find that the underlying BV-function $u_T = u$ and hence that $T \in \mathcal{T}_u$. $\square$

Proof of Theorem 7.4. Let $\{u_k\} \subset C^1(B^n, \mathcal{Y})$ be a sequence of smooth maps with equibounded energies, $\sup_k \|Du_k\|_{L^1} < \infty$, weakly converging to u in the BV-sense, see Theorem 7.3. By compactness, Proposition 2.18, possibly passing to a subsequence we find that $G_{u_k} \rightharpoonup T$ weakly in $\mathcal{Z}_{n,1}(B^n \times \mathcal{Y})$ to some $T \in \mathrm{cart}^{1,1}(B^n \times \mathcal{Y})$ satisfying $u_T = u$, *i.e.* $T \in \mathcal{T}_u$, see (7.4). Since by lower semicontinuity, Proposition 2.16,

$$\mathcal{E}_{1,1}(T) \leq \liminf_{k\to\infty} \int_{B^n} |Du_k|\, dx\,,$$

we readily conclude that

$$\inf\{\mathcal{E}_{1,1}(T) \mid T \in \mathcal{T}_u\} \leq \widetilde{\mathcal{E}_{TV}}(u)\,.$$

To prove the opposite inequality, by applying Theorem 2.13, for every $T \in \mathcal{T}_u$ we find a smooth sequence $\{u_k\} \subset C^1(B^n, \mathcal{Y})$ such that $G_{u_k} \rightharpoonup T$ weakly in $\mathcal{Z}_{n,1}(B^n \times \mathcal{Y})$ and $\|Du_k\|_{L^1} \to \mathcal{E}_{1,1}(T)$ as $k \to \infty$. Since the weak convergence $G_{u_k} \rightharpoonup T$ yields the convergence $u_k \rightharpoonup u_T$ weakly in the BV-sense, and $u_T = u$, we find that $\widetilde{\mathcal{E}_{TV}}(u) \leq \mathcal{E}_{1,1}(T)$, which proves the first equality in (7.5). The second equality in (7.5) follows from the definition of BV-energy, Definition 2.10. $\square$

The above results simplify if we specify them to $u \in W^{1,1}(B^n, \mathcal{Y})$ and/or $\mathcal{Y} = S^1$, recovering this way previous results, compare e.g. [13], [8], and [19].

The relaxed $W^{1,1}$-energy. The relaxed energy of $u \in W^{1,1}(B^n, \mathcal{Y})$ is of course given by

$$\widetilde{\mathcal{E}_{1,1}}(u) := \inf\left\{ \liminf_{k\to\infty} \int_{B^n} |Du_k|\, dx \,\middle|\, \{u_k\} \subset C^1(B^n, \mathcal{Y}),\ u_k \to u \text{ strongly in } L^1(B^n, \mathbb{R}^N) \right\},$$

see Remark 7.1. In this case, Theorem 7.4 reads as:

Corollary 7.5. *For any $u \in W^{1,1}(B^n, \mathcal{Y})$ we have $\widetilde{\mathcal{E}_{1,1}}(u) < \infty$. Every $T \in \mathcal{T}_u$ has the form*

$$T = G_u + \sum_{q \in H_1(\mathcal{Y})} \mathbb{L}_q \times C_q \qquad \text{on} \quad \mathcal{Z}^{n,1}(B^n \times \mathcal{Y}),$$

where $\mathbb{L}_q = \tau(\mathcal{L}_q, 1, \overrightarrow{\mathcal{L}}_q)$ is an integer multiplicity rectifiable current in $\mathcal{R}_{n-1}(B^n)$ and $C_q \in \mathcal{Z}_1(\mathcal{Y})$ is an integral 1-cycle in the homology class q, and its BV-energy is given by

$$\mathcal{E}_{1,1}(T) = \int_{B^n} |Du|\, dx + \sum_{q \in H_1(\mathcal{Y})} \int_{\mathcal{L}_q} \mathcal{L}_T(x)\, d\mathcal{H}^{n-1}(x)$$

where, for $x \in \mathcal{L}_q$, we have $\mathcal{L}_T(x) := \inf\{\mathcal{L}(\gamma) \mid \gamma \in \Gamma_q(x)\}$ and

$$\Gamma_q(x) := \{\gamma \in \mathrm{Lip}([0,1], \mathcal{Y}) \mid \gamma(0) = \gamma(1) = u(x), \quad \gamma_\# [\![(0,1)]\!] \in q\}.$$

The relaxed energy is given by

$$\widetilde{\mathcal{E}_{1,1}}(u) = \int_{B^n} |Du(x)|\, dx + \inf\left\{ \sum_{q \in H_1(\mathcal{Y})} \int_{\mathcal{L}_q} \mathcal{L}_T(x)\, d\mathcal{H}^{n-1}(x) \mid T \in \mathcal{T}_u \right\}.$$

The case $\mathcal{Y} = \mathcal{S}^1$. Further simplification arises if we assume $\mathcal{Y} = S^1$. In this case, in fact, $S_{T,\mathrm{sing}} = 0$, *i.e.* the equivalence classes of elements in $\mathrm{cart}^{1,1}(B^n \times S^1)$ have a unique representative, and the energies $\mathcal{E}_{1,1}(T)$ and $\mathcal{F}_{1,1}(T)$ are equal, *i.e.*, no gap phenomenon occurs. Moreover, if x belongs to the jump-concentration set $J_c(T)$, the 1-dimensional restriction has the form

$$\widehat{\pi}_\#(T \llcorner \{x\} \times S^1) = [\![\gamma_x]\!] + q\, [\![S^1]\!],$$

where $q \in \mathbb{Z}$ and $[\![\gamma_x]\!]$ is the current associated to a suitably oriented simple arc γ_x in S^1 connecting the points $u_T^-(x)$ and $u_T^+(x)$, where u_T is the function in $BV(B^n, S^1)$ associated to T, and $\gamma_x = 0$ if $x \notin J_{u_T}$. Consequently, in (7.5) we have

$$\mathcal{L}_T(x) = \mathcal{H}^1(\gamma_x) + 2\pi\, |q|$$

and hence in $\mathrm{cart}^{1,1}(B^n \times S^1)$ the BV-energy agrees with the energy obtained in [13], compare Theorem 1 of [14, Vol. II, Section 6.2.3].

The singular set. If $u \in W^{1,1}(B^n, S^1)$, its singular set is the current $\mathbb{P}(u) \in \mathcal{D}_{n-2}(B^n)$ given by

$$\mathbb{P}(u)(\phi) := -\frac{1}{2\pi}\, \partial G_u(\pi^\# \omega_{S^1} \wedge \pi^\# \phi) = \frac{1}{2\pi} \int_{B^n} u^\# \omega_{S^1} \wedge d\phi \tag{7.7}$$

for every $\phi \in \mathcal{D}^{n-2}(B^n)$, where

$$\omega_{S^1} := y^1 dy^2 - y^2 dy^1$$

is the volume 1-form in $S^1 \subset \mathbb{R}^2$. Therefore, $\mathbb{P}(u)$ is the boundary of the current $\mathbb{D}(u) \in \mathcal{D}_{n-1}(B^n)$ defined for any $\gamma \in \mathcal{D}^{n-1}(B^n)$ by

$$\mathbb{D}(u)(\gamma) := \frac{1}{2\pi} G_u(\pi^\# \omega_{S^1} \wedge \pi^\# \gamma) = \frac{1}{2\pi} \int_{B^n} u^\# \omega_{S^1} \wedge \gamma \,.$$

Proposition 7.6. *For every* $u \in W^{1,1}(B^n, S^1)$ *we have*

$$\mathbf{M}(\mathbb{D}(u)) \leq \frac{1}{2\pi} \int_{B^n} |Du| \, dx \,.$$

Proof. By the definition of mass we clearly infer

$$2\pi \, \mathbf{M}\big(\mathbb{D}(u)\big) \leq \int_{B^n} \|u^\# \omega_{S^1}\| \, dx \,.$$

Moreover, since $u^\# \omega_{S^1} = u^1 du^2 - u^2 du^1$, we estimate

$$\|u^\# \omega_{S^1}\|^2 \leq \sum_{i=1}^{n} |u^1 \, u^2_{x_i} - u^2 \, u^1_{x_i}|^2 \leq \sum_{i=1}^{n} (|u^1| \, |u^2_{x_i}| + |u^1| \, |u^2_{x_i}|)^2 \,.$$

Observe now that for any $a, b > 0$ and $\lambda, \mu > 0$ with $\lambda^2 + \mu^2 = 1$

$$\lambda \, a + \mu \, b \leq \sqrt{a^2 + b^2} \,.$$

Since $|u(x)| = 1$, this yields $(|u^1| \, |u^2_{x_i}| + |u^1| \, |u^2_{x_i}|)^2 \leq |D_{x_i} u|^2$ and hence the assertion. □

We now recover the following estimates about the relaxed energy, compare [8] and [19].

Proposition 7.7. *For every* $u \in W^{1,1}(B^n, S^1)$ *we have*

$$\widetilde{\mathcal{E}_{1,1}}(u) \leq 2 \, \mathcal{E}_{1,1}(u) \,, \qquad \textit{where} \qquad \mathcal{E}_{1,1}(u) := \int_{B^n} |Du| \, dx \,. \tag{7.8}$$

Moreover, for every $u \in BV(B^n, S^1)$ *we have*

$$\widetilde{\mathcal{E}_{TV}}(u) \leq 2 \, \mathcal{E}_{TV}(u) \,, \tag{7.9}$$

where $\mathcal{E}_{TV}(u)$ *is the total variation of* u*, given by* (6.1).

Proof. Let $u \in W^{1,1}(B^n, S^1)$. Proposition 7.6 yields that the real mass $m_{r,B^n}(\mathbb{P}(u)) \leq \mathcal{E}_{1,1}(u, B^n)/2\pi$ and hence, on account of Hardt-Pitts' result (7.1), the integral mass

$$m_{i,B^n}(\mathbb{P}(u)) \leq \frac{1}{2\pi} \mathcal{E}_{1,1}(u) \,,$$

see Definition 7.2. As a consequence, since for every $\varepsilon > 0$ we find a current $T \in \mathcal{T}_u$ such that

$$T = G_u + L \times [\![S^1]\!] \qquad \text{and} \qquad \mathcal{E}_{1,1}(T) = \mathcal{E}_{1,1}(u) + 2\pi\, \mathbf{M}(L)\,,$$

where $L \in \mathcal{R}_{n-1}(B^n)$ satisfies $\mathbf{M}(L) \le m_{i,B^n}(\mathbb{P}(u)) + \varepsilon$, taking into account Theorem 7.4 we obtain (7.8).

In the more general case $u \in BV(B^n, S^1)$, Theorem 6.7 yields the existence of a sequence $\{u_k\} \subset W^{1,1}(B^n, S^1)$ such that $u_k \rightharpoonup u$ weakly in the BV-sense and $\mathcal{E}_{1,1}(u_k) \to \mathcal{E}_{TV}(u)$. Also, for every k we find a smooth sequence $\{u_h^{(k)}\}_h \subset C^1(B^n, S^1)$ converging to u_k strongly in L^1 and such that $\mathcal{E}_{1,1}(u_h^{(k)}) \to \widetilde{\mathcal{E}_{1,1}}(u_k)$ $+1/k$ as $h \to \infty$. Finally, by (7.8) and by a diagonal argument we readily obtain (7.9). □

Remark 7.8. As in [20], since $\pi_1(\mathcal{Y})$ is commutative, if $u \in R_1^\infty(B^n, \mathcal{Y})$, for every $s = 1, \dots, \overline{s}$ we may find an integral current $L_s \in \mathcal{R}_{n-2}(B^n)$ satisfying

$$(-1)^n (\partial L_s) \llcorner B^n = \mathbb{P}_s(u) \qquad \text{and} \qquad \mathbf{M}(L_s) \le C \int_{B^n} |Du|\, dx$$

for some absolute constant $C > 0$ independent of u. Therefore, arguing as above it is not difficult to show that

$$\widetilde{\mathcal{E}_{1,1}}(u) \le C(n, \mathcal{Y}) \cdot \mathcal{E}_{1,1}(u) \qquad \forall\, u \in W^{1,1}(B^n, \mathcal{Y})\,, \tag{7.10}$$

where $C(n, \mathcal{Y}) > 0$ is an absolute constant, only depending on n and $\mathcal{Y}$. Finally, by Theorem 6.7 we conclude that

$$\widetilde{\mathcal{E}_{TV}}(u) \le C(n, \mathcal{Y}) \cdot \mathcal{E}_{TV}(u) \qquad \forall\, u \in BV(B^n, \mathcal{Y})\,,$$

where $\mathcal{E}_{TV}(u)$ is the total variation given by (6.1) and the optimal constant $C(n, \mathcal{Y})$ is the same as the optimal constant for $W^{1,1}$-functions in (7.10).

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