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On non-overdetermined inverse scattering at zero energy in three dimensions

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Abstract. We develop the $\bar\partial$-approach to inverse scattering at zero energy in dimensions $d \geq 3$ of [Beals, Coifman 1985], [Henkin, Novikov 1987] and [Novikov 2002]. As a result we give, in particular, uniqueness theorem, precise reconstruction procedure, stability estimate and approximate reconstruction for the problem of finding a sufficiently small potential v in the Schrödinger equation from a fixed non-overdetermined ("backscattering" type) restriction $h\big|_\Gamma$ of the Faddeev generalized scattering amplitude h in the complex domain at zero energy in dimension $d = 3$. For sufficiently small potentials v we formulate also a characterization theorem for the aforementioned restriction $h\big|_\Gamma$ and a new characterization theorem for the full Faddeev function h in the complex domain at zero energy in dimension $d = 3$. We show that the results of the present work have direct applications to the electrical impedance tomography via a reduction given first in [Novikov, 1987, 1988].

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1. Introduction

Consider the Schrödinger equation at zero energy

$$-\Delta\psi + v(x)\psi = 0, \quad x \in \mathbb{R}^d, \quad d \geq 2, \tag{1.1}$$

where

$$v \ \text{ is a sufficiently regular function on } \mathbb{R}^d \text{ with sufficient decay at infinity} \tag{1.2}$$

(precise assumptions on v are specified below in this introduction and in Sections 2 and 3). For equation (1.1), under assumptions (1.2), we consider the Faddeev generalized scattering amplitude $h(k, l)$, where $(k, l) \in \Theta$,

$$\Theta = \{k \in \mathbb{C}^d,\ l \in \mathbb{C}^d : k^2 = l^2 = 0, \quad \operatorname{Im} k = \operatorname{Im} l\}. \tag{1.3}$$

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For definitions of h see, for example, [HN, Section 2.2] and [No1, Section 2]. Given v, to determine h on Θ one can use, in particular, the formula

$$h(k,l) = H(k, k-l), \quad (k,l) \in \Theta, \tag{1.4}$$

and the linear integral equation

$$H(k,p) = \hat{v}(p) - \int\limits_{\mathbb{R}^d} \frac{\hat{v}(p+\xi)H(k,-\xi)\mathrm{d}\xi}{\xi^2 + 2k\xi}, \quad k \in \Sigma, \quad p \in \mathbb{R}^d, \tag{1.5}$$

where

$$\hat{v}(p) = (2\pi)^{-d} \int\limits_{\mathbb{R}^d} e^{ipx} u(x)\mathrm{d}x, \quad p \in \mathbb{R}^d, \tag{1.6}$$

$$\Sigma = \{k \in \mathbb{C}^d : k^2 = 0\}. \tag{1.7}$$

In the present work we consider, mainly, the three dimensional case $d = 3$. In addition, in the main considerations of the present work for $d = 3$ our basic assumption on v consists in the following condition on its Fourier transform

$$\hat{v} \in L^\infty_\mu(\mathbb{R}^3) \ \text{ for some real } \ \mu \geq 2, \tag{1.8}$$

where

$$\begin{aligned} L^\infty_\mu(\mathbb{R}^d) &= \{u \in L^\infty(\mathbb{R}^d) : \|u\|_\mu < +\infty\}, \\ \|u\|_\mu &= \operatorname*{ess\,sup}_{p \in \mathbb{R}^d} (1+|p|)^\mu |u(p)|, \quad \mu > 0. \end{aligned} \tag{1.9}$$

If v satisfies (1.8), then we consider (1.5) at fixed k as an equation for $H(k,\cdot) \in L^\infty_\mu(\mathbb{R}^3)$. An analysis of equation (1.5) for $d = 3$ and with (1.8) taken as a basic assumption on v is given in Section 3.

Note that, actually, h on Θ is a zero energy restriction of a function h introduced by Faddeev (see [F2, HN]) as an extention to the complex domain of the classical scattering amplitude for the Schrödinger equation at positive energies. In addition, the restriction $h\big|_\Theta$ was not considered in Faddeev's works. Note that $h\big|_\Theta$ was considered for the first time in [BC1] for $d = 3$ in the framework of Problem 1.1a formulated below. The Faddeev function h was, actually, rediscovered in [BC1]. The fact that $\bar{\partial}$- scattering data of [BC1] coincide with the Faddeev function h was observed, in particular, in [HN].

In the present work, in addition to h on Θ, we consider $h\big|_\Gamma$, $h\big|_{\Theta^\tau}$ and $h\big|_{\Gamma^\tau}$, where

$$\Gamma = \left\{ k = \frac{p}{2} + \frac{i|p|}{2}\gamma(p), \ \ l = -\frac{p}{2} + \frac{i|p|}{2}\gamma(p) : \ p \in \mathbb{R}^d \right\}, \tag{1.10a}$$

where γ is a piecewise continuous (or just measurable) function of $p \in \mathbb{R}^d$ with values in $\mathbb{S}^{d-1}$ and such that

$$\gamma(p)p = 0, \quad p \in \mathbb{R}^d, \tag{1.10b}$$

$$\Theta^\tau = \{(k,l) \in \Theta : |\operatorname{Im} k| = |\operatorname{Im} l| < \tau\}, \tag{1.11}$$

$$\Gamma^\tau = \Gamma \cap \Theta^\tau, \tag{1.12}$$

where $\tau > 0$. Note that

$$\Gamma \subset \Theta, \tag{1.13}$$

$$\dim \Theta = 3d - 4, \quad \dim \Gamma = \dim \mathbb{R}^d = d, \tag{1.14}$$

$$3d - 4 = d \quad \text{for} \quad d = 2, \qquad 3d - 4 > d \quad \text{for} \quad d \geq 3. \tag{1.15}$$

Using (1.4), (1.5) one can see that

$$h(k,l) \approx \hat{v}(p), \quad (k,l) \in \Theta, \quad k - l = p, \tag{1.16}$$

in the Born approximation (that is in the linear approximation near zero potential). Using (1.10), (1.13), (1.14), (1.16) one can see that, in general, $h\big|_\Gamma$ is a nonlinear analog of the Fourier transform $\hat{v}$. Note also that $h\big|_\Gamma$ is a zero energy analog of the reflection coefficient (backscattering amplitude) considered (in particular) in [Mos, P, HN, ER].

In the present work we consider, in particular, the following inverse scattering problems for equation (1.1) under assumptions (1.2).

Problem 1.1.

(a) Given h on Θ, find v on $\mathbb{R}^d$ (and characterize h on Θ);
(b) Given h on Θ^τ for some (sufficiently great) $\tau > 0$, find v on $\mathbb{R}^d$, at least, approximately.

Problem 1.2.

(a) Given h on Γ, find v on $\mathbb{R}^d$ (and characterize h on Γ);
(b) Given h on Γ^τ for some (sufficiently great) $\tau > 0$, find v on $\mathbb{R}^d$, at least, approximately.

Using (1.14), (1.15), (1.16) one can see that Problems 1.1a, 1.1b are strongly overdetermined for $d \geq 3$, whereas Problems 1.2a, 1.2b are nonoverdetermined for $d \geq 2$ (at least, in the sense of the dimension considerations and in the Born approximation). In addition, using (1.12), (1.13) one can see that any reconstruction method for Problems 1.2 is also a reconstruction method for Problems 1.1. The present work is focused on Problems 1.2a, 1.2b for the most important three-dimensional case $d = 3$. In addition, we are focused on potentials v with

$$\hat{v} \in L^\infty_\mu(\mathbb{R}^3) \text{ with sufficiently small } \|\hat{v}\|_\mu \text{ for some fixed } \mu \geq 2, \tag{1.17}$$

where $L^\infty_\mu(\mathbb{R}^3)$ and $\|\cdot\|_\mu$ are defined in (1.9). In some results we also still assume for simplicity that $\hat{v} \in \mathcal{C}(\mathbb{R}^3)$ (in addition to (1.8) or (1.17)), where $\mathcal{C}$ denotes the space of continuous functions. The main results of the present work include, in particular:

(I) uniqueness theorem, reconstruction procedure and stability estimate for Problem 1.2a for v satisfying (1.17) (with $\hat{v} \in \mathcal{C}(\mathbb{R}^3)$) (see Theorem 2.1) and

(II) approximate reconstruction method for Problem 1.2b for v satisfying (1.17) (with $\hat{v} \in \mathcal{C}(\mathbb{R}^3)$) (see Theorem 2.1 and Corollary 2.2).

These results are formulated and proved in Sections 2-12. In the present work we formulate also:

(III) characterization for Problem 1.2a for v satisfying (1.17) (see Theorem 2.3) and

(IV) new characterization for Problem 1.1a or more precisely a characterization for Problem 1.1a for v satisfying (1.17) (see Theorem 2.4).

We plan to give a complete proof of these characterizations in a separate work, where we plan to show also that the aforementioned results I and II remain valid without the additional assumption that $\hat{v} \in \mathcal{C}(\mathbb{R}^3)$. All these results I, II, III and IV are presented in detail in Section 2.

Note that Problem 1.1a was considered for the first time in [BC1] for $d = 3$ from pure mathematical point of view without any physical applications. No possibility to measure h on $\Theta\backslash\{(0,0)\}$ directly in some physical experiment is known at present. However, as it was shown in [No1] (see also [HN] (Note added in proof), [Na1, Na2, No4]), Problems 1.1 naturally arise in the electrical impedance tomography and, more generally, in the inverse boundary value problem (Problem 1.3) formulated as follows. Consider the equation

$$-\Delta\psi + v(x)\psi = 0, \quad x \in D, \tag{1.18}$$

where

$$\begin{aligned} &D \text{ is an open bounded domain in } \mathbb{R}^d, \quad d \geq 2, \\ &\text{with sufficiently regular boundary } \partial D, \\ &v \text{ is a sufficiently regular function on } \bar{D} = D \cup \partial D. \end{aligned} \tag{1.19}$$

We assume also that

$$\begin{aligned} &0 \text{ is not a Dirichlet eigenvalue for} \\ &\text{the operator } -\Delta + v \text{ in } D. \end{aligned} \tag{1.20}$$

Consider the map Φ such that

$$\frac{\partial \psi}{\partial \nu}\Big|_{\partial D} = \Phi\left(\psi|_{\partial D}\right) \tag{1.21}$$

for all sufficiently regular solutions of (1.18) in $\bar{D}$, where ν is the outward normal to ∂D. The map Φ is called the Dirichlet-to-Neumann map for equation (1.18). The aforementioned inverse boundary value problem is:

Problem 1.3. Given Φ, find v on D.

In addition, the simplest interpretation of D, v and Φ in the framework of the electrical impedance tomography consists in the following (see [SU, No1, Na1]): D is a body with isotropic conductivity $\sigma(x)$ (where $\sigma \geq \sigma_{\min} > 0$),

$$v(x) = (\sigma(x))^{-1/2}\, \Delta\, (\sigma(x))^{1/2}, \quad x \in D, \tag{1.22}$$

$$\Phi = \sigma^{-1/2}\left(\Lambda \sigma^{-1/2} + \frac{\partial \sigma^{1/2}}{\partial \nu}\right), \tag{1.23}$$

where Λ is the voltage-to-current map on ∂D and $\sigma^{1/2}$, $\partial\sigma^{1/2}/\partial\nu$ in (1.23) denote the multiplication operators by the functions $\sigma^{-1/2}|_{\partial D}$, $\left(\partial\sigma^{1/2}/\partial\nu\right)|_{\partial D}$, respectively.

Note that the formulation of Problem 1.3 goes back to Gelfand [G] and Calderon [C].

Returning to Problems 1.1, 1.2 and their relation to Problem 1.3 one can see that the Faddeev function h of Problems 1.1, 1.2 does not appear in Problem 1.3. However, as it was shown in [No1] (see also [HN] (where this result of [No1] was announced in Note added in proof), [Na1, Na2, No4]), if h corresponds to equation (1.1), where

$$\begin{aligned} &v \text{ of } (1.1) \text{ coincides on } D \text{ with } v \text{ of } (1.18)\\ &\text{and } v \text{ of } (1.1) \text{ is identically zero on } \mathbb{R}^d \backslash \bar{D}, \end{aligned} \tag{1.24}$$

then h on Θ can be determined from the Dirichlet-to-Neumann map Φ for equation (1.18) via the following formulas and equation:

$$h(k,l) = (2\pi)^{-d} \int\limits_{\partial D}\int\limits_{\partial D} e^{-ilx}(\Phi - \Phi_0)(x,y)\psi(y,k)\mathrm{d}y\mathrm{d}x \quad \text{for } (k,l)\in\Theta, \tag{1.25}$$

$$\psi(x,k) = e^{ikx} + \int\limits_{\partial D} A(x,y,k)\psi(y,k)\mathrm{d}y, \quad x \in \partial D, \tag{1.26}$$

$$A(x,y,k) = \int\limits_{\partial D} G(x-z,k)(\Phi-\Phi_0)(z,y)\mathrm{d}z, \quad x,y \in \partial D, \tag{1.27}$$

$$G(x,k) = -(2\pi)^{-d} e^{ikx} \int\limits_{\mathbb{R}^d} \frac{e^{i\xi x}\mathrm{d}\xi}{\xi^2 + 2k\xi}, \quad x \in \mathbb{R}^d, \tag{1.28}$$

where $k \in \mathbb{C}^d$, $k^2 = 0$ in (1.26)-(1.28), Φ_0 denotes the Dirichlet-to-Neumann map for equation (1.18) for $v \equiv 0$, and $(\Phi - \Phi_0)(x, y)$ is the Schwartz kernel of the integral operator $\Phi - \Phi_0$. Note that (1.25), (1.27), (1.28) are explicit formulas, whereas (1.26) is a linear integral equation (with parameter k) for ψ on ∂D. In addition, G of (1.28) is the Faddeev's Green function of [F1] for the Laplacian Δ. Note also that formulas and equation (1.25)- (1.27) are obtained and analyzed in [No1] for (1.19) specified as

$$\begin{aligned} &D \text{ is an open bounded domain in } \mathbb{R}^d, \ \ d \geq 2, \\ &\partial D \in C^2, \ \ v \in L^\infty(D). \end{aligned} \tag{1.29}$$

Formulas and equation (1.25)-(1.27) reduce Problem 1.3 to Problems 1.1, 1.2. In addition, from numerical point of view $h(k,l)$ for $(k,l) \in \Theta^\tau$ can be relatively easily determined from Φ via (1.27), (1.26), (1.25) if τ is sufficiently small. However, if $(k,l) \in \Theta\backslash\Theta^\tau$, where τ is sufficiently great, then the determination of $h(k,l)$ from Φ via (1.27), (1.26), (1.25) is very unstable (especially on the step (1.26)). The reason of this instability is that formulas and equation (1.25)-(1.28) involve the exponential functions e^{-ilx}, e^{ikx} and, actually, $e^{ik(x-z)}$ (arising in (1.27) in view of (1.28)), where $(k,l) \in \Theta$, $x \in \partial D$, $z \in \partial D$, which rapidly oscillate in x, z and may have exponentially great absolute values if $(k,l) \in \Theta\backslash\Theta^\tau$ (and, therefore, $|\mathrm{Re}\, k| = |\mathrm{Im}\, k| = |\mathrm{Re}\, l| = |\mathrm{Im}\, l| > \tau$) for sufficiently great τ.

These remarks show that Problems 1.1, 1.2 are especially important in their versions 1b, 2b as regards their applications to Problem 1.3 via (1.25)-(1.28) (or via similar reductions). In addition, in view of (1.13)-(1.15), one can see that it is much simpler to determine h on Γ (or on Γ^τ) only than completely on Θ (on on Θ^τ, respectively) from Φ via (1.25)-(1.28) for $d \geq 3$. Therefore, Problem 1.2b is of particular interest and importance in the framework of applications of Problems 1.1, 1.2 to Problem 1.3 for $d \geq 3$.

In the present work we consider, mainly, Problems 1.1 and 1.2 for $d = 3$. In addition, as it was already mentioned, we are focused on nonoverdetermined Problems 1.2a, 1.2b for v satisfying (1.17). The main results of the present work are presented in Section 2. (Some of these results were already mentioned above.) Note that only restrictions in time prevent us from generalizing all main results of the present work to the case $d > 3$. Actually, the results of the present work are obtained in the framework of a development of the $\bar\partial$-approach to inverse scattering at fixed energy in dimension $d \geq 3$ of [BC1, HN, No3, No5]. In particular, the central part of the present work consists in an analysis of the non-linear $\bar\partial$-equation (3.13) for the Faddeev function H on Θ for v satisfying (1.17) (with $\hat v \in \mathcal{C}(\mathbb{R}^3)$), see Sections 5, 6, 7.

Actually, in the present work we do not consider Problems 1.1 and 1.2 for $d = 2$: inverse scattering at fixed energy in dimension $d = 2$ differs considerably from inverse scattering at fixed energy in dimension $d \geq 3$. Note that a global reconstruction method for Problem 1.2a for $d = 2$ and for v of the form (1.2), (1.22), where $x \in \mathbb{R}^2$, $\sigma \geq \sigma_{\min} > 0$, was given in [Na2] in the framework of a development of the $\bar\partial$-approach to inverse scattering at fixed energy in dimension $d = 2$

(see references to [BLMP, GN, No2, T] given in [Na2] in connection with this approach). In addition, this result on Problem 1.2a is given in [Na2] in the framework of applications to the (two-dimensional) electrical impedance tomography via the reduction (1.25)-(1.27) for $d = 2$ (given first in [No1]). Besides, note that there is an essential similarity between the results of [Na2] on global reconstruction for Problem 1.2a for $d = 2$ and for v of the form (1.2), (1.22), where $x \in \mathbb{R}^2$, $\sigma \geq \sigma_{\min} > 0$, and results of [BC2] on global inverse scattering reconstruction for some 2×2 first order system on the plane (see also [BU] in this connection).

Applications of result of the present work to the electrical impedance tomography and more generally to Problem 1.3 will be analyzed in detail in a subsequent paper (where we plan to give, in particular, new stability estimates for Problem 1.3). Concerning results given in the literature on Problem 1.3, see [KV, SU], [HN] (note added in proof), [No1, A, Na1, Na2, BU, Ma, No4] and references therein.

2. Main results

As it was already mentioned in the introduction, the main results of the present work include, in particular:

(I) uniqueness theorem, reconstruction procedure and stability estimate for Problem 1.2a for v satisfying (1.17) (with $\hat{v} \in \mathcal{C}(\mathbb{R}^3)$) and
(II) approximate reconstruction method for Problem 1.2b for v satisfying (1.17) (with $\hat{v} \in \mathcal{C}(\mathbb{R}^3)$),

see Theorem 2.1 and Corollary 2.2 formulated below in this section (and proved by means of analysis developed in Sections 3-12).

We identify $h\big|_{\Gamma}$ and $h\big|_{\Gamma^\tau}$ with R and $R_{2\tau}$ on $\mathbb{R}^d$, where

$$R(p) = h\left(\frac{p}{2} + \frac{i|p|}{2}\gamma(p), -\frac{p}{2} + \frac{i|p|}{2}\gamma(p)\right), \quad p \in \mathbb{R}^d, \tag{2.1}$$

$$\begin{aligned} R_{2\tau}(p) &= R(p) \ \text{ for } |p| < 2\tau, \quad p \in \mathbb{R}^d, \\ R_{2\tau}(p) &= 0 \ \text{ for } |p| \geq 2\tau, \quad p \in \mathbb{R}^d, \end{aligned} \tag{2.2}$$

where γ is the function of (1.10).

Theorem 2.1. *Let*

$$\hat{v} \in L^\infty_\mu(\mathbb{R}^3) \ \textit{for some} \ \mu \geq 2, \tag{2.3}$$

$$\|\hat{v}\|_\mu \leq C < \frac{1}{c_1(\mu) + 8c_6(\mu)}, \tag{2.4}$$

where $L^\infty_\mu(\mathbb{R}^3)$ *and* $\|\cdot\|_\mu$ *are defined in* (1.9), $c_1(\mu)$ *and* $c_6(\mu)$ *are the positive constants of Lemmas* 3.1 *and* 6.4. (*For simplicity we also still assume that* $\hat{v} \in$

$\mathcal{C}(\mathbb{R}^3)$.) *Let* R *be defined by* (2.1) (*for some given* γ *of* (1.10) *for* $d = 3$). *Then*

$$R \in L^{\infty}_{\mu}(\mathbb{R}^3), \quad \|R\|_{\mu} \le \frac{C}{1 - c_1(\mu)C}, \tag{2.5}$$

and R *uniquely determines* $\hat{v}$ *via the following reconstruction procedure*

$$R \xrightarrow[\substack{\text{by successive}\\ \text{approximations}}]{(6.6)} H \xrightarrow{(7.1b),(7.2b)} \hat{v}, \tag{2.6}$$

where (6.6) *is a nonlinear integral equation of Proposition* 6.1 *of Section* 6, (7.1b), (7.2b) *are explicit formulas of Section* 7 *and where we solve* (6.6) *by the method of successive approximations* (*see Proposition* 6.7 *and Lemma* 6.5). *In addition, if* R_{appr} *is an arbitrary approximation to* R, *where* R_{appr} *also satisfies* (2.5), *and* $\hat{v}_{\text{appr}}$ *is determined from* R_{appr} *via* (2.6) (*with* H *replaced by* H_{appr}), *then the following stability estimate holds:*

$$\|\hat{v} - \hat{v}_{\text{appr}}\|_{\mu} \le \frac{1 - c_1(\mu)C}{1 - (c_1(\mu) + 8c_6(\mu))C} \|R - R_{\text{appr}}\|_{\mu}. \tag{2.7}$$

One can see that Theorem 2.1 includes uniqueness theorem, reconstruction procedure and stability estimate for Problem 1.2a (of the introduction) for v satisfying (1.17) (with $\hat{v} \in \mathcal{C}(\mathbb{R}^3)$).

Theorem 2.1 follows from Proposition 3.2, Lemmas 6.5, 6.6, Propositions 6.1, 6.7 and formulas (7.1), (7.2) (of Sections 3,6 and 7). In particular, condition (2.4) of Theorem 2.1 implies condition (6.20) of Proposition 6.7 and condition (3.6) of (part I of) Proposition 3.2.

Corollary 2.2. *Let* v *satisfy* (2.3), (2.4) *and, in addition,*

$$\hat{v} \in L^{\infty}_{\mu^*}(\mathbb{R}^3) \ \text{ for some } \ \mu^* > \mu. \tag{2.8}$$

(*For simplicity we also still assume that* $\hat{v} \in \mathcal{C}(\mathbb{R}^3)$.) *Let* $\hat{v}_{2\tau}$ *denotes* $\hat{v}_{\text{appr}}$ *reconstructed from* R_{appr} *via* (2.6) (*as in Theorem* 2.1), *where* $R_{\text{appr}} = R_{2\tau}$ (*defined by* (2.1), (2.2) *for* $d = 3$). *Then*

$$R \in L^{\infty}_{\mu^*}(\mathbb{R}^3) \tag{2.9}$$

and

$$\|\hat{v} - \hat{v}_{2\tau}\|_{\mu} \le \frac{1 - c_1(\mu)C}{1 - (c_1(\mu) + 8c_6(\mu))C} \frac{\|R\|_{\mu^*}}{(1 + 2\tau)^{\mu^* - \mu}} \ \text{for } \ \tau > 0. \tag{2.10}$$

One can see that Theorem 2.1 and Corollary 2.2 give an approximate reconstruction method for Problem 1.2b (of the introduction) for v satisfying (1.17) (with $\hat{v} \in \mathcal{C}(\mathbb{R}^3)$).

Note that (2.9) follows from the property that $R \in L^{\infty}_{\mu}(\mathbb{R}^3)$, the assumption (2.8) and the part II of Proposition 3.2 with $\mu = \mu^*$. Further, Corollary 2.2 follows

from Theorem 2.1 and estimates (6.25), (6.26). The approximate reconstruction of Corollary 2.2 is presented in more detail in Proposition 6.8 complemented by formulas (7.5)-(7.8).
One can see that Theorem 2.1 and Corollary 2.2 give also reconstruction results for Problem 1.1a and Problem 1.1b (of the introduction) for $d = 3$ and v satisfying (1.17) (with $\hat{v} \in \mathcal{C}(\mathbb{R}^3)$). Let us compare these results with the reconstructions for Problems 1.1a and 1.1b for $d = 3$ via formulas (2.11), (2.12), (2.14) presented below. From formula (1.4), equation (1.5) and Proposition 3.2 (of Section 3) it follows that if v satisfies (2.3), then

$$\hat{v}(p) = \lim_{\substack{(k,l)\in\Theta,\ k-l=p \\ |\operatorname{Im} k|=|\operatorname{Im} l|=\tau\to\infty}} h(k,l) \ \text{ for any } \ p \in \mathbb{R}^3, \tag{2.11}$$

$$\begin{gathered} |\hat{v}(p) - h(k,l)| \le \frac{2c_2(\mu)C^2}{(1+|p|)^{\mu}} \frac{(\ln \tau)^2}{\tau} \\ \text{for } \ (k,l)\in\Theta, \ \ p=k-l, \ \ |\operatorname{Im} k|=|\operatorname{Im} l|=\tau \ge \tau(C,\mu), \ \ \|\hat{v}\|_{\mu} \le C, \end{gathered} \tag{2.12}$$

where $c_2(\mu)$ is the constant of Lemma 3.1 and $\tau(C,\mu)$ is the smallest number such that

$$c_2(\mu)C\frac{(\ln \tau(C,\mu))^2}{\tau(C,\mu)} \le \frac{1}{2}, \ \ \ln \tau(C,\mu) \ge 2.$$

Actually, for sufficiently regular v on $\mathbb{R}^3$ with sufficient decay at infinity formula (2.11) and some results of the type (2.12) (with less precise right-hand side) were given first in [HN]. Note also that if

$$\begin{gathered} v \in L^{\infty}(\mathbb{R}^3), \ \ \operatorname*{ess\,sup}_{x\in\mathbb{R}^3}(1+|x|)^{3+\varepsilon}|v(x)| \le C \\ \text{for some positive } \ \varepsilon \ \text{ and } \ C, \end{gathered} \tag{2.13}$$

then

$$\begin{gathered} |\hat{v}(p) - h(k,l)| \le \frac{2\tilde{c}_2(\varepsilon)C^2}{\tau} \ \text{ for } \ (k,l) \in \Theta, \ \ p = k - l, \\ |\operatorname{Im} k| = |\operatorname{Im} l| = \tau \ge \tilde{\tau}(C,\varepsilon), \end{gathered} \tag{2.14}$$

where $\tilde{c}_2(\varepsilon)$ and $\tilde{\tau}(C,\varepsilon)$ are some positive constants (similar to constants $c_2(\mu)$ and $\tau(C,\mu)$ of (2.12)) (see [Na1] and [No3] as regards estimate (2.14) under assumption (2.13)). One can see that for $d = 3$ already the simple formulas (2.11), (2.12), (2.14) give a reconstruction method for Problem 1.1a and an approximate reconstruction method for Problem 1.1b. However, for this approximate reconstruction of the Fourier transform $\hat{v}$ from h on Θ^{τ} via (2.12), (2.14) the error decaies rather slowly as $\tau \to +\infty$: even for v of the Schwartz class on $\mathbb{R}^3$ the decay rate of this error, for example, in the uniform norm on the ball $\mathcal{B}_{\tau} = \{p \in \mathbb{R}^3 : \ |p| \le r\}$, where $r > 0$ is fixed, is not faster than $O(1/\tau)$ as $\tau \to +\infty$. An important advantage of the approximation $\hat{v}_{2\tau}$ of Corollary 2.2 in comparison with the approximate reconstruction based on (2.12), (2.14) consists in a fast decay of the error norm $\|\hat{v} - \hat{v}_{2\tau}\|_\mu = O(1/\tau^{\mu^*-\mu})$ as $\tau \to +\infty$ (see estimate (2.10)), at least, if $\mu^* - \mu$ is sufficiently great. For example, if v belongs to the Schwartz class on $\mathbb{R}^3$ and, as in Theorem 2.1 and Corollary 2.2, is sufficiently small in the sense (2.4) for some μ, then estimate (2.10) holds for any $\mu^* > \mu$ and $\|\hat{v} - \hat{v}_{2\tau}\|_\mu = O(\tau^{-\infty})$ as $\tau \to +\infty$. This fast convergence of $\hat{v}_{2\tau}$ to $\hat{v}$ as $\tau \to +\infty$ is in particular important in the framework of applications to Problem 1.3 (of the introduction) via the reduction (1.25)-(1.27): the point is that the determination of $h\big|_{\Theta^\tau}$ from Φ via (1.25)-(1.27) is sufficiently stable for sufficiently small τ only (see related discussion of the introduction), but $\hat{v}_{2\tau}$ reconstructed from $h\big|_{\Gamma^\tau}$ (as described in Corollary 2.2) well approximates $\hat{v}$ even if τ is relatively small (due to the rapid decay of the error $\hat{v} - \hat{v}_{2\tau}$ as $\tau \to +\infty$). An obvious disadvantage of Theorem 2.1 and Corollary 2.2 in comparison with formulas (2.11), (2.12), (2.14) consists in the small norm assumption (2.4). In a subsequent work we plan to propose an approximate reconstruction of $\hat{v}$ from h on Θ^τ (for $d = 3$) with a similar (fast) decay of the error for $\tau \to +\infty$ as in Corollary 2.2 but without the assumption that v is small in some sense.

As it was already mentioned in the introduction, in the present work we formulate also:

(III) characterization for Problem 1.2a for v satisfying (1.17) and

(IV) new characterization for Problem 1.1a or more precisely a characterization for Problem 1.1a for v satisfying (1.17),

see Theorems 2.2 and 2.3 presented next.

Theorem 2.3. *Let v satisfy* (2.3) *and*

$$\|\hat{v}\|_\mu \le C < 1/c_1(\mu), \tag{2.15}$$

where $c_1(\mu)$ is the constant of Lemma 3.1. *Then R* (*defined according to* (2.1), (1.4), (1.5)) *satisfies* (2.5). *Conversely, let*

$$R \in L^\infty_\mu(\mathbb{R}^3) \text{ for some } \mu \ge 2 \tag{2.16}$$

and

$$\|R\|_\mu \le r/2, \quad r < c_7(\mu), \tag{2.17}$$

where $c_7(\mu)$ is some positive constant. Then R is the scattering data (*defined according to* (2.1), (1.4), (1.5)) *for some potential v, where*

$$\hat{v} \in L^\infty_\mu(\mathbb{R}^3), \quad \|\hat{v}\|_\mu \le r. \tag{2.18}$$

One can see that Theorem 2.4 gives a characterization for Problem 1.2a (of the introduction) for v satisfying (1.17).

Consider

$$\Omega = \{k \in \mathbb{C}^3,\ p \in \mathbb{R}^3 :\ k^2 = 0,\ p^2 = 2kp\}, \tag{2.19}$$

$$\Xi = \{(k, p) :\ k = \frac{p}{2} + \frac{i|p|}{2}\gamma(p),\ p \in \mathbb{R}^3\} \tag{2.20}$$

where γ is the function of (1.10).

Note that

$$\Omega \approx \Theta, \quad \Xi \approx \Gamma \tag{2.21}$$

or more precisely

$$\begin{aligned} &(k, p) \in \Omega \Rightarrow (k, k - p) \in \Theta, \quad (k, l) \in \Theta \Rightarrow (k, k - l) \in \Omega, \\ &(k, p) \in \Xi \Rightarrow (k, k - p) \in \Gamma, \quad (k, l) \in \Gamma \Rightarrow (k, k - l) \in \Xi, \end{aligned} \tag{2.22}$$

where Θ and Γ are defined by (1.3) and (1.10a) for $d = 3$. Due to (2.21), (2.22), h on Θ in Problem 1.1 for $d = 3$ can be considered as H on Ω and h on Γ in Problem 1.2 for $d = 3$ can be considered as H on Ξ, where h and H are related by (1.4).

Consider

$$\begin{aligned} &L^\infty_\mu(\Omega) = \{U \in L^\infty(\Omega) :\ |||U|||_\mu < +\infty\}, \\ &|||U|||_\mu = \operatorname*{ess\,sup}_{(k,p)\in\Omega} (1 + |p|)^\mu |U(k, p)|, \quad \mu > 0. \end{aligned} \tag{2.23}$$

Theorem 2.4. *Let v satisfy* (2.3), (2.15) *and H be defined on Ω by means of* (1.5). *Then*

$$H \in L^\infty_\mu(\Omega), \quad |||H|||_\mu \le \frac{C}{1 - c_1(\mu)C}, \tag{2.24}$$

and for almost any $p \in \mathbb{R}^3\backslash 0$ the $\bar\partial$- equation (3.13) *for H on Ω holds.*

Conversely, let

$$H \in L^\infty_\mu(\Omega) \ \text{for some} \ \mu \ge 2, \tag{2.25}$$

$$|||H|||_\mu \le r, \quad r < c_8(\mu), \tag{2.26}$$

where c_8 is a positive constant, and for almost any $p \in \mathbb{R}^3\backslash 0$ the $\bar\partial$- equation (3.13) *holds. Then H on Ω is the scattering data* (*defined using* (1.5)) *for some potential v, where*

$$\hat v \in L^\infty_\mu(\mathbb{R}^3), \quad \|\hat v\|_\mu \le r. \tag{2.27}$$

One can see that Theorem 2.4 gives a characterization for Problem 1.1a (of the introduction) for v satisfying (1.17) (and where h on Θ is considered as H on Ω). In a separate work we plan to give a detailed comparison of Theorem 2.4 with related results of [BC1] and [HN]. In particular, Theorem 2.4 develops and simplifies the results of [BC1] on the range characterization of H on Ω.

The scheme of proof of Theorems 2.3 and 2.4 consists in the following:

(1) The result that (2.3), (2.15) imply (2.5) and (2.24) follows from Proposition 3.2.

(2) It is a separate lemma that the $\bar{\partial}$- equation (3.13) remains valid for almost any $p \in \mathbb{R}^3 \backslash 0$ if v satisfies (2.3) and (2.15).

(3) To prove the sufficiency parts of Theorems 2.3 and 2.4, we use Proposition 3.2, the aforementioned separate lemma concerning the $\bar{\partial}$- equation (3.13), and the analysis developed in Sections 4, 5, 6 and 7. In addition, in the framework of this proof we obtain that the constants $c_7(\mu)$ and $c_8(\mu)$ of Theorems 2.3 and 2.4 can be defined as follows:

$$c_7(\mu) = \min\left(\tfrac{1}{4c_6(\mu)},\ \tfrac{1}{c_1(\mu)+2c_6(\mu)}\right), \tag{2.28}$$

$$c_8(\mu) = \tfrac{1}{c_1(\mu)+2c_6(\mu)}, \tag{2.29}$$

where c_1 and c_6 are the constants of Lemmas 3.1 and 6.4.

On the basis of this scheme we plan to give a complete proof of Theorems 2.3 and 2.4 in a separate work, where we plan to show also that Theorem 2.1 and Corollary 2.2 remain valid without the additional assumption that $\hat{v} \in \mathcal{C}(\mathbb{R}^3)$.

3. Some results on direct scattering

In this section we give some results on direct scattering at zero energy in three dimensions or, more precisely, some results concerning equation (1.5) and the function H of (1.5) under assumption (1.8).

Consider the operator $A(k)$ from (1.5) for $d = 3$:

$$(A(k)U)(p) = \int\limits_{\mathbb{R}^3} \frac{\hat{v}(p+\xi)U(-\xi)d\xi}{\xi^2 + 2k\xi}, \quad p \in \mathbb{R}^3, \quad k \in \Sigma, \tag{3.1}$$

where U is a test function, Σ is defined by (1.7) for $d = 3$. Let $\mathcal{C}$ stand for continuous functions.

Lemma 3.1. *Let v satisfy* (1.8), *$A(k)$ be defined by* (3.1) *and $U \in L^\infty_\mu(\mathbb{R}^3)$. Then:*

$$A(k)U \in \mathcal{C}(\mathbb{R}^3), \tag{3.2}$$

$$\|A(k)U\|_\mu \le c_1(\mu)\|\hat{v}\|_\mu \|U\|_\mu, \tag{3.3a}$$

$$\|A(k)U\|_\mu \le c_2(\mu)\|\hat{v}\|_\mu \|U\|_\mu \frac{(\ln(|\operatorname{Im} k|))^2}{|\operatorname{Im} k|}, \quad \ln|\operatorname{Im} k| \ge 2, \tag{3.3b}$$

for $k \in \Sigma$ *(defined by* (1.7) *for* $d = 3$*), where* $c_1(\mu)$*,* $c_2(\mu)$ *and* $\rho(\mu)$ *are some positive constants; in addition,*

$$\|(A(k') - A(k))U\|_\mu \le \Delta(k, k')\|\hat{v}\|_\mu \|U\|_\mu \tag{3.4a}$$

for some $\Delta(k, k')$ *such that*

$$\lim_{k' \to k} \Delta(k, k') = 0, \tag{3.4b}$$

where $k, k' \in \Sigma$*; in addition,*

$$(A(k)U)(p) \in \mathcal{C}(\Sigma \times \mathbb{R}^3) \ \text{ as a function of } \ k \ \text{ and } \ p. \tag{3.5}$$

Lemma 3.1 is proved in Section 8.

Proposition 3.2. *Let* v *satisfy* (1.8) *and* $\|\hat{v}\|_\mu \le C$*. Then the following statements are valid:*
(I) *if*

$$\eta_1(C) \stackrel{\text{def}}{=} c_1(\mu)C < 1, \tag{3.6}$$

then equation (1.5) *is uniquely solvable for* $H(k, \cdot) \in L^\infty_\mu(\mathbb{R}^3)$ *for any* $k \in \Sigma$ *(by the method of successive approximations) and*

$$\|H(k, \cdot)\|_\mu \le \frac{C}{1 - c_1(\mu)C}, \quad k \in \Sigma, \tag{3.7}$$

$$H - \hat{v} \in \mathcal{C}(\Sigma \times \mathbb{R}^3), \tag{3.8a}$$

$$|H(k, p) - \hat{v}(p)| \le \frac{c_1(\mu)C^2}{(1 - c_1(\mu)C)(1 + |p|)^\mu}, \quad k \in \Sigma, \quad p \in \mathbb{R}^3; \tag{3.8b}$$

(II) *if*

$$\eta_2(C, \tau) \stackrel{\text{def}}{=} c_2(\mu)C\frac{(\ln \tau)^2}{\tau} < 1, \quad \ln \tau \ge 2, \tag{3.9}$$

then equation (1.5) *is uniquely solvable (by the method of successive approximations) for* $H(k, \cdot) \in L^\infty_\mu(\mathbb{R}^3)$ *for any* $k \in \Sigma \backslash \Sigma^\tau$*, where*

$$\Sigma^\tau = \{k \in \Sigma : \ |\operatorname{Im} k| < \tau\}, \tag{3.10}$$

and

$$|H(k, \cdot)|_\mu \le \frac{C}{1 - \eta_2(C, |\operatorname{Im} k|)}, \quad k \in \Sigma \backslash \Sigma^\tau, \tag{3.11}$$

$$H - \hat{v} \in \mathcal{C}((\Sigma \backslash \Sigma^\tau) \times \mathbb{R}^3), \tag{3.12a}$$

$$|H(k,p) - \hat{v}(p)| \le \frac{\eta_2(C, |\operatorname{Im} k|)C}{(1 - \eta_2(C, |\operatorname{Im} k|))(1 + |p|)^\mu}, \quad k \in \Sigma \backslash \Sigma^\tau, \quad p \in \mathbb{R}^3. \tag{3.12b}$$

Proposition 3.2 is proved in Section 8.

Further, note that if v satisfies (1.8) and $\|\hat{v}\|_\mu \le C$, where C satisfies (3.6), and also $\hat{v} \in \mathcal{C}(\mathbb{R}^3)$, then the Faddeev function H (of the part I of Proposition 3.2) satisfies the following $\bar{\partial}$-equation on Ω:

$$\bar{\partial}_k H(k,p)\big|_{Z_p} = \sum_{j=1}^{3} \left(-2\pi \int_{\xi \in S_k} \xi_j H(k,-\xi) H(k+\xi, p+\xi) \frac{\mathrm{d}s}{|\operatorname{Im} k|^2} \right) \mathrm{d}\bar{k}_j\big|_{Z_p} \tag{3.13}$$

for any $p \in \mathbb{R}^3 \backslash 0$, where

$$Z_p = \{k \in \mathbb{C}^3 : (k,p) \in \Omega\}, \quad p \in \mathbb{R}^3 \backslash 0, \tag{3.14}$$

$$S_k = \{\xi \in \mathbb{R}^3 : \xi^2 + 2k\xi = 0\}, \quad k \in Z_p, \tag{3.15}$$

$\mathrm{d}s$ is arc-length measure on the circle S_k in $\mathbb{R}^3$. Note also that, under the assumptions of the part II of Proposition 3.2 with $\hat{v} \in \mathcal{C}(\mathbb{R}^3)$, the $\bar{\partial}$-equation (3.13) remains valid with Z_p replaced by $Z_p \cap (\Sigma \backslash \Sigma^\tau)$. Actually, at least under somewhat stronger assumptions on v than in the part I of Proposition 3.2 with $\hat{v} \in \mathcal{C}(\mathbb{R}^3)$, the $\bar{\partial}$-equation (3.13) was obtained for the first time in [BC1].

4. Coordinates on Ω

Consider Ω defined by (2.19). For our considerations we introduce some convinient coordinates on Ω. Let

$$\Omega_\nu = \{k \in \mathbb{C}^3,\ p \in \mathbb{R}^3 \backslash \mathcal{L}_\nu :\ k^2 = 0,\ p^2 = 2kp\}, \tag{4.1}$$

where

$$\mathcal{L}_\nu = \{p \in \mathbb{R}^3 :\ p = t\nu,\ t \in \mathbb{R}\}, \quad \nu \in \mathbb{S}^2. \tag{4.2}$$

Note that Ω_ν is an open and dense subset of Ω.

For $p \in \mathbb{R}^3 \backslash \mathcal{L}_\nu$ consider $\theta(p)$ and $\omega(p)$ such that

$$\begin{aligned} &\theta(p), \omega(p) \ \text{ smoothly depend on } \ p \in \mathbb{R}^3 \backslash \mathcal{L}_\nu, \\ &\text{take values in } \ \mathbb{S}^2, \ \text{ and} \\ &\theta(p)p = 0,\ \omega(p)p = 0,\ \theta(p)\omega(p) = 0. \end{aligned} \tag{4.3}$$

Note that (4.3) implies that

$$\omega(p) = \frac{p \times \theta(p)}{|p|} \ \text{ for } \ p \in \mathbb{R}^3 \backslash \mathcal{L}_\nu \tag{4.4a}$$

or

$$\omega(p) = -\frac{p \times \theta(p)}{|p|} \ \text{ for } \ p \in \mathbb{R}^3 \backslash \mathcal{L}_\nu, \tag{4.4b}$$

where $\times$ denotes vector product.

To satisfy (4.3), (4.4a) we can take

$$\theta(p) = \frac{\nu \times p}{|\nu \times p|}, \ \omega(p) = \frac{p \times \theta(p)}{|p|}, \ p \in \mathbb{R}^3 \backslash \mathcal{L}_\nu. \tag{4.5}$$

Lemma 4.1. *Let* θ, ω *satisfy* (4.3). *Then the following formulas give a diffeomorphism between* Ω_ν *and* $(\mathbb{C}\backslash 0) \times (\mathbb{R}^3 \backslash \mathcal{L}_\nu)$*:*

$$(k,p) \to (\lambda,p), \ \ where \ \ \lambda = \lambda(k,p) = \frac{2k(\theta(p) + i\omega(p))}{i|p|}, \tag{4.6}$$

$$(\lambda,p) \to (k,p), \ \ where \ \ k = k(\lambda,p) = \kappa_1(\lambda, p)\theta(p) + \kappa_2(\lambda,p)\omega(p) + \frac{p}{2}, \tag{4.7}$$

$$\kappa_1(\lambda, p) = \frac{i|p|}{4}\left(\lambda + \frac{1}{\lambda}\right), \ \ \kappa_2(\lambda, p) = \frac{|p|}{4}\left(\lambda - \frac{1}{\lambda}\right),$$

where $(k, p) \in \Omega_\nu$, $(\lambda, p) \in (\mathbb{C}\backslash 0) \times (\mathbb{R}^3 \backslash \mathcal{L}_\nu)$.

Actually, Lemma 4.1 follows from properties (4.3) and the result that formulas (4.6), (4.7) for $\lambda(k)$ and $k(\lambda)$ at fixed $p \in \mathbb{R}^3 \backslash \mathcal{L}_\nu$ give a diffeomorphism between $\{k \in \mathbb{C}^3 : \ k^2 = 0, \ p^2 = 2kp\}$ and $\mathbb{C}\backslash 0$. The latter result follows from the fact (see [GN, No2]) that the following formulas

$$\lambda = \frac{k_1 + ik_2}{i|E|^{1/2}}, \ \ k_1 = \frac{i|E|^{1/2}}{2}\left(\lambda + \frac{1}{\lambda}\right), \ \ k_2 = \frac{|E|^{1/2}}{2}\left(\lambda - \frac{1}{\lambda}\right)$$

give a diffeomorphism between $\{k \in \mathbb{C}^2 : \ k^2 = E\}$, $E < 0$, and $\mathbb{C}\backslash 0$.

Note that for k and λ of (4.6), (4.7) the following formulas hold:

$$|\operatorname{Im} k| = \frac{|p|}{4}\left(|\lambda| + \frac{1}{|\lambda|}\right), \ \ |\operatorname{Re} k| = \frac{|p|}{4}\left(|\lambda| + \frac{1}{|\lambda|}\right), \tag{4.8}$$

where $(k, p) \in \Omega_\nu$, $(\lambda, p) \in (\mathbb{C}\backslash 0) \times (\mathbb{R}^3 \backslash \mathcal{L}_\nu)$.

We consider λ, p of Lemma 4.1 as coordinates on Ω_ν and on Ω.

5. $\bar\partial$-equation for H on Ω in the coordinates λ, p

Lemma 5.1. *Let the assumptions of the part I of Proposition* 3.1 *be fulfilled and* $\hat{v} \in \mathcal{C}(\mathbb{R}^3)$. *Let* λ, p *be the coordinates of Lemma* 4.1, *where* θ, ω *satisfy* (4.3), (4.4a). *Then*

$$\begin{aligned}\frac{\partial}{\partial\bar\lambda} H(k(\lambda,p), p) = -\frac{\pi}{4}\int_{-\pi}^{\pi}\left(\frac{|p|}{2}\frac{(|\lambda|^2-1)}{\bar\lambda|\lambda|}(\cos\varphi - 1) - |p|\frac{1}{\bar\lambda}\sin\varphi\right) \\ \times H(k(\lambda,p), -\xi(\lambda,p,\varphi))H(k(\lambda,p,\varphi), p+\xi(\lambda,p,\varphi))d\varphi\end{aligned} \tag{5.1}$$

for $\lambda \in \mathbb{C}\backslash 0$, $p \in \mathbb{R}^3\backslash\mathcal{L}_\nu$, *where* $k(\lambda, p)$ *is defined in* (4.7) (*and also depends on* ν, θ, ω),

$$\xi(\lambda, p, \varphi) = \operatorname{Re} k(\lambda, p)(\cos\varphi - 1) + k^\perp(\lambda, p)\sin\varphi, \tag{5.2}$$

$$k^\perp(\lambda, p) = \frac{\operatorname{Im} k(\lambda, p) \times \operatorname{Re} k(\lambda, p)}{|\operatorname{Im} k(\lambda, p)|}, \tag{5.3}$$

where $\times$ *in* (5.3) *denotes vector product.*

Proof of Lemma 5.1 is given in Section 9. In this proof we deduce (5.1) from (3.13).

Note that (5.1) can be written as

$$\frac{\partial}{\partial\bar\lambda} H(k(\lambda, p), p) = \{H, H\}(\lambda, p), \quad \lambda \in \mathbb{C}\backslash 0, \quad p \in \mathbb{R}^3\backslash\mathcal{L}_\nu, \tag{5.4}$$

where

$$\begin{aligned}\{U_1, U_2\}(\lambda, p) = -\frac{\pi}{4}\int_{-\pi}^{\pi} &\left(\frac{|p|}{2}\frac{|\lambda|^2 - 1}{\bar\lambda|\lambda|}(\cos\varphi - 1) - \frac{|p|}{\bar\lambda}\sin\varphi\right)\\ &\times U_1(k(\lambda, p), -\xi(\lambda, p, \varphi))\\ &\times U_2(k(\lambda, p) + \xi(\lambda, p, \varphi), p + \xi(\lambda, p, \varphi))d\varphi,\end{aligned} \tag{5.5}$$

where U_1, U_2 are test functions on Ω (defined by (2.19)) and $k(\lambda, p)$, $\xi(\lambda, p, \varphi)$ are defined by (4.7), (5.2), $(\lambda, p) \in (\mathbb{C}\backslash 0) \times (\mathbb{R}^3\backslash\mathcal{L}_\nu)$. Note that in the left-hand side of (5.1), (5.4)

$$(k(\lambda, p), p) \in \Omega_\nu \tag{5.6a}$$

and in the right-hand side of (5.1), (5.5)

$$\begin{aligned}&(k(\lambda, p), -\xi(\lambda, p, \varphi)) \in \Omega\backslash(0, 0),\\ &(k(\lambda, p) + \xi(\lambda, p, \varphi), p + \xi(\lambda, p, \varphi)) \in \Omega\backslash(0, 0),\end{aligned}$$

where $\lambda \in \mathbb{C}\backslash 0$, $p \in \mathbb{R}^3\backslash\mathcal{L}_\nu$, $\varphi \in [-\pi, \pi]$ (and (0,0) denotes the point $\{k = 0, p = 0\}$).

Lemma 5.2. *Let the assumptions of Lemma* 4.1 *be fulfilled. Let* $U_1, U_2 \in L^\infty_\mu(\Omega)$ *for some* $\mu \geq 2$, *where* $L^\infty_\mu(\Omega)$ *is defined by* (2.23). *Let* $\{U_1, U_2\}$ *be defined by* (5.5). *Then:*

$$\{U_1, U_2\} \in L^\infty_{\text{local}}((\mathbb{C}\backslash 0) \times (\mathbb{R}^3\backslash\mathcal{L}_\nu)) \tag{5.7}$$

and

$$\begin{aligned}|\{U_1, U_2\}(\lambda, p)| \leq{}& \frac{|||U_1|||_\mu |||U_2|||_\mu}{(1 + |p|)^\mu}\\ &\times\left(\frac{c_3(\mu)|\lambda|}{(|\lambda|^2 + 1)^2} + \frac{c_4(\mu)|p|||\lambda|^2 - 1|}{|\lambda|^2(1+|p|(|\lambda|+|\lambda|^{-1}))^2} + \frac{c_5(\mu)|p|}{|\lambda|(1+|p|(|\lambda|+|\lambda|^{-1}))}\right)\end{aligned} \tag{5.8}$$

for almost all $(\lambda, p) \in (\mathbb{C}\backslash 0) \times (\mathbb{R}^3\backslash\mathcal{L}_\nu)$.

Proof of Lemma 5.2 is given in Section 10.

6. Finding H on Ω from its nonredundant restrictions $H\big|_{\Xi}$

Our next purpose is to give an integral equation for finding H on Ω from $R = H\big|_{\Xi}$, where Ω and Ξ are defined by (2.19), (2.20). Actually, we will give an integral equation for finding H on Ω_ν from $R = H\big|_{\Xi_\nu}$, where Ω_ν is defined by (4.1) and $\Xi_\nu = \Xi \cap \Omega_\nu$. In the coordinates of Lemma 4.1 this means that we will give an integral equation for finding

$$H(\lambda, p) = H(k(\lambda, p), p), \ \ \lambda \in \mathbb{C}\backslash 0, \ \ p \in \mathbb{R}^3\backslash\mathcal{L}_\nu, \tag{6.1}$$

from

$$R(p) = H(\lambda_0(p), p) = H(k(\lambda_0(p), p), p), \ \ p \in \mathbb{R}^3\backslash\mathcal{L}_\nu, \tag{6.2}$$

where λ_0 of (6.2) is a piecewise continuous function of $p \in \mathbb{R}^3\backslash\mathcal{L}_\nu$ with values in

$$T = \{\lambda \in \mathbb{C} : \ |\lambda| = 1\}. \tag{6.3}$$

These properties of λ_0 of (6.2) follow from the properties of γ of (1.10a) and from (4.6). Note that if, for example, $\gamma = \theta$, where θ, ω are defined by (4.5), then $\lambda_0(p) \equiv 1$ for $p \in \mathbb{R}^3\backslash\mathcal{L}_\nu$.

We will use the following formula

$$u(\lambda) = u(\lambda_0) - \frac{1}{\pi}\int_{\mathbb{C}} \frac{\partial u(\zeta)}{\partial\bar\zeta}\left(\frac{1}{\zeta-\lambda} - \frac{1}{\zeta-\lambda_0}\right) d\,\mathrm{Re}\,\zeta\, d\ \mathrm{Im}\ \zeta, \quad \lambda \in \mathbb{C}\backslash 0, \ \ \lambda_0 \in \mathbb{C}\backslash 0, \tag{6.4}$$

where $u(\lambda)$ is continuous and bounded for $\lambda \in \mathbb{C}\backslash 0$, $\partial u(\lambda)/\partial\bar\lambda$ is bounded for $\lambda \in \mathbb{C}\backslash 0$, and $\partial u(\lambda)/\partial\bar\lambda = O(|\lambda|^{-2})$ as $|\lambda| \to \infty$. Note that the aforementioned assumptions on $\partial u(\lambda)/\partial\bar\lambda$ in (6.4) can be somewhat weakened. One can prove (6.4) using the formula

$$\frac{\partial}{\partial\bar\lambda}\frac{1}{\pi\lambda} = \delta(\lambda) \tag{6.5}$$

(where δ is the Dirac function), the Liouville theorem and the property that (6.4) holds for $\lambda = \lambda_0$.

Proposition 6.1. *Let the assumptions of Lemma* 5.1 *be fulfilled. Let* $H = H(\lambda, p)$, $R = R(p)$ *be defined by* (6.1), (6.2). *Then* $H = H(\lambda, p)$, $(\lambda, p) \in (\mathbb{C}\backslash 0) \times (\mathbb{R}^3\backslash\mathcal{L}_\nu)$, *satisfies the following nonlinear integral equation*

$$H(\lambda, p) = R(p) + M(H)(\lambda, p), \ \ \lambda \in \mathbb{C}\backslash 0, \ \ p \in \mathbb{R}^3\backslash\mathcal{L}_\nu, \tag{6.6}$$

where

$$M(U)(\lambda,p) = -\frac{1}{\pi}\int_{\mathbb{C}} (U,U)(\zeta,p)\left(\frac{1}{\zeta-\lambda} - \frac{1}{\zeta-\lambda_0(p)}\right) d\,\mathrm{Re}\,\zeta\, d\ \mathrm{Im}\ \zeta, \quad \lambda \in \mathbb{C}\backslash 0, \ \ p \in \mathbb{R}^3\backslash\mathcal{L}_\nu, \tag{6.7}$$

$$(U_1, U_2)(\zeta, p) = \{U_1', U_2'\}(\zeta, p), \quad \zeta \in \mathbb{C}\backslash 0, \quad p \in \mathbb{R}^3\backslash\mathcal{L}_\nu, \tag{6.8a}$$

$$U_j'(k, p) = U_j(\lambda(k, p), p), \quad (k, p) \in \Omega_\nu, \quad j = 1, 2, \tag{6.8b}$$

where U, U_1, U_2 *are test functions on* $(\mathbb{C}\backslash 0) \times (\mathbb{R}^3\backslash\mathcal{L}_\nu)$, $\{U_1', U_2'\}$ *is defined by* (5.5), $\lambda_0 = \lambda_0(p)$ *is the function of* (6.2), $\lambda(k, p)$ *is defined in* (4.6).

Remark 6.2. In addition to (6.8), note that definition of (U_1, U_2) can be also written as

$$\begin{aligned}(U_1, U_2)(\lambda, p) = -\frac{\pi}{4}\int_{-\pi}^{\pi} &\left(\frac{|p|}{2}\frac{|\lambda|^2 - 1}{\bar{\lambda}|\lambda|}(\cos\varphi - 1) - \frac{|p|}{\bar{\lambda}}\sin\varphi\right) \\ &\times U_1(z_1(\lambda, p, \varphi), -\xi(\lambda, p, \varphi)) \\ &\times U_2(z_2(\lambda, p, \varphi), p + \xi(\lambda, p, \varphi))d\varphi,\end{aligned} \tag{6.9}$$

where

$$\begin{aligned} z_1(\lambda,p,\varphi) &= \frac{2k(\lambda, p)(\theta(-\xi(\lambda,p,\varphi)) + i\omega(-\xi(\lambda,p,\varphi)))}{i|p|}, \\ z_2(\lambda,p,\varphi) &= \frac{2(k(\lambda,p)+\xi(\lambda,p,\varphi))(\theta(p+\xi(\lambda,p,\varphi)) + i\omega(p+\xi(\lambda,p,\varphi)))}{i|p|}, \end{aligned} \tag{6.10}$$

$\lambda \in \mathbb{C}\backslash 0$, $p \in \mathbb{R}^3\backslash\mathcal{L}_\nu$, $\varphi \in [-\pi, \pi]$, $k(\lambda, p)$ is defined in (4.7), $\xi(\lambda, p, \varphi)$ is defined by (5.2), θ, ω are the vector functions of (4.3), (4.4a).

Remark 6.3. Under the assumptions of Proposition 6.1, equation (6.6) holds, at least, for almost any $(\lambda, p) \in (\mathbb{C}\backslash 0) \times (\mathbb{R}^3\backslash\mathcal{L}_\nu)$.

Proposition 6.1 follows from Lemmas 4.1, 5.1, 5.2 and formula (6.4) for $u(\lambda) = H(\lambda, p)$ (defined by (6.1)).

Consider

$$\begin{aligned} &L^\infty_\mu((\mathbb{C}\backslash 0)\times(\mathbb{R}^3\backslash\mathcal{L}_\nu)) = \{U \in L^\infty((\mathbb{C}\backslash 0)\times(\mathbb{R}^3\backslash\mathcal{L}_\nu)) : \ |||U|||_\mu < \infty\}, \\ &|||U|||_\mu = \operatorname*{ess\,sup}_{\lambda\in\mathbb{C}\backslash 0,\ p\in\mathbb{R}^3\backslash\mathcal{L}_\nu} (1 + |p|)^\mu |U(\lambda, p)|, \quad \mu > 0. \end{aligned} \tag{6.11}$$

Under the assumptions of Proposition 6.1, from the part I of Proposition 3.2 and formulas (6.1), (6.2) it follows that

$$H, R \in L^\infty_\mu((\mathbb{C}\backslash 0) \times (\mathbb{R}^3\backslash\mathcal{L}_\nu)) \tag{6.12}$$

(where R is independent of $\lambda \in \mathbb{C}\backslash 0$).

Note that

$$M(U)(\lambda, p) = N(U)(\lambda, p) - N(U)(\lambda_0(p), p), \tag{6.13a}$$

$$N(U)(\lambda, p) = I(U, U)(\lambda, p), \tag{6.13b}$$

$$I(U_1, U_2)(\lambda, p) = -\frac{1}{\pi}\int_{\mathbb{C}}(U_1, U_2)(\zeta, p)\frac{d\,\mathrm{Re}\,\zeta\ d\ \mathrm{Im}\ \zeta}{\zeta - \lambda}, \tag{6.13c}$$

where $(\lambda, p) \in (\mathbb{C}\backslash 0)\times(\mathbb{R}^3\backslash\mathcal{L}_\nu)$, U, U_1, U_2 are test functions on $(\mathbb{C}\backslash 0)\times(\mathbb{R}^3\backslash\mathcal{L}_\nu)$, (U_1, U_2) is defined by (6.8).

To deal with nonlinear integral equation (6.6) we use Lemmas 6.4, 6.5 and 6.6 given below.

Lemma 6.4. *Let $U, U_1, U_2 \in L^\infty_\mu((\mathbb{C}\backslash 0)\times(\mathbb{R}^3\backslash\mathcal{L}_\nu))$ for some $\mu \geq 2$. Let $M(U)$, $N(U)$, $I(U_1, U_2)$ be defined by* (6.7), (6.13), *where λ, p are the coordinates of Lemma* 4.1 *under assumption* (4.4*a*). *Then*

$$I(U_1, U_2), N(U), M(U) \in L^\infty_\mu((\mathbb{C}\backslash 0)\times(\mathbb{R}^3\backslash\mathcal{L}_\nu)), \tag{6.14a}$$

$$I(U_1, U_2)(\cdot, p), N(U)(\cdot, p), M(U)(\cdot, p) \in C(\mathbb{C}\backslash 0)\cap L^\infty(\mathbb{C}\backslash 0) \quad \textit{for almost any } p \in \mathbb{R}^3\backslash\mathcal{L}_\nu, \tag{6.14b}$$

$$|||I(U_1, U_2)|||_\mu \leq c_6(\mu)|||U_1|||_\mu|||U_2|||_\mu, \tag{6.15a}$$

$$|||N(U)|||_\mu \leq c_6(\mu)|||U|||^2_\mu, \tag{6.15b}$$

$$|||M(U)|||_\mu \leq 2c_6(\mu)|||U|||^2_\mu, \tag{6.15c}$$

$$|||N(U_1) - N(U_2)|||_\mu \leq c_6(\mu)(|||U_1|||_\mu + |||U_2|||_\mu)|||U_1 - U_2|||_\mu, \tag{6.16a}$$

$$|||M(U_1) - M(U_2)|||_\mu \leq 2c_6(\mu)(|||U_1|||_\mu + |||U_2|||_\mu)|||U_1 - U_2|||_\mu. \tag{6.16b}$$

Lemma 6.4 is proved in Section 11.

Lemma 6.5. *Let $\mu \geq 2$ and $0 < r < (4c_6(\mu))^{-1}$. Let M be defined by* (6.7) (*where λ, p are the coordinates of Lemma* 4.1 *under assumption* (4.4a)). *Let $U_0 \in L^\infty_\mu((\mathbb{C}\backslash 0)\times(\mathbb{R}^3\backslash\mathcal{L}_\nu))$ and $|||U_0|||_\mu \leq r/2$. Then the equation*

$$U = U_0 + M(U) \tag{6.17}$$

is uniquely solvable for $U \in L^\infty_\mu((\mathbb{C}\backslash 0)\times(\mathbb{R}^3\backslash\mathcal{L}_\nu))$, $|||U|||_\mu \leq r$, and U can be found by the method of successive approximations, in addition,

$$|||U - (M_{U_0})^n(0)|||_\mu \leq \frac{r(4c_6(\mu)r)^n}{2(1 - 4c_6(\mu)r)}, \quad n \in \mathbb{N}, \tag{6.18}$$

where M_{U_0} denotes the map $V \to U_0 + M(V)$.

Lemma 6.5 is proved in Section 12 (using Lemma 6.4 and the lemma about contraction maps).

Lemma 6.6. *Let the assumptions of Lemma* 6.5 *be fulfilled. Let also $\tilde{U}_0 \in L^\infty_\mu((\mathbb{C}\backslash 0)\times(\mathbb{R}^3\backslash\mathcal{L}_\nu))$, $|||\tilde{U}_0|||_\mu \leq r/2$ and $\tilde{U}$ denote the solution of* (6.17) *with U_0 replaced by $\tilde{U}_0$, where $\tilde{U} \in L^\infty_\mu((\mathbb{C}\backslash 0)\times(\mathbb{R}^3\backslash\mathcal{L}_\nu))$, $|||\tilde{U}|||_\mu \leq r$. Then*

$$|||U - \tilde{U}|||_\mu \leq \frac{|||U_0 - \tilde{U}_0|||_\mu}{1 - 4c_6(\mu)r}. \tag{6.19}$$

Lemma 6.6 is proved in Section 12.

As a corollary of Proposition 6.1 and Lemmas 6.5 and 6.6, we obtain the following result.

Proposition 6.7. *Let the assumptions of Lemma* 5.1 *be fulfilled. Let*

$$r \overset{\text{def}}{=} \frac{2C}{1 - c_1(\mu)C} < \frac{1}{4c_6(\mu)}, \tag{6.20}$$

where C is the constant of Proposition 3.2. *Let* $H = H(\lambda, p)$, $R = R(p)$ *be defined by* (6.1), (6.2). *Then*

$$|||H|||_\mu \le r/2, \quad |||R|||_\mu \le r/2 \tag{6.21}$$

and R uniquely and stably determines H via nonlinear integral equation (6.6) *considered for* $|||H||| \le r$. *In addition, this equation is solvable by the method of successive approximations according to* (6.18) *(of Lemma* 6.5*) and the stability estimate holds according to* (6.19) *(of Lemma* 6.6*) (where* U_0, U, $\tilde{U}_0$, $\tilde{U}$ *should be replaced by* R, H, $\tilde{R}$, $\tilde{H}$*, respectively).*

Finally in this section, we apply Propositions 6.1, 6.7 and Lemmas 6.5, 6.6 to approximate finding H on Ω from $H\big|_{\Xi^\tau}$, where

$$\Omega^\tau = \{(k, p) \in \Omega : \; |\operatorname{Im} k| < \tau\}, \tag{6.22}$$

$$\Xi^\tau = \Xi \cap \Omega^\tau, \tag{6.23}$$

where Ω and Ξ are defined by (2.19), (2.20). In the coordinates of Lemma 4.1 this means that we deals with approximate finding $H = H(\lambda, p)$ defined by (6.1) from $R_{2\tau} = \chi_{2\tau} R$, where $R = R(p)$ is defined by (6.2) and χ_s denotes the multiplication operator by the function $\chi_r(p)$, where

$$\chi_s(p) = 1 \text{ for } |p| < s, \quad \chi_s(p) = 0 \text{ for } |p| \ge s, \quad \text{where } p \in \mathbb{R}^3, \; s > 0. \tag{6.24}$$

One can see that $R_{2\tau}$ is a low-frequency part of R and, thus, $H\big|_{\Xi^\tau}$ is a low-frequency part of $H\big|_{\Xi}$. One can see also that Ω^τ is a low-imaginary part of Ω and, therefore, Ξ^τ is a low-imaginary part of Ξ.

Note that

$$|||\chi_{2\tau} R|||_\mu \le |||R|||_\mu, \tag{6.25}$$

$$|||R - \chi_{2\tau} R|||_\mu \le \frac{|||R|||_{\mu^*}}{(1 + 2\tau)^{\mu^* - \mu}} \tag{6.26}$$

for $R \in L^\infty_{\mu^*}((\mathbb{C}\backslash 0) \times (\mathbb{R}^3 \backslash \mathcal{L}_\nu))$, where $0 \le \mu \le \mu^*$, $\tau > 0$.

Using Propositions 6.1, 6.7, Lemmas 6.5, 6.6 and estimates (6.25), (6.26) we obtain the following result.

Proposition 6.8. *Let the assumptions of Proposition* 6.7 *be fulfilled. Let also*

$$\hat{v} \in L^{\infty}_{\mu^*}(\mathbb{R}^3) \ \text{for some} \ \mu^* > \mu. \tag{6.27}$$

Let $\tau > 0$*. Then:*

$$|||\chi_{2\tau} R|||_{\mu} \le r/2, \tag{6.28a}$$

$$R \in L^{\infty}_{\mu^*}(\mathbb{R}^3 \backslash \mathcal{L}_{\nu}); \tag{6.28b}$$

$\chi_{2\tau} R$ *uniquely and stably determines* $H_{2\tau}$*, where* $H_{2\tau}$ *denotes the solution of the nonlinear integral equation*

$$H_{2\tau} = \chi_{2\tau} R + M(H_{2\tau}), \ \ |||H_{2\tau}|||_{\mu} \le r, \tag{6.29}$$

see Lemmas 6.5, 6.6*; the following estimate holds:*

$$|||H - H_{2\tau}|||_{\mu} \le \frac{|||R|||_{\mu^*}}{(1+2\tau)^{\mu^*-\mu}(1-4c_6(\mu)r)}. \tag{6.30}$$

Note that (6.28b) follows from the property that $R \in L^{\infty}_{\mu}(\mathbb{R}^3 \backslash \mathcal{L}_{\nu})$, the assumption (6.27), the part II of Proposition 3.2 for $\mu = \mu^*$ and definition (6.2). Estimate (6.30) follows from Proposition 6.7, Lemma 6.6 (where U_0, U, $\tilde{U}_0$, $\tilde{U}$ are replaced by R, H, $\chi_{2r} R$, $H_{2\tau}$, respectively) and from (6.28), (6.29), (6.26).

Actually, in Proposition 6.8, $H_{2\tau}$ is a low-frequency approximation to H. In addition, estimate (6.30) shows that the error between $H_{2\tau}$ and H rapidly decays in the norm $|||\cdot|||_{\mu}$ as $\tau \to +\infty$ if $\mu^* - \mu$ is sufficiently great.

7. Finding $\hat{v}$ on $\mathbb{R}^3$ from H on Ω and some related results

Actually, in this section we consider finding $\hat{v}$ on $\mathbb{R}^3 \backslash \mathcal{L}_{\nu}$ from H on Ω_{ν} in the coordinates of Lemma 4.1 under assumption (4.4a). In addition, under the assumptions of Proposition 6.8, we consider also approximate finding $\hat{v}$ on $\mathbb{R}^3 \backslash \mathcal{L}_{\nu}$ from $H_{2\tau}$ introduced in Proposition 6.8 as a low-frequency approximation to H.

Under assumption (2.3), formulas (2.11), (4.7), (4.8) imply that

$$H(\lambda, p) \to \hat{v}(p) \ \text{as} \ \lambda \to 0, \tag{7.1a}$$

$$H(\lambda, p) \to \hat{v}(p) \ \text{as} \ \lambda \to \infty, \tag{7.1b}$$

where $\lambda \in \mathbb{C} \backslash 0$, $p \in \mathbb{R}^3 \backslash \mathcal{L}_{\nu}$ and $H(\lambda, p)$ is defined by (6.1). In addition, under the assumptions of Proposition 6.1, formulas (6.6), (7.1) (and estimates (3.7), (3.8), (5.7), (5.8)) imply that

$$\hat{v}(p) = R(p) + M(H)(0, p), \tag{7.2a}$$

$$\hat{v}(p) = R(p) - N(H)(\lambda_0(p), p) \tag{7.2b}$$

for $p \in \mathbb{R}^3\backslash\mathcal{L}_\nu$, where M, N are defined by (6.7), (6.8), (6.13), and λ_0 is the function of (6.2). In addition, due to (6.13a), we have that

$$M(H)(0, p) = N(H)(0, p) - N(\lambda_0(p), p), \quad p \in \mathbb{R}^3\backslash\mathcal{L}_\nu, \tag{7.3}$$

and, as a corollary of (7.2), (7.3), we have that

$$N(H)(0, p) = 0, \quad p \in \mathbb{R}^3\backslash\mathcal{L}_\nu. \tag{7.4}$$

Further, under the assumptions of Proposition 6.8, using (6.29) we obtain that

$$H_{2\tau}(\lambda, p) \to \hat{v}^+_{2\tau}(p) \text{ as } \lambda \to 0, \tag{7.5a}$$

$$H_{2\tau}(\lambda, p) \to \hat{v}^-_{2\tau}(p) \text{ as } \lambda \to \infty, \tag{7.5b}$$

where

$$\hat{v}^+_{2\tau}(p) = \chi_{2\tau} R(p) + M(H_{2\tau})(0, p), \tag{7.6a}$$

$$\hat{v}^-_{2\tau}(p) = \chi_{2\tau} R(p) - N(H_{2\tau})(\lambda_0(p), p), \tag{7.6b}$$

for $p \in \mathbb{R}^3\backslash\mathcal{L}_\nu$, where M, N are defined by (6.7), (6.8), (6.13) and λ_0 is the function of (6.2). In addition, formulas (1.9), (6.11), (7.1), (7.5) imply that

$$\|\hat{v} - \hat{v}^\pm_{2\tau}\|_\mu \le |||H - H_{2\tau}|||_\mu. \tag{7.7}$$

Under the assumptions of Proposition 6.8, formulas (6.30), (7.7) imply that $\hat{v}$ on $\mathbb{R}^3$ can be approximately determined from $H_{2\tau}$ as $\hat{v}^\pm_{2\tau}$ of (7.5), (7.6) and

$$\|\hat{v} - \hat{v}^\pm_{2\tau}\|_\mu = O\left(\frac{1}{\tau^{\mu^*-\mu}}\right) \text{ as } \tau \to +\infty. \tag{7.8}$$

8. Proofs of Lemma 3.1 and Proposition 3.2

Proof of Lemma 3.1

Proof of (3.3). We have that

$$|A(k)U(p)| \le I(k, p)\|\hat{v}\|_\mu \|U\|_\mu, \tag{8.1}$$

where

$$I(k, p) = \int\limits_{\mathbb{R}^3} \frac{d\xi}{(1 + |p + \xi|)^\mu (1 + |\xi|)^\mu |\xi^2 + 2k\xi|}, \quad k \in \Sigma, \quad p \in \mathbb{R}^3. \tag{8.2}$$

To prove (3.3) it is sufficient to prove that

$$I(k, p) \le \frac{c_1(\mu)}{(1+|p|)^\mu}, \tag{8.3a}$$

$$I(k, p) \le \frac{c_2(\mu)(\ln(|\operatorname{Im} k|))^2}{|\operatorname{Im} k|(1+|p|)^\mu}, \quad \ln|\operatorname{Im} k| \ge 2, \tag{8.3b}$$

where $k \in \Sigma$, $p \in \mathbb{R}^3$. Note that

$$I(k, p) \le \left(\int\limits_{|\xi|\le|p+\xi|} + \int\limits_{|\xi|\ge|p+\xi|} \right) \frac{d\xi}{(1+|p+\xi|)^\mu(1+|\xi|)^\mu|\xi^2+2k\xi|}, \tag{8.4}$$

where $k \in \Sigma$, $p \in \mathbb{R}^3$. Note also that

$$|\xi| \le |p+\xi| \Rightarrow |p+\xi| \ge |p|/2, \quad |\xi| \ge |p+\xi| \Rightarrow |\xi| \ge |p|/2, \tag{8.5}$$

where $\xi, p \in \mathbb{R}^3$. Using (8.4), (8.5) we obtain that

$$I(k, p) \le (1+|p|/2)^{-\mu}(I_1(k) + I_2(k, p)), \tag{8.6}$$

where

$$\begin{aligned} I_1(k) &= \int\limits_{\mathbb{R}^3} \frac{d\xi}{(1+|\xi|)^\mu|\xi^2+2k\xi|}, \\ I_2(k, p) &= \int\limits_{\mathbb{R}^3} \frac{d\xi}{(1+|p+\xi|)^\mu|\xi^2+2k\xi|}, \end{aligned} \tag{8.7}$$

where $k \in \Sigma$, $p \in \mathbb{R}^3$. Note that

$$I_1(k) = I_2(k, 0), \quad k \in \Sigma. \tag{8.8}$$

Note further that

$$\begin{aligned} &I_2(k, p) = \\ &\int\limits_{\mathbb{R}^3} \frac{d\xi}{(1+|(\xi+\operatorname{Re}k)-(\operatorname{Re}k-p)|)^\mu|(\xi+\operatorname{Re}k)^2-(\operatorname{Re}k)^2+2i\operatorname{Im}k(\xi+\operatorname{Re}k)|} \\ &= I_3(k, \operatorname{Re}k-p), \end{aligned} \tag{8.9}$$

$$I_3(k, p) = \int\limits_{\mathbb{R}^3} \frac{d\xi}{(1+|\xi-p|)^\mu|\xi^2-(\operatorname{Re}k)^2+2i\operatorname{Im}k\xi|}, \tag{8.10}$$

where $k \in \Sigma$, $p \in \mathbb{R}^3$.

In view of (8.6)-(8.10), to prove (8.3) it is sufficient to prove that

$$I_3(k,p) \le \tilde c_1(\mu), \tag{8.11a}$$

$$I_3(k,p) \le \frac{\tilde c_2(\mu)(\ln(|\operatorname{Im} k|))^2}{|\operatorname{Im} k|}, \quad |\operatorname{Im} k| \ge \rho(\mu), \tag{8.11b}$$

where $k \in \Sigma$, $p \in \mathbb{R}^3$.
Consider $p_\parallel = p_\parallel(p, \operatorname{Im} k)$, $p_\perp = p_\perp(p, \operatorname{Im} k)$, where

$$p_\parallel = \frac{p \operatorname{Im} k}{|\operatorname{Im} k|}\frac{\operatorname{Im} k}{|\operatorname{Im} k|} \ \text{ for } |\operatorname{Im} k| \ne 0, \quad p_\parallel = 0 \ \text{ for } |\operatorname{Im} k| = 0, \ p_\perp = p - p_\parallel, \tag{8.12}$$

where p, $\operatorname{Im} k \in \mathbb{R}^3$. Using the properties

$$\operatorname{Im} k \operatorname{Re} k = 0, \quad (\operatorname{Re} k)^2 = (\operatorname{Im} k)^2 \ \text{ for } k \in \Sigma \tag{8.13}$$

and changing variables in the integral of (8.10), we obtain that

$$I_3(k,p) = \int\limits_{\mathbb{R}^3} \frac{d\xi}{(1+((\xi_1-|p_\perp|)^2+\xi_2^2+(\xi_3-\operatorname{sgn}(p_\parallel \operatorname{Im} k)|p_\parallel|)^2)^{1/2})^{\mu}|\xi^2-(\operatorname{Im} k)^2+2i|\operatorname{Im} k|\xi_3|}, \tag{8.14}$$

where $k \in \Sigma$, $p \in \mathbb{R}^3$. Further, using (8.14) we obtain that

$$I_3(k,p) \le \sqrt{2} I_4(|\operatorname{Im} k|, |p_\parallel|, |p_\perp|), \tag{8.15}$$

$$I_4(\rho,s,t) = \int\limits_{\mathbb{R}^3} \frac{d\xi}{(1+(\xi_1-t)^2+\xi_2^2+(\xi_3-s)^2)^{\mu/2}(|\xi_1^2+\xi_2^2+\xi_3^2-\rho^2|+2\rho|\xi_3|)}, \tag{8.16}$$

where $k \in \Sigma$, $p \in \mathbb{R}^3$, $\rho, s, t \in [0, +\infty[$. Due to (8.15), to prove (8.11) it is sufficient to prove that

$$I_4(\rho,s,t) \le \tilde c_1(\mu)/\sqrt{2}, \tag{8.17a}$$

$$I_4(\rho,s,t) \le \frac{\tilde c_2(\mu)(\ln \rho)^2}{\sqrt{2}\rho}, \quad \ln \rho \ge 2, \tag{8.17b}$$

where $\rho, s, t \in [0, +\infty[$. Note that

$$I_4(\rho,s,t) \le \left(\int\limits_{|\xi_3|\le|\xi_3-s|} + \int\limits_{|\xi_3|\ge|\xi_3-s|} \right)$$
$$\times \frac{d\xi}{(1+(\xi_1-t)^2+\xi_2^2+(\xi_3-s)^2)^{\mu/2}(|\xi_1^2+\xi_2^2+\xi_3^2-\rho^2|+2\rho|\xi_3|)}$$
$$\le \int\limits_{|\xi_3|\le|\xi_3-s|} \frac{d\xi}{(1+(\xi_1-t)^2+\xi_2^2+\xi_3^2)^{\mu/2}(|\xi_1^2+\xi_2^2+\xi_3^2-\rho^2|+2\rho|\xi_3|)}$$
$$+ \int\limits_{|\xi_3|\ge|\xi_3-s|} \frac{d\xi}{(1+(\xi_1-t)^2+\xi_2^2+(\xi_3-s)^2)^{\mu/2}(|\xi_1^2+\xi_2^2+(\xi_3-s)^2-\rho^2|+2\rho|\xi_3-s|)}$$
$$\le 2\int\limits_{\mathbb{R}^3} \frac{d\xi}{(1+(\xi_1-t)^2+\xi_2^2+\xi_3^2)^{\mu/2}(|\xi_1^2+\xi_2^2+\xi_3^2-\rho^2|+2\rho|\xi_3|)}$$
$$= 2I_4(\rho,0,t), \tag{8.18}$$

where $\rho, s, t \in [0, +\infty[$. In addition, in (8.18) we used, in particular, that

$$\begin{aligned} &|\xi_1^2+\xi_2^2+\xi_3^2-\rho^2|+2\rho|\xi_3| \\ &\ge |\xi_1^2+\xi_2^2+(\xi_3-s)^2-\rho^2|+2\rho|\xi_3-s| \ \text{ if } \ |\xi_3|\ge|\xi_3-s|. \end{aligned} \tag{8.19}$$

To prove (8.19) we rewrite it as

$$\begin{aligned} &\rho^2-\xi_1^2-\xi_2^2-\xi_3^2+2\rho|\xi_3| \ge \rho^2-\xi_1^2-\xi_2^2-(\xi_3-s)^2+2\rho|\xi_3-s| \\ &\text{for } \ |\xi_3|\ge|\xi_3-s|, \ \ \xi_1^2+\xi_2^2+\xi_3^2\le\rho^2, \end{aligned} \tag{8.20a}$$

$$\begin{aligned} &\xi_1^2+\xi_2^2+\xi_3^2-\rho^2+2\rho|\xi_3| \ge \xi_1^2+\xi_2^2+(\xi_3-s)^2-\rho^2+2\rho|\xi_3-s| \\ &\text{for } \ |\xi_3|\ge|\xi_3-s|, \ \ \xi_1^2+\xi_2^2+\xi_3^2\ge\rho^2, \ \ \xi_1^2+\xi_2^2+(\xi_3-s)^2\ge\rho^2, \end{aligned} \tag{8.20b}$$

$$\begin{aligned} &\xi_1^2+\xi_2^2+\xi_3^2-\rho^2+2\rho|\xi_3| \ge \rho^2-\xi_1^2-\xi_2^2-(\xi_3-s)^2+2\rho|\xi_3-s| \\ &\text{for } \ |\xi_3|\ge|\xi_3-s|, \ \ \xi_1^2+\xi_2^2+\xi_3^2\ge\rho^2, \ \ \xi_1^2+\xi_2^2+(\xi_3-s)^2\le\rho^2. \end{aligned} \tag{8.20c}$$

Inequality (8.20a) follows from the inequalities

$$-x^2+2\rho x \ge -y^2+2\rho y \ \text{ for } \ 0\le y\le x\le\rho, \tag{8.21}$$

$$y = |\xi_3-s| \le x = |\xi_3| \le \sqrt{\rho^2-\xi_1^2-\xi_2^2} \le \rho. \tag{8.22}$$

Inequality (8.20b) is obvious. Inequality (8.20c) follows from the inequalities

$$x^2-\delta^2+2\rho x \ge \delta^2-y^2+2\rho y \quad \text{for } 0\le\delta\le\rho,\ \ 0\le y\le\delta\le x, \tag{8.23}$$

$$y=|\xi_3-s|\le\delta=\sqrt{\rho^2-\xi_1^2-\xi_2^2}\le x=|\xi_3|,\ \ \delta=\sqrt{\rho^2-\xi_1^2-\xi_2^2}\le\rho. \tag{8.24}$$

In turn, inequality (8.23) follows from the inequalities

$$\begin{aligned} &x^2-\delta^2+2\rho x\ge 2\rho\delta \ \text{ for } 0\le\delta\le x,\\ &\delta^2-y^2+2\rho y \overset{(8.21)}{\le} 2\rho\delta \ \text{ for } 0\le y\le\delta\le\rho. \end{aligned} \tag{8.25}$$

Thus formulas (8.19), (8.18) are proved.

Due to (8.18), to prove (8.17) it is sufficient to prove that

$$I_4(\rho,0,t)\le\frac{\tilde c_1(\mu)}{2\sqrt2}, \tag{8.26a}$$

$$I_4(\rho,0,t)\le\frac{\tilde c_2(\mu)(\ln\rho)^2}{2\sqrt2\rho},\quad \ln\rho\ge2, \tag{8.26b}$$

where $\rho, t\in[0,+\infty[$. Using spherical coordinates we obtain that

$$\begin{aligned} &I_4(\rho,0,t)\\ &=\int_0^{+\infty}\int_{-\pi}^{\pi}\int_0^{\pi}\frac{r^2\sin\psi\,\mathrm d\psi\,\mathrm d\varphi\,\mathrm dr}{(1+r^2+t^2-2rt\sin\psi\cos\varphi)^{\mu/2}(|r^2-\rho^2|+2\rho\, r|\cos\psi|)}\\ &=2\int_0^{+\infty}\int_{-\pi}^{\pi}\int_0^{\pi/2}\frac{r^2\sin\psi\,\mathrm d\psi\,\mathrm d\varphi\,\mathrm dr}{(1+r^2+t^2-2rt\sin\psi\cos\varphi)^{\mu/2}(|r^2-\rho^2|+2\rho\, r\cos\psi)}\\ &\le4\int_0^{+\infty}\int_{-\pi/2}^{\pi/2}\int_0^{\pi/2}\frac{r^2\sin\psi\,\mathrm d\psi\,\mathrm d\varphi\,\mathrm dr}{(1+r^2+t^2-2rt\sin\psi\cos\varphi)^{\mu/2}(|r^2-\rho^2|+2\rho\, r\cos\psi)}\\ &\le4\int_0^{+\infty}\int_{-\pi/2}^{\pi/2}\int_0^{\pi/2}\frac{r^2\sin\psi\,\mathrm d\psi\,\mathrm d\varphi\,\mathrm dr}{(1+r^2+t^2-2rt\cos\varphi)^{\mu/2}(|r^2-\rho^2|+2\rho\, r\cos\psi)}\\ &=4\int_0^{+\infty}\left(\int_{-\pi/2}^{\pi/2}\frac{\mathrm d\varphi}{(1+r^2+t^2-2rt\cos\varphi)^{\mu/2}}\int_0^{\pi/2}\frac{\sin\psi\,\mathrm d\psi}{(|r^2-\rho^2|+2\rho\, r\cos\psi)}\right)r^2\mathrm dr, \end{aligned} \tag{8.27}$$

where $\rho, t \in [0, +\infty[$. Further, we obtain that:

$$
\begin{aligned}
&\int\limits_{-\pi/2}^{\pi/2} \frac{d\varphi}{1+r^2+t^2-2rt\cos\varphi} \\
&= \int\limits_{-\pi/2}^{\pi/2} \frac{d\varphi}{1+r^2+t^2-2rt(1-2(\sin(\varphi/2))^2)} \\
&= 2\int\limits_{-\pi/4}^{\pi/4} \frac{d\varphi}{1+(r-t)^2+4rt(\sin\varphi)^2} \\
&\le 2\int\limits_{-\pi/4}^{\pi/4} \frac{\sqrt{2}\cos\varphi d\varphi}{1+(r-t)^2+4rt(\sin\varphi)^2} = 4\int\limits_{0}^{1/\sqrt{2}} \frac{\sqrt{2}du}{1+(r-t)^2+4rtu^2} \\
&= \frac{2\sqrt{2}}{\sqrt{rt}} \int\limits_{0}^{\sqrt{2rt}} \frac{du}{1+(r-t)^2+u^2} \le \frac{4\sqrt{2}}{\sqrt{rt}} \int\limits_{0}^{\sqrt{2rt}} \frac{du}{(\sqrt{1+(r-t)^2}+u)^2} \\
&= \frac{4\sqrt{2}}{\sqrt{rt}} \left(\frac{1}{\sqrt{1+(r-t)^2}} - \frac{1}{\sqrt{1+(r-t)^2}+\sqrt{2rt}} \right) \\
&= \frac{8}{\sqrt{1+(r-t)^2}(\sqrt{1+(r-t)^2}+\sqrt{2rt})};
\end{aligned}
\tag{8.28}
$$

$$
\begin{aligned}
&\int\limits_{0}^{\pi/2} \frac{\sin\psi d\psi}{|r^2-\rho^2|+2\rho r\cos\psi} = \int\limits_{0}^{1} \frac{du}{|r^2-\rho^2|+2\rho r u} \\
&= \frac{1}{2\rho r} \ln\left(|r^2-\rho^2|+2\rho r u\right)\Big|_0^1 = \frac{1}{2\rho r} \ln\left(1+\frac{2\rho r}{|r^2-\rho^2|}\right),
\end{aligned}
\tag{8.29}
$$

where $\rho, t \in [0, +\infty[$. Using (8.27)-(8.29) we obtain that

$$
\begin{aligned}
I_4(\rho, 0, t) &\overset{\mu\ge 2}{\le} 32 \int\limits_{0}^{+\infty} \frac{dr}{\sqrt{1+(r-t)^2}(\sqrt{1+(r-t)^2}+\sqrt{2rt})} \\
&\le 32\int\limits_{0}^{+\infty} \frac{dr}{1+(r-t)^2} \le 32\int\limits_{-\infty}^{+\infty} \frac{dr}{1+r^2} = 32\pi \quad \text{for} \quad \rho = 0, \quad t \ge 0,
\end{aligned}
\tag{8.30a}
$$

$$I_4(\rho,0,t) \overset{\mu\geq 2}{\leq} \frac{16}{\rho}\int\limits_0^{+\infty} \frac{\ln\left(1+\frac{2\rho r}{|r^2-\rho^2|}\right) r\,\mathrm{d}r}{\sqrt{1+(r-t)^2}(\sqrt{1+(r-t)^2}+\sqrt{2rt})} \tag{8.30b}$$
$$= I_5(\rho, t/\rho) \ \text{ for } \ \rho > 0, \quad t \geq 0,$$

where

$$I_5(\rho,\varepsilon) = 16\int\limits_0^{+\infty} \frac{\rho\ln\left(1+\frac{2\tau}{|\tau^2-1|}\right)\tau\,\mathrm{d}\tau}{\sqrt{1+\rho^2(\tau-\varepsilon)^2}(\sqrt{1+\rho^2(\tau-\varepsilon)^2}+\rho\sqrt{2\tau\varepsilon})}, \quad \rho>0, \ \varepsilon \geq 0.$$

As regards $I_5(\rho,\varepsilon)$, we will estimate it separately for $\varepsilon \in [0, 1/4]$, $\varepsilon \in [1/4, 2]$ and $\varepsilon \in [2, +\infty[$. For $\varepsilon \in [0, 1/4]$, $\rho > 0$, we start with the partition:

$$I_5(\rho,\varepsilon) = 16\left(\int\limits_0^{1/2} + \int\limits_{1/2}^{3/2} + \int\limits_{3/2}^{+\infty}\right) \frac{\rho\ln\left(1+\frac{2r}{|r^2-1|}\right) r\,\mathrm{d}r}{1+\rho^2(r-\varepsilon)^2+\rho\sqrt{2r\varepsilon}\sqrt{1+\rho^2(r-\varepsilon)^2}}$$
$$= 16(I_{5,1}(\rho,\varepsilon) + I_{5,2}(\rho,\varepsilon) + I_{5,3}(\rho,\varepsilon)), \tag{8.31}$$

where $I_{5,1}$, $I_{5,2}$, $I_{5,3}$ correspond to $\int\limits_0^{1/2}$, $\int\limits_{1/2}^{3/2}$, $\int\limits_{3/2}^{+\infty}$, respectively. Further,

$$I_{5,1}(\rho,\varepsilon) \leq \ln(7/3)\int\limits_0^{1/2} \frac{\rho\, r\,\mathrm{d}r}{1+\rho^2(r-\varepsilon)^2+\rho\sqrt{r\varepsilon}(1+\rho|r-\varepsilon|)} \tag{8.32}$$
$$= \ln(7/3)\tilde{I}_{5,1}(\rho,\varepsilon),$$

where $\rho > 0$, $\varepsilon \in [0, 1/4]$. In addition:

$$\tilde{I}_{5,1}(\rho,\varepsilon) = \int\limits_0^{1/2} \frac{\rho\, r\,\mathrm{d}r}{1+\rho^2 r^2} = \frac{1}{2}\int\limits_0^{1/4} \frac{\rho\,\mathrm{d}\tau}{1+\rho^2\tau} = \frac{\ln\left(1+\rho^2/4\right)}{2\rho} \tag{8.33a}$$

for $\rho > 0$, $\varepsilon = 0$;

$$\tilde{I}_{5,1}(\rho,\varepsilon)$$
$$= \left(\int\limits_0^{1/2} + \int\limits_{1/2}^{3/2} + \int\limits_{3/2}^{1/(2\varepsilon)}\right) \frac{\rho\,\varepsilon^2\tau\,\mathrm{d}\tau}{1+(\rho\varepsilon)^2(\tau-1)^2+\rho\varepsilon\sqrt{\tau}(1+\rho\varepsilon|\tau-1|)} \tag{8.33b}$$
$$= \tilde{I}_{5,1,1}(\rho,\varepsilon) + \tilde{I}_{5,1,2}(\rho,\varepsilon) + \tilde{I}_{5,1,\varepsilon}(\rho,\varepsilon) \ \text{ for } \ \rho > 0, \quad \varepsilon \in]0, 1/4],$$

where $\tilde{I}_{5,1,1}$, $\tilde{I}_{5,1,2}$, $\tilde{I}_{5,1,\varepsilon}$ correspond to $\int\limits_0^{1/2}$, $\int\limits_{1/2}^{3/2}$, $\int\limits_{3/2}^{1/(2\varepsilon)}$, respectively. In addition:

$$\tilde{I}_{5,1,1}(\rho,\varepsilon) \le \frac{\rho\varepsilon^2/4}{1+(\rho\varepsilon)^2/4} = \frac{(\rho\varepsilon)^2}{\rho\,(4+(\rho\varepsilon)^2)} \le \min\left(\frac{1}{\rho}, \frac{\rho}{4^3}\right), \tag{8.34}$$

$$\begin{aligned}\tilde{I}_{5,1,2}(\rho,\varepsilon) &\le 2\int\limits_1^{3/2} \frac{\rho\,\varepsilon^2(3/2)\mathrm{d}\tau}{1+\rho\varepsilon\sqrt{1/2}(1+\rho\varepsilon(\tau-1))} = 3\int\limits_0^{1/2} \frac{\rho\,\varepsilon^2\mathrm{d}\tau}{1+\rho\varepsilon/\sqrt{2}+(\rho\varepsilon)^2\tau/\sqrt{2}}\\ &= \frac{3\sqrt{2}\rho\varepsilon^2}{(\rho\varepsilon)^2}\ln\left(\sqrt{2}+\rho\varepsilon+(\rho\varepsilon)^2\tau\right)\Big|_0^{1/2}\\ &= \frac{3\sqrt{2}}{\rho}\ln\left(1+\frac{(\rho\varepsilon)^2}{2(\sqrt{2}+\rho\varepsilon)}\right) \le \frac{3\sqrt{2}}{\rho}\ln\left(1+\frac{\rho^2}{32\sqrt{2}}\right),\end{aligned} \tag{8.35}$$

$$\begin{aligned}\tilde{I}_{5,1,3}(\rho,\varepsilon) &\le \int\limits_{1/2}^{(2\varepsilon)^{-1}-1} \frac{\rho\,\varepsilon^2(\tau+1)\mathrm{d}\tau}{1+(\rho\varepsilon)^2\tau^2} \le 3\int\limits_{1/2}^{(2\varepsilon)^{-1}} \frac{\rho\,\varepsilon^2\tau\,\mathrm{d}\tau}{1+(\rho\varepsilon)^2\tau^2}\\ &= \frac{3\rho\varepsilon^2}{2(\rho\varepsilon)^2}\ln\left(1+(\rho\varepsilon)^2 x\right)\Big|_{1/4}^{1/(2\varepsilon)^2} \le \frac{3}{2\rho}\ln\left(1+\rho^2/4\right),\end{aligned} \tag{8.36}$$

where $\rho > 0$, $\varepsilon \in]0, 1/4]$. Further,

$$I_{5,2}(\rho,\varepsilon) \le \frac{\rho}{1+\rho^2/16}\int\limits_{1/2}^{3/2} \ln\left(1+\frac{2r}{|r^2-1|}\right) r\,\mathrm{d}r, \tag{8.37}$$

$$\begin{aligned}I_{5,3}(\rho,\varepsilon) &\le \int\limits_{3/2}^{+\infty} \frac{\rho(2r/|r^2-1|)r\,\mathrm{d}r}{1+\rho^2(r-\varepsilon)^2} \le \int\limits_{3/2}^{+\infty} \frac{4\rho(1+(r^2-1)^{-1})\mathrm{d}r}{(1+\rho\,(r-\varepsilon))^2}\\ &\le \frac{8}{1+\rho\,5/4},\end{aligned} \tag{8.38}$$

where $\rho > 0$, $\varepsilon \in]0, 1/4]$.

For $\varepsilon \in [1/4, 2]$, $\rho > 0$ we use the partition:

$$\begin{aligned}I_5(\rho,\varepsilon) &= 16\left(\int\limits_0^{1/8} + \int\limits_{1/8}^{3} + \int\limits_3^{+\infty}\right) \frac{\rho\ln\left(1+\frac{2r}{|r^2-1|}\right) r\,\mathrm{d}r}{1+\rho^2(r-\varepsilon)^2+\rho\sqrt{2r\varepsilon}\sqrt{1+\rho^2(r-\varepsilon)^2}}\\ &= 16(I_{5,4}(\rho,\varepsilon)+I_{5,5}(\rho,\varepsilon)+I_{5,6}(\rho,\varepsilon)),\end{aligned} \tag{8.39}$$

where $I_{5,4}$, $I_{5,5}$, $I_{5,6}$ correspond to $\int\limits_0^{1/8}$, $\int\limits_{1/8}^{3}$, $\int\limits_3^{+\infty}$, respectively. In addition:

$$I_{5,4}(\rho,\varepsilon) \le \ln(3/2)\int\limits_0^{1/8}\frac{\rho\, r\mathrm{d}r}{1+\rho^2(r-\varepsilon)^2} \le \frac{\ln(3/2)\rho}{64+\rho^2}, \tag{8.40}$$

$$I_{5,5}(\rho,\varepsilon) \le \rho\int\limits_{1/8}^{3}\ln\left(1+\frac{2r}{|r^2-1|}\right) r\mathrm{d}r, \tag{8.41a}$$

$$\begin{aligned}
I_{5,5}(\rho,\varepsilon) &\le \int\limits_{1/8}^{3}\frac{3\rho\ln\left(1+\frac{2}{|r-1|}\right)\mathrm{d}r}{1+\rho\sqrt{1/32}(1+\rho\,|r-\varepsilon|)}\\
&\le \left(\int\limits_{|r-\varepsilon|\le|r-1|,\ 1/8<r<3} + \int\limits_{|r-\varepsilon|\ge|r-1|,\ 1/8<r<3}\right)\frac{3\rho\ln\left(1+\frac{2}{|r-1|}\right)\mathrm{d}r}{1+\rho\sqrt{1/32}+\rho^2\sqrt{1/32}|r-\varepsilon|)}\\
&\le 12\sqrt{2}\left(\int\limits_{1/8}^{3}\frac{\rho\ln\left(1+\frac{2}{|r-\varepsilon|}\right)\mathrm{d}r}{4\sqrt{2}+\rho+\rho^2|r-\varepsilon|} + \int\limits_{1/8}^{3}\frac{\rho\ln\left(1+\frac{2}{|r-1|}\right)\mathrm{d}r}{4\sqrt{2}+\rho+\rho^2|r-1|}\right)\\
&\le 48\sqrt{2}\int\limits_0^{3}\frac{\rho\ln\left(1+\frac{2}{r}\right)\mathrm{d}r}{4\sqrt{2}+\rho+\rho^2 r} \le 48\sqrt{2}\int\limits_0^{3}\frac{\ln\left(1+\frac{2}{r}\right)\mathrm{d}r}{1+\rho\, r}\\
&= \frac{48\sqrt{2}}{\rho}\left(\int\limits_0^{1}+\int\limits_1^{3\rho}\right)\frac{\ln\left(1+\frac{2\rho}{\tau}\right)\mathrm{d}\tau}{1+\tau}\\
&\overset{\rho\ge1}{\le} \frac{48\sqrt{2}}{\rho}\left(\int\limits_0^{1}\frac{(\ln(3\rho)+\ln(1/\tau))\mathrm{d}\tau}{1+\tau} + \int\limits_1^{3\rho}\frac{\ln(1+2\rho)\mathrm{d}\tau}{1+\tau}\right)\\
&= \frac{48\sqrt{2}}{\rho}\left(\ln(1+2\rho)\ln(1/2+(3/2)\rho)+\ln(3\rho)\ln 2+\int\limits_0^{1}\frac{\ln(1/\tau)\mathrm{d}\tau}{1+\tau}\right),\quad \rho\ge1,
\end{aligned} \tag{8.41b}$$

$$I_{5,6}(\rho,\varepsilon) \le \int\limits_3^{+\infty}\frac{\rho(2r/|r^2-1|)r\mathrm{d}r}{1+\rho^2(r-\varepsilon)^2} \le \int\limits_3^{+\infty}\frac{4\rho(1+(r^2-1)^{-1})\mathrm{d}r}{(1+\rho\,(r-\varepsilon))^2} \le \frac{5}{1+\rho}, \tag{8.42}$$

where $\rho > 0$, $\varepsilon \in [1/4, 2]$.

For $\varepsilon \in [2, +\infty[$, $\rho > 0$ we use the partition:

$$I_5(\rho, \varepsilon) = 16(I_{5,7}(\rho, \varepsilon) + I_{5,8}(\rho, \varepsilon)), \tag{8.43}$$

where $I_{5,7} = I_{5,1} + I_{5,2}$, $I_{5,8} = I_{5,3}$, where $I_{5,1}$, $I_{5,2}$, $I_{5,3}$ are defined as in (8.31). In addition,

$$I_{5,7}(\rho, \varepsilon) \le \frac{\rho}{1 + \rho^2/4} \int_0^{3/2} \ln\left(1 + \frac{2r}{|r^2 - 1|}\right) r\,\mathrm{d}r, \tag{8.44}$$

$$\begin{aligned}
I_{5,8}(\rho, \varepsilon) &\le \int_{3/2}^{+\infty} \frac{\rho(2r/|r^2-1|)r\,\mathrm{d}r}{1 + \rho^2(r-\varepsilon)^2 + \rho\sqrt{3}(1 + \rho\,|r - \varepsilon|)} \\
&= \int_{3/2}^{+\infty} \frac{2\rho(1 + (r^2-1)^{-1})\mathrm{d}r}{1 + \sqrt{3}\rho + \sqrt{3}\rho^2|r - \varepsilon| + \rho^2(r-\varepsilon)^2} \\
&\le \int_{-\infty}^{+\infty} \frac{4\rho\,\mathrm{d}r}{1 + \sqrt{3}\rho + \sqrt{3}\rho^2|r| + \rho^2 r^2} \\
&\le \int_0^1 \frac{8\rho\,\mathrm{d}r}{\sqrt{3}\rho + \sqrt{3}\rho^2 r} + \int_1^{+\infty} \frac{8\rho\,\mathrm{d}r}{1 + \rho^2 r^2} \\
&\le \frac{8}{\sqrt{3}} \int_0^1 \frac{\mathrm{d}r}{1 + \rho r} + \int_1^{+\infty} \frac{16\rho\,\mathrm{d}r}{(1 + \rho r)^2} = \frac{8}{\sqrt{3}} \frac{\ln(1+\rho)}{\rho} + \frac{16}{1+\rho},
\end{aligned} \tag{8.45}$$

where $\rho > 0$, $\varepsilon \in [2, +\infty[$.

Estimates (8.26) follow from (8.30)-(8.45). Thus, estimates (8.17), (8.11), (8.3) are proved. The proof of (3.3) is completed.

Proof of (3.2). Let

$$f_1(\xi) = \hat{v}(\xi), \quad f_2(\xi) = \frac{U(\xi)}{\xi^2 - 2k\xi}, \tag{8.46}$$

where $\xi \in \mathbb{R}^3$, $k \in \Sigma$. We have, in particular, that

$$f_1 \in L^\infty(\mathbb{R}^3), \quad f_2 \in L^1(\mathbb{R}^3). \tag{8.47}$$

Property (3.2) follows from (8.46), (8.47) and the following lemma.

Lemma 8.1. *Let f_1, f_2 satisfy* (8.47). *Then the convolution*

$$f_1 * f_2 \in \mathcal{C}(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3), \tag{8.48}$$

where

$$(f_1 * f_2)(p) = \int_{\mathbb{R}^3} f_1(p-\xi) f_2(\xi) \mathrm{d}\xi, \quad p \in \mathbb{R}^3. \tag{8.49}$$

Lemma 8.1 follows from the following properties of (fixed) $f_1 \in L^\infty(\mathbb{R}^3)$, $f_2 \in L^1(\mathbb{R}^3)$:

$$\int_{\mathbb{R}^3} |f_2(\xi)| \mathrm{d}\xi < \infty, \tag{8.50a}$$

$$\int_{\mathcal{B}_r} |f_2(\xi)| \mathrm{d}\xi \to 0 \text{ as } r \to +\infty, \tag{8.50b}$$

$$\sup_{\mathrm{mes}\, \mathcal{A} \le \varepsilon} \int_{\mathcal{A}} |f_2(\xi)| \mathrm{d}\xi \to 0 \text{ as } \varepsilon \to 0, \tag{8.50c}$$

$$\begin{aligned} &\forall\, r > 0, \varepsilon > 0, \lambda > 1\ \exists\, u \in \mathcal{C}(\mathcal{B}_{r+1}) \text{ such that} \\ &\mathrm{mes\,supp}\,(f_1 - u) < \varepsilon \text{ in } \mathcal{B}_{r+1}, \quad \|u\|_{\mathcal{C}(\mathcal{B}_{r+1})} \le \lambda \|f_1\|_{L^\infty(\mathbb{R}^3)}, \end{aligned} \tag{8.50d}$$

where

$$\mathcal{B}_r = \{\xi \in \mathbb{R}^3 : \ |\xi| < r\}. \tag{8.51}$$

The proof of (3.2) is completed.

Proof of (3.4). Due to (3.3a), we have that

$$\|(A(k) - A(l))U\|_\mu \le 2c_1(\mu)\|\hat{v}\|_\mu \|U\|_\mu, \quad k, l \in \Sigma. \tag{8.52}$$

Besides, we have that

$$\begin{aligned} &|(A(k) - A(l))U(p)| \\ &\le (\Delta_1(l, \varepsilon, p) + \Delta_2(k, l, \varepsilon, p) + \Delta_3(k, l, \varepsilon, r, p) + \Delta_4(k, l, r, p)) \\ &\quad \times \|\hat{v}\|_\mu \|U\|_\mu, \end{aligned} \tag{8.53}$$

where

$$\Delta_1(l, \varepsilon, p) = \int_{\mathcal{D}(l,\varepsilon)} \frac{\mathrm{d}\xi}{(1 + |p+\xi|)^\mu (1+|\xi|)^\mu |\xi^2 + 2l\xi|}, \tag{8.54}$$

$$\Delta_2(k, l, \varepsilon, p) = \int_{\mathcal{D}(l,\varepsilon)} \frac{\mathrm{d}\xi}{(1 + |p+\xi|)^\mu (1+|\xi|)^\mu |\xi^2 + 2k\xi|}, \tag{8.55}$$

$$\Delta_3(k,l,\varepsilon,r,p) = \int\limits_{\mathcal{B}_r\backslash\mathcal{D}(l,\varepsilon)} \frac{2|(k-l)\xi|d\xi}{(1+|p+\xi|)^{\mu}(1+|\xi|)^{\mu}|\xi^2+2k\xi||\xi^2+2l\xi|}, \quad (8.56)$$

$$\Delta_4(k,l,r,p) = \int\limits_{\mathbb{R}^3\backslash\mathcal{B}_r} \frac{2|(k-l)\xi|d\xi}{(1+|p+\xi|)^{\mu}(1+|\xi|)^{\mu}|\xi^2+2k\xi||\xi^2+2l\xi|}, \quad (8.57)$$

where

$$\mathcal{D}(l,\varepsilon) = \{\xi\in\mathbb{R}^3:\ \ |\xi^2+2l\xi|\le\varepsilon\}, \quad (8.58)$$

$\mathcal{B}_r$ is defined by (8.51),

$$0<\varepsilon\le 1,\ 2|l|+2\le r,\ |k-l|\le 1,\ k,l\in\Sigma,\ \ p\in\mathbb{R}^3, \quad (8.59)$$

where $|z| = (|\mathrm{Re}\, z|^2 + |\,\mathrm{Im}\ z|^2)^{1/2}$ for $z\in\mathbb{C}^d$. Note that

$$\begin{aligned}&|\xi^2+2l\xi|\ge|\xi^2+2\mathrm{Re}\, l\xi|\ge|\xi|(|\xi|-2|\mathrm{Re}\, l|)\ge 2|\xi|\ge 4>\varepsilon\\ &\text{for}\ \ \xi\in\mathbb{R}^3\backslash\mathcal{B}_r\end{aligned} \quad (8.60)$$

and, therefore,

$$\mathcal{D}(l,\varepsilon)\subset\mathcal{B}_r \quad (8.61)$$

under conditions (8.59). Further, we estimate separately Δ_1, Δ_2, Δ_3 and Δ_4.

Estimate of Δ_1. In a similar way with (8.6), (8.7) we obtain that

$$\Delta_1(l,\varepsilon,p)\le(1+|p|/2)^{-\mu}\left(\Delta_{1,1}(l,\varepsilon)+(\Delta_{1,2}(l,\varepsilon,p)\right), \quad (8.62)$$

$$\begin{aligned}\Delta_{1,1}(l,\varepsilon) &= \int\limits_{\mathcal{D}(l,\varepsilon)} \frac{d\xi}{(1+|\xi|)^{\mu}|\xi^2+2l\xi|},\\ \Delta_{1,2}(l,\varepsilon,p) &= \int\limits_{\mathcal{D}(l,\varepsilon)} \frac{d\xi}{(1+|p+\xi|)^{\mu}|\xi^2+2l\xi|},\end{aligned} \quad (8.63)$$

where $0<\varepsilon\le 1$, $l\in\Sigma$, $p\in\mathbb{R}^3$. In addition,

$$\Delta_{1,1}(l,\varepsilon)\le\Delta_{1,3}(l,\varepsilon),\ \ \Delta_{1,2}(l,\varepsilon,p)\le\Delta_{1,3}(l,\varepsilon), \quad (8.64)$$

where

$$\Delta_{1,3}(l,\varepsilon)=\int\limits_{\mathcal{D}(l,\varepsilon)}\frac{\mathrm{d}\xi}{|\xi^2+2l\xi|}\le\int\limits_{\mathcal{D}(l,\varepsilon)}\frac{\sqrt{2}\mathrm{d}\xi}{|\xi^2+2\mathrm{Re}\,l\xi|+2|\,\mathrm{Im}\;l\xi|}$$

$$\overset{(8.13)}{=}\int\limits_{|(\xi+\mathrm{Re}l)^2-(\mathrm{Re}l)^2+2i\,\mathrm{Im}\,l(\xi+\mathrm{Re}\,l)|\le\varepsilon}\frac{\sqrt{2}\mathrm{d}\xi}{|(\xi+\mathrm{Re}l)^2-(\mathrm{Re}l)^2|+2|\,\mathrm{Im}\,l(\xi+\mathrm{Re}l)|}$$

$$\le\int\limits_{|\xi^2-(\mathrm{Re}\,l)^2|\le\varepsilon}\frac{\sqrt{2}\mathrm{d}\xi}{|\xi^2-(\mathrm{Re}l)^2|+2|\,\mathrm{Im}\,l\xi|}\overset{\rho_l=|\mathrm{Re}l|,(8.13)}{\le}\int\limits_{|\xi^2-\rho_l^2|\le\varepsilon}\frac{\sqrt{2}\mathrm{d}\xi}{|\xi^2-\rho_l^2|+2\rho_l|\xi_3|}$$

$$\le\int\limits_{(\max(\rho_l^2-\varepsilon,0))^{1/2}}^{(\rho_l^2+\varepsilon)^{1/2}}\int\limits_{-\pi}^{\pi}\int\limits_{0}^{\pi}\frac{\sqrt{2}r^2\sin\psi\,\mathrm{d}\psi\,\mathrm{d}\varphi\,\mathrm{d}r}{|r^2-\rho_l^2|+2\rho_l r|\cos\psi|}=4\pi\sqrt{2}\Delta_{1,4}(\rho_l,\varepsilon),\tag{8.65}$$

$$\Delta_{1,4}(\rho,\varepsilon)=\int\limits_{(\max(\rho^2-\varepsilon,0))^{1/2}}^{(\rho^2+\varepsilon)^{1/2}}\int\limits_{0}^{\pi/2}\frac{r^2\sin\psi\,\mathrm{d}\psi\,\mathrm{d}r}{|r^2-\rho^2|+2\rho\,r\cos\psi}$$

$$\overset{(8.29)}{=}\frac{1}{2}\int\limits_{(\max(\rho^2-\varepsilon,0))^{1/2}}^{(\rho^2+\varepsilon)^{1/2}}\ln\left(1+\frac{2\rho\,r}{|r^2-\rho^2|}\right)\frac{r}{\rho}\mathrm{d}r\tag{8.66}$$

$$\le\frac{1}{2}\left(\int\limits_{(\max(\rho^2-\varepsilon,0))^{1/2}}^{\rho}+\int\limits_{\rho}^{(\rho^2+\varepsilon)^{1/2}}\right)\ln\left(1+\frac{2\rho}{|r-\rho|}\right)\left(1+\frac{r-\rho}{\rho}\right)\mathrm{d}r$$

$$=\Delta_{1,4,1}(\rho,\varepsilon)+\Delta_{1,4,2}(\rho,\varepsilon),$$

where $\Delta_{1,4,1}$, $\Delta_{1,4,2}$ correspond to $\int\limits_{(\max(\rho^2-\varepsilon,0))^{1/2}}^{\rho}$, $\int\limits_{\rho}^{(\rho^2+\varepsilon)^{1/2}}$ respectively, $0<\varepsilon\le 1$, $l\in\Sigma$, $\rho_l=|\mathrm{Re}\,l|=|l|/\sqrt{2}$, $0\le\rho$. In addition,

$$\Delta_{1,4}(\rho,\varepsilon)=\int\limits_{0}^{\varepsilon^{1/2}}\mathrm{d}r=\varepsilon^{1/2}\ \text{ for }\ \rho=0,\ 0<\varepsilon\le 1,\tag{8.67a}$$

$$\begin{aligned}
\Delta_{1,4,1}(\rho,\varepsilon) &\le \frac{1}{2}\int\limits_{(\max(\rho^2-\varepsilon,0))^{1/2}}^{\rho} \ln\left(1+\frac{2\rho}{|r-\rho|}\right) dr \\
&\overset{(8.68)}{\le} \frac{1}{2}\int\limits_{(\max(\rho^2-\varepsilon,0))^{1/2}}^{\rho} \frac{1}{\alpha}\left(\frac{2\rho}{\rho-r}\right)^{\alpha} dr \\
&= -\frac{(2\rho)^{\alpha}}{2\alpha(1-\alpha)}(\rho-r)^{1-\alpha}\Big|_{(\max(\rho^2-\varepsilon,0))^{1/2}}^{\rho} \\
&= \frac{(2\rho)^{\alpha}}{2\alpha(1-\alpha)}\left(\rho-(\max(\rho^2-\varepsilon,0))^{1/2}\right)^{1-\alpha} \\
&\overset{(8.69)}{\le} \frac{(2\rho)^{\alpha}}{2\alpha(1-\alpha)}\varepsilon^{(1-\alpha)/2}
\end{aligned} \tag{8.67b}$$

for $\rho>0$, $0<\varepsilon\le 1$, $0<\alpha<1$,

$$\begin{aligned}
\Delta_{1,4,2}(\rho,\varepsilon) &\overset{(8.68)}{\le} \int\limits_{\rho}^{(\rho^2+\varepsilon)^{1/2}} \frac{1}{2\alpha}\left(\frac{2\rho}{\eta-\rho}\right)^{\alpha} dr + \int\limits_{\rho}^{(\rho^2+\varepsilon)^{1/2}} dr \\
&= \frac{(2\rho)^{\alpha}}{2\alpha(1-\alpha)}\left((\rho^2+\varepsilon)^{1/2}-\rho\right)^{1-\alpha} + \left((\rho^2+\varepsilon)^{1/2}-\rho\right) \\
&\overset{(8.69)}{\le} \left(\frac{(2\rho)^{\alpha}}{2\alpha(1-\alpha)}+1\right)\varepsilon^{(1-\alpha)/2}
\end{aligned} \tag{8.67c}$$

for $\rho>0$, $0<\varepsilon\le 1$, $0<\alpha<1$.

Note that in (8.67b), (8.67c) we used the inequalities

$$\ln(1+x)\le \alpha^{-1}x^{\alpha} \quad \text{for } x\ge 0,\quad 0<\alpha\le 1, \tag{8.68}$$

$$(x+\varepsilon)^{1/2}-x^{1/2}\le \varepsilon^{1/2} \quad \text{for } x\ge 0,\quad \varepsilon\ge 0. \tag{8.69}$$

Due to (8.62)-(8.67) we have that

$$\Delta_1(l,\varepsilon,p)\le 8\pi\sqrt{2}(1+|p|/2)^{-\mu}\left(1+\frac{(2|\mathrm{Re}\, l|)^{\alpha}}{\alpha(1-\alpha)}\right)\varepsilon^{(1-\alpha)/2} \tag{8.70}$$

for $l\in\Sigma$, $0<\varepsilon\le 1$, $p\in\mathbb{R}^3$, $0<\alpha<1$.

Estimate of Δ_2. In a similar way with (8.62)-(8.65) we obtain that

$$\Delta_2(k,l,\varepsilon,p)\le 2(1+|p|/2)^{-\mu}\tilde{\Delta}_2(k,l,\varepsilon), \tag{8.71}$$

$$\begin{aligned}\tilde{\Delta}_2(k,l,\varepsilon) &= \int\limits_{\mathcal{D}(l,\varepsilon)} \frac{\sqrt{2}\mathrm{d}\xi}{|\xi^2+2\mathrm{Re}\,k\xi|+2|\operatorname{Im}\,k\xi|}\\
&\le \int\limits_{|((\xi+\mathrm{Re}\,k)-(\mathrm{Re}\,k-\mathrm{Re}\,l))^2-(\mathrm{Re}\,l)^2|\le\varepsilon} \frac{\sqrt{2}\mathrm{d}\xi}{|(\xi+\mathrm{Re}\,k)^2-(\mathrm{Re}\,k)^2|+2|\operatorname{Im}\,k(\xi+\mathrm{Re}\,k)|}\\
&= \int\limits_{|(\xi-(\mathrm{Re}\,k-\mathrm{Re}\,l))^2-(\mathrm{Re}\,l)^2|\le\varepsilon} \frac{\sqrt{2}\mathrm{d}\xi}{|\xi^2-(\mathrm{Re}\,k)^2|+2|\operatorname{Im}\,k\xi|},\end{aligned} \tag{8.72}$$

where $k, l \in \Sigma$, $0 < \varepsilon \le 1$, $p \in \mathbb{R}^3$. Note that

$$|(\xi-\zeta)^2-\rho^2| \le \varepsilon \Leftrightarrow \max(\rho^2-\varepsilon,0) \le (\xi-\zeta)^2 \le \rho^2+\varepsilon \Rightarrow \tag{8.73}$$

$$\max((\max(\rho^2-\varepsilon,0))^{1/2}-|\zeta|,0) \le |\xi| \le |\zeta|+(\rho^2+\varepsilon)^{1/2} \tag{8.74}$$

$$\overset{(8.69)}{\Rightarrow} \max(\rho-\varepsilon^{1/2}-|\zeta|,0) \le |\xi| \le \rho+\varepsilon^{1/2}+|\zeta|, \tag{8.75}$$

where $\xi, \zeta \in \mathbb{R}^3$, $\rho \ge 0$, $0 < \varepsilon \le 1$. Using (8.72) and (8.73)-(8.75) for $\zeta = \mathrm{Re}\,k - \mathrm{Re}\,l$, $\rho = |\mathrm{Re}\,l|$, in a similar way with (8.65), (8.66) we obtain that

$$\begin{aligned}&\tilde{\Delta}_2(k,l,\varepsilon)\\
&= 2\pi\sqrt{2}\left(\int\limits_{\max(\rho_l-\delta,0)}^{\rho_k} + \int\limits_{\rho_k}^{\rho_l+\delta}\right) \ln\left(1+\frac{2\rho_k}{|r-\rho_k|}\right)\left(1+\frac{r-\rho_k}{\rho_k}\right)\mathrm{d}r\\
&= 2\pi\sqrt{2}(\tilde{\Delta}_{2,1}(\rho_k,\rho_l,\delta)+\tilde{\Delta}_{2,2}(\rho_k,\rho_l,\delta)),\\
&\rho_k=|\mathrm{Re}k|\ne 0,\ \rho_l=|\mathrm{Re}l|,\ \delta=\varepsilon^{1/2}+|\mathrm{Re}k-\mathrm{Re}l|,\ k,l\in\Sigma,\ 0<\varepsilon\le 1,\end{aligned} \tag{8.76}$$

where $\tilde{\Delta}_{2,1}$, $\tilde{\Delta}_{2,2}$ correspond to $\int\limits_{\max(\rho_l-\delta,0)}^{\rho_k}$, $\int\limits_{\rho_k}^{\rho_l+\delta}$ respectively. In addition, in a similar way with (8.67) we obtain that

$$\tilde{\Delta}_2(k,l,\varepsilon) \le 4\pi\sqrt{2}(\delta+\rho_l-\rho_k) \le 4\pi\sqrt{2}(\varepsilon^{1/2}+2|\mathrm{Re}\,k-\mathrm{Re}\,l|)\ \text{for}\ k=0, \tag{8.77a}$$

$$\begin{aligned}\tilde{\Delta}_{2,1}(\rho_k,\rho_l,\delta) &\le \frac{(2\rho_k)^\alpha}{\alpha(1-\alpha)}(\rho_k-\max(\rho_l-\delta,0))^{1-\alpha}\\
&\le \frac{(2\rho_k)^\alpha}{\alpha(1-\alpha)}(\delta+|\rho_k-\rho_l|)^{1-\alpha}\\
&\le \frac{(2|\mathrm{Re}\,k|)^\alpha}{\alpha(1-\alpha)}(\varepsilon^{1/2}+2|\mathrm{Re}\,k-\mathrm{Re}\,l|)^{1-\alpha},\end{aligned} \tag{8.77b}$$

$$\begin{aligned}\tilde{\Delta}_{2,2}(\rho_k,\rho_l,\delta) &\le \frac{(2\rho_k)^{\alpha}}{\alpha(1-\alpha)}(\rho_l+\delta-\rho_k)^{1-\alpha}+2(\rho_l+\delta-\rho_k)\\ &\le \frac{(2|\mathrm{Re}\,k|)^{\alpha}}{\alpha(1-\alpha)}(\varepsilon^{1/2}+2|\mathrm{Re}\,k-\mathrm{Re}\,l|)^{1-\alpha}\\ &\quad+2(\varepsilon^{1/2}+2|\mathrm{Re}\,k-\mathrm{Re}\,l|),\end{aligned} \tag{8.77c}$$

where $k, l, \varepsilon, \rho_k, \rho_l, \delta$ are the same as in (8.76) and $0<\alpha<1$.

Due to (8.71), (8.76), (8.77) we have that

$$\begin{aligned}&\Delta_2(k,l,\varepsilon,p)\\ &\le 8\pi\sqrt{2}(1+|p|/2)^{-\mu}\left(\frac{(2|\mathrm{Re}\,k|)^{\alpha}}{\alpha(1-\alpha)}+3^{\alpha}\right)(\varepsilon^{1/2}+2|\mathrm{Re}k-\mathrm{Re}l|)^{1-\alpha}\end{aligned} \tag{8.78}$$

for $k,l\in\Sigma,\ |k-l|\le 1,\ 0<\varepsilon\le 1,\ p\in\mathbb{R}^3,\ 0<\alpha<1.$

Estimate of Δ_3. We have that

$$\begin{aligned}\Delta_3(k,l,\varepsilon,r,p) &\overset{(8.51),(8.58)}{\le} \int\limits_{\mathcal{B}_r}\frac{2|k-l|r\,\mathrm{d}\xi}{(1+|p+\xi|)^{\mu}(1+|\xi|)^{\mu}|\xi^2+2k\xi|\varepsilon}\\ &\overset{(8.3a)}{\le} \frac{2|k-l|r}{\varepsilon}\,\frac{c_1(\mu)}{(1+|p|)^{\mu}}\end{aligned} \tag{8.79}$$

under conditions (8.59).

Estimate of Δ_4. We have that

$$\begin{aligned}\Delta_4(k,l,r,p) &\overset{(8.57),(8.59),(8.60)}{\le} \int\limits_{\mathbb{R}^3\setminus\mathcal{B}_r}\frac{|k-l|\mathrm{d}\xi}{(1+|p+\xi|)^{\mu}(1+|\xi|)^{\mu}|\xi^2+2k\xi|}\\ &\overset{(8.3a)}{\le} |k-l|\frac{c_1(\mu)}{(1+|p|)^{\mu}}\end{aligned} \tag{8.80}$$

under conditions (8.59).

Now formulas (3.4) follow from (8.52), (8.53) and estimates (8.70), (8.78)-(8.80) with $\varepsilon=|k-l|^{\beta}$, $0<|k-l|\le 1$ for fixed $k\in\Sigma$, $r\ge 2(|k|+\sqrt{2})+2$, $\alpha\in]0,1[$ and $\beta\in]0,1[$.

The proof of (3.4) is completed. □

Finally, property (3.5) follows from the presentation

$$\begin{aligned}&(A(k)U)(p)-(A(k')U)(p')\\ &=((A(k)U)(p)-(A(k)U)(p'))+((A(k)U)(p')-(A(k')U)(p'))\end{aligned} \tag{8.81}$$

and properties (3.2), (3.4). The proof of Lemma 3.1 is completed. □

Proof of Proposition 3.2. Proposition 3.2 follows from equation (1.5) written as

$$H(k,\cdot) = \hat{v} - A(k)H(k,\cdot) \tag{8.82}$$

and Lemma 3.1. In addition, to obtain (3.8a), (3.12a) we use the presentation

$$\tilde{H}(k,p) - \tilde{H}(k',p') = (\tilde{H}(k,p) - \tilde{H}(k,p')) + (\tilde{H}(k,p') - \tilde{H}(k',p')), \tag{8.83}$$

where

$$\tilde{H}(k,\cdot) \overset{\text{def}}{=} H(k,\cdot) - \hat{v} \overset{(8.82)}{=} -A(k)H(k,\cdot), \tag{8.84}$$

$$\tilde{H}(k,\cdot) \overset{(3.2),(8.84)}{\in} \mathcal{C}(\mathbb{R}^3) \text{ as soon as } H(k,\cdot) \in L^\infty_\mu(\mathbb{R}^3), \tag{8.85}$$

$$\begin{aligned}\tilde{H}(k,\cdot) - \tilde{H}(k',\cdot) &= H(k,\cdot) - H(k',\cdot)\\ &\overset{(8.82)}{=} ((I+A(k))^{-1} - (I+A(k'))^{-1})\hat{v}\\ &= (I+A(k))^{-1}((I+A(k'))^{-1}\\ &\qquad - (I+A(k)))(I+A(k'))^{-1})\hat{v}\\ &= (I+A(k))^{-1}(A(k')-A(k))(I+A(k'))^{-1})\hat{v},\end{aligned} \tag{8.86}$$

$$\tilde{H}(k,\cdot) - \tilde{H}(k',\cdot) \overset{(8.85)}{\in} \mathcal{C}(\mathbb{R}^3) \text{ as soon as } H(k,\cdot), H(k',\cdot) \in L^\infty_\mu(\mathbb{R}^3), \tag{8.87a}$$

$$\begin{gathered}\|\tilde{H}(k,\cdot) - \tilde{H}(k',\cdot)\|_\mu \overset{(3.4),(8.86)}{\to} 0 \text{ as } k' \to k\\ \text{as soon as } (I+A(k'))^{-1} \text{ is uniformly bounded in a neighborhood of } k,\end{gathered} \tag{8.87b}$$

$$\begin{gathered}\sup_{p'\in\mathbb{R}^3} (1+|p'|)^\mu |\tilde{H}(k,p') - \tilde{H}(k',p')| \overset{(8.87)}{\to} 0 \text{ as } k' \to k\\ \text{as soon as } (I+A(k'))^{-1} \text{ is uniformly bounded in a neighborhood of } k,\end{gathered} \tag{8.88}$$

where $k, k' \in \Sigma$, $p, p' \in \mathbb{R}^3$.

The proof of Proposition 3.2 is completed. □

9. Proof of Lemma 5.1

The proof of Lemma 5.1 of the present work is similar to the proof of Lemma 4.1 of [No5]. Proceeding from (3.13), (4.3), (4.4a), (4.7), (5.2), (5.3) in a similar way with the proof of Lemma 4.1 of [No5] we obtain that:

$$\begin{aligned}\frac{\partial}{\partial\bar{\lambda}} H(k(\lambda,p),p) = -\frac{\pi}{2} &\int\limits_{\{\xi\in\mathbb{R}^3:\ \xi^2+2k\xi=0\}} \left(\frac{\partial\bar{\kappa}_1}{\partial\bar{\lambda}}\theta\xi + \frac{\partial\bar{\kappa}_2}{\partial\bar{\lambda}}\omega\xi\right)\\ &\times H(k,-\xi)H(k+\xi,p+\xi)\frac{ds}{|\operatorname{Im} k|^2}, \quad \lambda \in \mathbb{C}\backslash 0, \quad p \in \mathbb{R}^3\backslash\mathcal{L}_\nu,\end{aligned} \tag{9.1}$$

where $k = k(\lambda, p)$, $\kappa_1 = \kappa_1(\lambda, p)$, $\kappa_2 = \kappa_2(\lambda, p)$ are defined in (4.7), $\theta = \theta(p)$, $\omega = \omega(p)$ are the vector-functions of (4.3), (4.4a), ds is arc-length measure on the circle $\{\xi \in \mathbb{R}^3 : \xi^2 + 2k\xi = 0\}$ and, in addition,

$$\mathrm{d}s = |\mathrm{Re}\, k| \mathrm{d}\varphi, \tag{9.2}$$

$$\begin{aligned}\frac{\partial \bar{\kappa}_1}{\partial \bar{\lambda}} \theta\xi + \frac{\partial \bar{\kappa}_2}{\partial \bar{\lambda}} \omega\xi &= \left(\frac{\partial \bar{\kappa}_1}{\partial \bar{\lambda}} \mathrm{Re}\, \kappa_1 + \frac{\partial \bar{\kappa}_2}{\partial \bar{\lambda}} \mathrm{Re}\, \kappa_2 \right) (\cos\varphi - 1) \\ &\quad + \frac{|p|}{2|\mathrm{Im}\, k|} \left(\frac{\partial \bar{\kappa}_1}{\partial \bar{\lambda}} \mathrm{Im}\, \kappa_2 - \frac{\partial \bar{\kappa}_2}{\partial \bar{\lambda}} \mathrm{Im}\, \kappa_1 \right) \sin\varphi\end{aligned} \tag{9.3}$$

under the assumption that the circle $\{\xi \in \mathbb{R}^3 : \xi^2 + 2k\xi = 0\}$ is parametrized by $\varphi \in]-\pi, \pi[$ according to (5.2). (Note that in the proof of Lemma 4.1 of [No5] the $\bar{\partial}$-equation similar to (9.1) is not valid for $|\lambda| = 1$ but it is not indicated because of a misprint.)

The difinition of κ_1, κ_2 (see (4.7)) implies that

$$\frac{\partial \bar{\kappa}_1}{\partial \bar{\lambda}} = -\frac{i|p|}{4} \left(1 - \frac{1}{\bar{\lambda}^2}\right), \quad \frac{\partial \bar{\kappa}_2}{\partial \bar{\lambda}} = \frac{|p|}{4} \left(1 + \frac{1}{\bar{\lambda}^2}\right), \tag{9.4}$$

$$\begin{aligned}\mathrm{Re}\, \kappa_1 &= \frac{i|p|}{8} \left(\lambda - \bar{\lambda} + \frac{1}{\lambda} - \frac{1}{\bar{\lambda}}\right), \quad \mathrm{Im}\, \kappa_1 = \frac{|p|}{8} \left(\lambda + \bar{\lambda} + \frac{1}{\lambda} + \frac{1}{\bar{\lambda}}\right), \\ \mathrm{Re}\, \kappa_2 &= \frac{|p|}{8} \left(\lambda + \bar{\lambda} - \frac{1}{\lambda} - \frac{1}{\bar{\lambda}}\right), \quad \mathrm{Im}\, \kappa_2 = \frac{|p|}{8i} \left(\lambda - \bar{\lambda} - \frac{1}{\lambda} + \frac{1}{\bar{\lambda}}\right),\end{aligned} \tag{9.5}$$

where $\lambda \in \mathbb{C}\backslash 0$, $p \in \mathbb{R}^3$. Due to (9.4), (9.5) we have that

$$\begin{aligned}&\frac{\partial \bar{\kappa}_1}{\partial \bar{\lambda}} \mathrm{Re}\, \kappa_1 + \frac{\partial \bar{\kappa}_2}{\partial \bar{\lambda}} \mathrm{Re}\, \kappa_2 \\ &= \frac{|p|^2}{32} \left(1 - \frac{1}{\bar{\lambda}^2}\right) \left(\lambda - \bar{\lambda} + \frac{1}{\lambda} - \frac{1}{\bar{\lambda}} + \left(1 + \frac{1}{\bar{\lambda}^2}\right) \left(\lambda + \bar{\lambda} - \frac{1}{\lambda} - \frac{1}{\bar{\lambda}}\right)\right) \\ &= \frac{|p|^2}{32} \left(\lambda - \bar{\lambda} + \frac{1}{\lambda} - \frac{1}{\bar{\lambda}} - \frac{\lambda}{\bar{\lambda}^2} + \frac{1}{\bar{\lambda}} - \frac{1}{\lambda\bar{\lambda}^2} + \frac{1}{\bar{\lambda}^3} \right. \\ &\quad \left. + \lambda + \bar{\lambda} - \frac{1}{\lambda} - \frac{1}{\bar{\lambda}} + \frac{\lambda}{\bar{\lambda}^2} + \frac{1}{\bar{\lambda}} - \frac{1}{\lambda\bar{\lambda}^2} - \frac{1}{\bar{\lambda}^3}\right) \\ &= \frac{|p|^2}{16} \left(\lambda - \frac{1}{\lambda\bar{\lambda}^2}\right) = \frac{|p|^2}{16} \lambda \left(1 - \frac{1}{|\lambda|^4}\right),\end{aligned} \tag{9.6}$$

$$
\begin{aligned}
&\frac{\partial\bar\kappa_1}{\partial\bar\lambda}\operatorname{Im}\kappa_2-\frac{\partial\bar\kappa_2}{\partial\bar\lambda}\operatorname{Im}\kappa_1\\
&=-\frac{|p|^2}{32}\left(1-\frac{1}{\bar\lambda^2}\right)\left(\lambda-\bar\lambda-\frac{1}{\lambda}+\frac{1}{\bar\lambda}+\left(1+\frac{1}{\bar\lambda^2}\right)\left(\lambda+\bar\lambda+\frac{1}{\lambda}+\frac{1}{\bar\lambda}\right)\right)\\
&=-\frac{|p|^2}{32}\left(\lambda-\bar\lambda-\frac{1}{\lambda}+\frac{1}{\bar\lambda}-\frac{\lambda}{\bar\lambda^2}+\frac{1}{\bar\lambda}+\frac{1}{\lambda\bar\lambda^2}-\frac{1}{\bar\lambda^3}\right.\\
&\qquad\left.+\lambda+\bar\lambda+\frac{1}{\lambda}+\frac{1}{\bar\lambda}+\frac{\lambda}{\bar\lambda^2}+\frac{1}{\bar\lambda}+\frac{1}{\lambda\bar\lambda^2}+\frac{1}{\bar\lambda^3}\right)\\
&=-\frac{|p|^2}{16}\left(\lambda+\frac{2}{\bar\lambda}+\frac{1}{\lambda\bar\lambda^2}\right)=-\frac{|p|^2}{16}\frac{(|\lambda|^2+1)^2}{|\lambda|^2\bar\lambda}.
\end{aligned}\tag{9.7}
$$

Due to (9.6), (9.7), (4.8) we have that

$$
\left(\frac{\partial\bar\kappa_1}{\partial\bar\lambda}\operatorname{Re}\kappa_1+\frac{\partial\bar\kappa_2}{\partial\bar\lambda}\operatorname{Re}\kappa_2\right)\frac{1}{|\operatorname{Im}k|}=\frac{|p|}{4}\frac{(|\lambda|^2-1)}{\bar\lambda|\lambda|},\tag{9.8}
$$

$$
\left(\frac{\partial\bar\kappa_1}{\partial\bar\lambda}\operatorname{Im}\kappa_2-\frac{\partial\bar\kappa_2}{\partial\bar\lambda}\operatorname{Im}\kappa_1\right)\frac{|p|}{2|\operatorname{Im}k|^2}=-\frac{|p|}{2\bar\lambda},\tag{9.9}
$$

where $(\lambda,p)\in(\mathbb{C}\backslash 0)\times(\mathbb{R}^3\backslash\mathcal{L}_\nu)$.

The $\bar\partial$-equation (5.1) follows from (9.1), (9.2), (9.3), (9.8), (9.9) and the property that $|\operatorname{Re}k|=|\operatorname{Im}k|$ for $k\in\Sigma$ defined by (1.7).

Lemma 5.1 is proved. □

10. Proof of Lemma 5.2

Let us show, first, that

$$
\{U_1,U_2\}\in L^\infty_{\text{local}}((\mathbb{C}\backslash 0)\times(\mathbb{R}^3\backslash\mathcal{L}_\nu)).\tag{10.1}
$$

Property (10.1) follows from definition (5.5), the properties

$$
\begin{aligned}
&U_1(k,-\xi(k,\varphi))\in L^\infty(\Sigma\times[0,2\pi])\ \text{(as a function of } k,\varphi),\\
&U_1(k,-\xi(k,\varphi))\in L^\infty(\Omega\times[0,2\pi])\ \text{(as a function of } k,p,\varphi\\
&\text{(with no dependence on } p)),
\end{aligned}\tag{10.2}
$$

$$
U_2(k+\xi(k,\varphi),p+\xi(k,\varphi))\in L^\infty(\Omega\times[0,2\pi])\ \text{(as a function of } k,p,\varphi),\tag{10.3}
$$

where

$$
\Sigma=\{k\in\mathbb{C}^3:\ k^2=0\},\quad \Omega=\{k\in\mathbb{C}^3,\ p\in\mathbb{R}^3:\ k^2=0,\ p^2=2kp\},\tag{10.4}
$$

$$
\xi(k,\varphi)=\operatorname{Re}k(\cos\varphi-1)+k^\perp\sin\varphi,\quad k^\perp=\frac{\operatorname{Im}k\times\operatorname{Re}k}{|\operatorname{Im}k|}\tag{10.5}
$$

(where $\times$ in (10.5) denotes vector product), and from Lemma 4.1. In turn, (10.2) follows from $U_1 \in L^\infty(\Omega)$, definition (10.4) and the fact that $p = -\xi(k, \varphi)$, $\varphi \in [0, 2\pi]$, is a parametrization of the set $\{p \in \mathbb{R}^3 : \ p^2 = 2kp\}$, $k \in \Sigma\backslash\{0\}$. To prove (10.3), consider

$$\Theta = \{k \in \mathbb{C}^3,\ l \in \mathbb{C}^3 :\ k^2 = l^2 = 0, \quad \operatorname{Im} k = \operatorname{Im} l\}. \tag{10.6}$$

Note that

$$\begin{aligned} &\Theta \approx \Omega, \\ &(k, l) \in \Theta \Rightarrow (k, k - l) \in \Omega, \quad (k, p) \in \Omega \Rightarrow (k, k - p) \in \Theta. \end{aligned} \tag{10.7}$$

Consider

$$u_2(k, l) = U_2(k, k - l), \quad (k, l) \in \Theta. \tag{10.8}$$

The property $U_2 \in L^\infty(\Omega)$ is equivalent to the property $u_2 \in L^\infty(\Theta)$. Property (10.3) is equivalent to the property

$$u_2(k + \xi(k, \varphi), l) \in L^\infty(\Theta \times [0, 2\pi]) \ \text{ (as a function of } k, l, \varphi). \tag{10.9}$$

Property (10.9) follows from the property

$$\begin{aligned} &u_2(\zeta(l, \psi, \varphi) + i \operatorname{Im} l, l) \in L^\infty(\Sigma \times [0, 2\pi] \times [0, 2\pi]) \\ &\text{(as a function of } l, \psi, \varphi), \end{aligned} \tag{10.10}$$

where

$$\zeta(l, \psi, \varphi) = \operatorname{Re} l \cos(\varphi - \psi) + l^\perp \sin(\varphi - \psi), \quad l^\perp = \frac{\operatorname{Im} l \times \operatorname{Re} l}{|\operatorname{Im} l|} \tag{10.11}$$

(where $\times$ in (10.11) denotes vector product). Note that $k = \zeta(l, \psi, \varphi)$, $\varphi \in [0, 2\pi]$ at fixed $\psi \in [0, 2\pi]$ is a parametrization of the set $S_l = \{k \in \mathbb{C}^3 :\ k^2 = l^2, \ \operatorname{Im} k = \operatorname{Im} l\}$, $l \in \Sigma\backslash 0$. In turn, (10.10) follows from $u_2 \in L^\infty(\Theta)$, definition (10.6) and the aforementioned fact concerning the parametrization of S_l. Thus, properties (10.10), (10.9), (10.3) are proved. This completes the proof of (10.1).

Let us prove now (5.8).

We have that

$$\{U_1, U_2\} = \{U_1, U_2\}_1 + \{U_1, U_2\}_2, \tag{10.12}$$

where

$$\{U_1, U_2\}_1(\lambda, p) = -\frac{\pi |p|(|\lambda|^2 - 1)}{8\bar{\lambda}|\lambda|}\{U_1, U_2\}_3(\lambda, p), \tag{10.13a}$$

$$\begin{aligned} \{U_1, U_2\}_3(\lambda, p) = &\int_{-\pi}^{\pi} (\cos\varphi - 1) \\ \times\, U_1(k(\lambda, p), -\xi(\lambda, p, \varphi))&U_2(k(\lambda, p) + \xi(\lambda, p, \varphi), p + \xi(\lambda, p, \varphi))\mathrm{d}\varphi, \end{aligned} \tag{10.13b}$$

$$\{U_1, U_2\}_2 = \frac{\pi |p|}{4\bar{\lambda}} \{U_1, U_2\}_4(\lambda, p), \tag{10.14a}$$

$$\begin{aligned}\{U_1, U_2\}_4(\lambda, p) = \int_{-\pi}^{\pi} \sin\varphi \\ \times U_1(k(\lambda, p), -\xi(\lambda, p, \varphi))U_2(k(\lambda, p)+\xi(\lambda, p, \varphi), p+\xi(\lambda, p, \varphi))\mathrm{d}\varphi,\end{aligned} \tag{10.14b}$$

$\lambda \in \mathbb{C}\backslash 0$, $p \in \mathbb{R}^3\backslash \mathcal{L}_\nu$.

Formulas (5.2), (5.3) imply that

$$|\xi|^2 = |\mathrm{Re}\, k|^2((\cos\varphi - 1)^2 + (\sin\varphi)^2) = 4|\mathrm{Re}\, k|^2(\sin(\varphi/2))^2, \tag{10.15}$$

where $\xi = \xi(\lambda, p, \varphi)$, $k = k(\lambda, p)$.

The relation $p^2 = 2k(\lambda, p)p$, $\lambda \in \mathbb{C}\backslash 0$, $p \in \mathbb{R}^3\backslash \mathcal{L}_\nu$, implies that

$$p = -\mathrm{Re}\, k(\lambda, p)(\cos\psi - 1) - k^\perp(\lambda, p)\sin\psi \tag{10.16}$$

for some $\psi = \psi(\lambda, p) \in [-\pi, \pi]$, where $k^\perp(\lambda, p)$ is defined by (5.3). Formulas (5.2), (5.3), (10.16) imply that

$$\begin{aligned}|p + \xi|^2 &= |\mathrm{Re}\, k|^2((\cos\varphi - \cos\psi)^2 + (\sin\varphi - \sin\psi)^2) \\ &= 4|\mathrm{Re}\, k|^2 \left(\sin\frac{\varphi - \psi}{2}\right)^2, \\ |p|^2 &= 4|\mathrm{Re}\, k|^2 \left(\sin\frac{\psi}{2}\right)^2,\end{aligned} \tag{10.17}$$

where $\xi = \xi(\lambda, p, \varphi)$, $k = k(\lambda, p)$, $\psi = \psi(\lambda, p)$.

Using the assumptions of Lemma 5.2 and formulas (10.13b), (10.14b), (10.15), (10.17) we obtain that

$$\begin{aligned}|\{U_1, U_2\}_3(\lambda, p)| &\le A(r, \psi, \mu, \mu)|||U_1|||_\mu |||U_2|||_\mu, \\ |\{U_1, U_2\}_4(\lambda, p)| &\le B(r, \psi, \mu, \mu)|||U_1|||_\mu |||U_2|||_\mu\end{aligned} \tag{10.18}$$

for $r = |\mathrm{Re}\, k(\lambda, p)|$, $\psi = \psi(\lambda, p)$ (of (10.16)) and almost all $(\lambda, p) \in (\mathbb{C}\backslash 0) \times (\mathbb{R}^3\backslash \mathcal{L}_\nu)$, where

$$A(r, \psi, \alpha, \beta) = \int_{-\pi}^{\pi} \frac{(1 - \cos\varphi)\mathrm{d}\varphi}{(1 + 2r|\sin(\varphi/2)|)^\alpha (1 + 2r|\sin(\frac{\varphi-\psi}{2})|)^\beta}, \tag{10.19a}$$

$$B(r, \psi, \alpha, \beta) = \int_{-\pi}^{\pi} \frac{|\sin\varphi|\mathrm{d}\varphi}{(1 + 2r|\sin(\varphi/2)|)^\alpha (1 + 2r|\sin(\frac{\varphi-\psi}{2})|)^\beta}, \tag{10.19b}$$

for $r \geq 0$, $\psi \in [-\pi, \pi]$, $\alpha \geq 2$, $\beta \geq 2$. In addition, in (10.18) we used also that, in view of Lemma 4.1, properties (10.2), (10.3) and definitions (10.13), (10.14), the variations of U_1, U_2 on the sets of zero measure in Ω imply variations of $\{U_1, U_2\}_3$ and $\{U_1, U_2\}_4$ on sets of zero measure, only, in $(\mathbb{C}\backslash 0) \times (\mathbb{R}^3 \backslash \mathcal{L}_\nu)$.

Further, we use the following lemma of [No5].

Lemma 10.1. **[No5].** *Let* $r \geq 0$, $\psi \in [-\pi, \pi]$, $\rho = 2r|\sin(\psi/2)|$, $\alpha \geq 2$, $\beta \geq 2$. *Then*

$$A(r, \psi, \alpha, \beta) \leq \sum_{j=1}^{4} A_j(r, \psi, \alpha, \beta), \tag{10.20}$$

$$A_1(r, \psi, \alpha, \beta) \leq \min\left(\frac{\rho^3}{6r^3}, \frac{\rho}{r^3}\right) \frac{1}{(1+\rho/2)^\beta}, \tag{10.21}$$

$$A_2(r, \psi, \alpha, \beta) \leq \frac{\rho^3}{r^3} \frac{1}{(1+\rho/2)^{\alpha+1}}, \tag{10.22}$$

$$A_3(r, \psi, \alpha, \beta) \leq \frac{4\rho^3}{r^3} \frac{1}{(1+\rho)^\alpha (1+\rho/2)}, \tag{10.23}$$

$$A_4(r, \psi, \alpha, \beta) \leq \left(\frac{3}{1+r^2} + \frac{2\pi}{(1+\sqrt{2}r)^\alpha}\right) \frac{1}{(1+\rho/2)^\beta}, \tag{10.24}$$

$$B(r, \psi, \alpha, \beta) \leq \sum_{j=1}^{4} B_j(r, \psi, \alpha, \beta), \tag{10.25}$$

$$B_1(r, \psi, \alpha, \beta) \leq \min\left(\frac{\rho^2}{2r^2}, \frac{\sqrt{2}\rho}{r^2}\right) \frac{1}{(1+\rho/2)^\beta}, \tag{10.26}$$

$$B_2(r, \psi, \alpha, \beta) \leq \frac{2\rho^2}{r^2} \frac{1}{(1+\rho/2)^{\alpha+1}}, \tag{10.27}$$

$$B_3(r, \psi, \alpha, \beta) \leq \frac{4\rho^2}{r^2} \frac{1}{(1+\rho)^\alpha (1+\rho/2)}, \tag{10.28}$$

$$B_4(r, \psi, \alpha, \beta) \leq \left(\frac{5}{1+r} + \frac{3}{(1+\sqrt{2}r)^\alpha}\right) \frac{1}{(1+\rho/2)^\beta}. \tag{10.29}$$

Lemma 10.2. *Let*

$$r = r(\lambda, p) = \frac{\rho}{4}\left(|\lambda| + \frac{1}{|\lambda|}\right), \quad |\sin(\psi/2)| = \frac{\rho}{2r}, \tag{10.30}$$

where $\lambda \in \mathbb{C}\backslash 0$, $\rho \geq 0$, $\psi \in [-\pi, \pi]$. *Then:*

$$\frac{\rho\,||\lambda|^2-1|}{|\lambda|^2}A_1 \leq \frac{4^3|\lambda|}{\sqrt{6}(|\lambda|^2+1)^2(1+\rho/2)^{\beta}}, \tag{10.31}$$

$$\frac{\rho\,||\lambda|^2-1|}{|\lambda|^2}A_2 \leq \frac{2\cdot 4^3|\lambda|}{(|\lambda|^2+1)^2(1+\rho/2)^{\alpha}}, \tag{10.32}$$

$$\frac{\rho\,||\lambda|^2-1|}{|\lambda|^2}A_3 \leq \frac{2\cdot 4^4|\lambda|}{(|\lambda|^2+1)^2(1+\rho)^{\alpha}}, \tag{10.33}$$

$$\frac{\rho\,||\lambda|^2-1|}{|\lambda|^2}A_4 \leq \frac{4\pi\rho||\lambda|^2-1|}{|\lambda|^2(1+(\rho/4)(|\lambda|+|\lambda|^{-1}))^2(1+\rho/2)^{\beta}}, \tag{10.34}$$

$$\frac{\rho}{|\lambda|}B_1 \leq \frac{16\sqrt{2}|\lambda|}{(|\lambda|^2+1)^2(1+\rho/2)^{\beta}}, \tag{10.35}$$

$$\frac{\rho}{|\lambda|}B_2 \leq \frac{4^3|\lambda|}{(|\lambda|^2+1)^2(1+\rho/2)^{\alpha}}, \tag{10.36}$$

$$\frac{\rho}{|\lambda|}B_3 \leq \frac{2\cdot 4^3|\lambda|}{(|\lambda|^2+1)^2(1+\rho)^{\alpha}}, \tag{10.37}$$

$$\frac{\rho}{|\lambda|}B_4 \leq \frac{8\rho}{|\lambda|(1+(\rho/4)(|\lambda|+|\lambda|^{-1}))(1+\rho/2)^{\beta}}, \tag{10.38}$$

where $A_j = A_j(r, |\psi|, \alpha, \beta)$, $B_j = B_j(r, |\psi|, \alpha, \beta)$ *are the same as in Lemma* 10.1, $j = 1, 2, 3, 4$, $\alpha \geq 2$, $\beta \geq 2$.

Proof of Lemma 10.2. Using (10.30) we obtain that

$$\rho\min\left(\frac{\rho^3}{6r^3}, \frac{\rho}{r^3}\right) = \frac{4^3|\lambda|^3}{(|\lambda|^2+1)^3}\min\left(\frac{\rho}{6}, \frac{1}{\rho}\right) \leq \frac{4^3|\lambda|^3}{\sqrt{6}(|\lambda|^2+1)^3}, \tag{10.39}$$

$$\rho\min\left(\frac{\rho^2}{2r^2}, \frac{\sqrt{2}\rho}{r^2}\right) = \frac{16|\lambda|^2}{(|\lambda|^2+1)^2}\min\left(\frac{\rho}{2}, \sqrt{2}\right) \leq \frac{16\sqrt{2}|\lambda|^2}{(|\lambda|^2+1)^2}, \tag{10.40}$$

where $\lambda \in \mathbb{C}\backslash 0$, $\rho \geq 0$. Estimates (10.31), (10.35) follow from (10.21), (10.26) and (10.39), (10.40). Estimates (10.32), (10.33), (10.36), (10.37) follow from

(10.22), (10.23), (10.27), (10.28) and (10.30). Estimates (10.34), (10.38) follow from (10.24), (10.29), the inequalities

$$\begin{aligned} &\frac{3}{1+r^2} + \frac{2\pi}{(1+\sqrt{2}r)^\alpha} \leq \frac{4\pi}{(1+r)^2}, \\ &\frac{5}{1+r} + \frac{3}{(1+\sqrt{2}r)^\alpha} \leq \frac{8}{1+r}, \end{aligned} \tag{10.41}$$

where $r \geq 0$, $\alpha \geq 2$, and from (10.30). Lemma 10.2 is proved. □

Estimate (5.8) follows from (10.12)-(10.14), (10.18), (4.8), (10.17) (for $|p|$) and Lemmas 10.1, 10.2.

Lemma 5.2 is proved. □

11. Proof of Lemma 6.4

Let

$$J_1(\lambda) = \int_{\mathbb{C}} \frac{|\zeta|}{(|\zeta|^2+1)^2} \frac{d\,\mathrm{Re}\zeta\, d\,\mathrm{Im}\,\zeta}{|\zeta-\lambda|}, \tag{11.1}$$

$$J_2(\lambda,\rho) = \int_{\mathbb{C}} \frac{(|\zeta|^2+1)\rho}{|\zeta|^2(1+\rho(|\zeta|+|\zeta|^{-1}))^2} \frac{d\,\mathrm{Re}\zeta\, d\,\mathrm{Im}\,\zeta}{|\zeta-\lambda|}, \tag{11.2}$$

$$J_3(\lambda,\rho) = \int_{\mathbb{C}} \frac{\rho}{|\zeta|(1+\rho(|\zeta|+|\zeta|^{-1}))} \frac{d\,\mathrm{Re}\zeta\, d\,\mathrm{Im}\,\zeta}{|\zeta-\lambda|}, \tag{11.3}$$

where $\lambda \in \mathbb{C}$, $\rho > 0$.

Lemma 11.1. *The following estimates hold:*

$$J_1(\lambda) \leq n_1, \quad \lambda \in \mathbb{C}, \tag{11.4}$$

$$J_2(\lambda,\rho) \leq n_2, \quad \lambda \in \mathbb{C}, \quad \rho > 0, \tag{11.5}$$

$$J_3(\lambda,\rho) \leq n_3, \quad \lambda \in \mathbb{C}, \quad \rho > 0, \tag{11.6}$$

for some positive constants n_1, n_2, n_3 (where J_1, J_2, J_3 are defined by (11.1)-(11.3)).

Proof of Lemma 11.1.

Proof of (11.4). We have that

$$
\begin{aligned}
J_1(\lambda) &\le \left(\int\limits_{|\zeta|\le|\zeta-\lambda|} + \int\limits_{|\zeta|\ge|\zeta-\lambda|} \right) \frac{2|\zeta|}{(|\zeta|^2+1)(|\zeta|+1)^2} \frac{d\,\mathrm{Re}\zeta\, d\,\mathrm{Im}\,\zeta}{|\zeta-\lambda|} \\
&\le \int\limits_{|\zeta|\le|\zeta-\lambda|} \frac{2d\,\mathrm{Re}\zeta\, d\,\mathrm{Im}\,\zeta}{(|\zeta|^2+1)(|\zeta|+1)^2} \\
&\quad + \int\limits_{|\zeta|\ge|\zeta-\lambda|} \frac{2d\,\mathrm{Re}\zeta\, d\,\mathrm{Im}\,\zeta}{(|\zeta-\lambda|^2+1)(|\zeta-\lambda|+1)|\zeta-\lambda|} \\
&\le \int_0^{+\infty} \frac{4\pi r\, \mathrm{d}r}{(r^2+1)(r+1)^2} + \int_0^{+\infty} \frac{4\pi r\, \mathrm{d}r}{(r^2+1)(r+1)r} \le n_1,
\end{aligned}
\tag{11.7}
$$

where $\lambda \in \mathbb{C}$. Estimate (11.4) is proved. □

Proof of (11.5). We have that

$$
J_2(\lambda,\rho) = J_{2,1}(\lambda,\rho) + J_{2,2}(\lambda,\rho), \tag{11.8a}
$$

$$
\begin{aligned}
J_{2,1}(\lambda,\rho) &= \int\limits_{|\zeta|<1} \frac{(|\zeta|^2+1)\rho\, d\,\mathrm{Re}\zeta\, d\,\mathrm{Im}\,\zeta}{|\zeta|^2(1+\rho(|\zeta|+|\zeta|^{-1}))^2|\zeta-\lambda|} \\
&= \int\limits_{|\zeta|<1} \frac{\rho(|\zeta|^2+1)d\,\mathrm{Re}\zeta\, d\,\mathrm{Im}\,\zeta}{(|\zeta|+\rho(|\zeta|^2+1))^2|\zeta-\lambda|},
\end{aligned}
\tag{11.8b}
$$

$$
J_{2,2}(\lambda,\rho) = \int\limits_{|\zeta|>1} \frac{(|\zeta|^2+1)\rho\, d\,\mathrm{Re}\zeta\, d\,\mathrm{Im}\,\zeta}{|\zeta|^2(1+\rho(|\zeta|+|\zeta|^{-1}))^2|\zeta-\lambda|}, \tag{11.8c}
$$

where $\lambda \in \mathbb{C}$, $\rho > 0$. In addition,

$$
\begin{aligned}
J_{2,1}(\lambda,\rho) &\le \left(\int\limits_{\substack{|\zeta|<1\\ |\zeta|\le|\zeta-\lambda|}} + \int\limits_{\substack{|\zeta|<1\\ |\zeta|\ge|\zeta-\lambda|}} \right) \frac{2\rho\, d\,\mathrm{Re}\zeta\, d\,\mathrm{Im}\,\zeta}{(|\zeta|+\rho)^2|\zeta-\lambda|} \\
&\le \int\limits_{|\zeta|<1} \frac{2\rho\, d\,\mathrm{Re}\zeta\, d\,\mathrm{Im}\,\zeta}{(|\zeta|+\rho)^2|\zeta|} + \int\limits_{|\zeta|<1} \frac{2\rho\, d\,\mathrm{Re}\zeta\, d\,\mathrm{Im}\,\zeta}{(|\zeta-\lambda|+\rho)^2|\zeta-\lambda|} \\
&\le \int\limits_{\mathbb{C}} \frac{4\rho\, d\,\mathrm{Re}\zeta\, d\,\mathrm{Im}\,\zeta}{(|\zeta|+\rho)^2|\zeta|} = \int_0^{\infty} \frac{8\pi\rho\, \mathrm{d}r}{(r+\rho)^2} = 8\pi,
\end{aligned}
\tag{11.9a}
$$

$$
\begin{aligned}
J_{2,2}(\lambda,\rho) &\le \left(\int\limits_{\substack{|\zeta|>1\\ |\zeta|\le|\zeta-\lambda|}} + \int\limits_{\substack{|\zeta|>1\\ |\zeta|\ge|\zeta-\lambda|}} \right) \frac{2\rho\, d\,\mathrm{Re}\zeta\; d\;\mathrm{Im}\;\zeta}{(1+\rho|\zeta|)^2|\zeta-\lambda|} \\
&\le \int\limits_{|\zeta|>1} \frac{2\rho\, d\,\mathrm{Re}\zeta\; d\;\mathrm{Im}\;\zeta}{(1+\rho\,|\zeta|)^2|\zeta|} + \int\limits_{|\zeta|>1} \frac{2\rho\, d\,\mathrm{Re}\zeta\; d\;\mathrm{Im}\;\zeta}{(1+\rho\,|\zeta-\lambda|)^2|\zeta-\lambda|} \qquad (11.9\mathrm{b}) \\
&\le \int\limits_{\mathbb{C}} \frac{4\rho\, d\,\mathrm{Re}\zeta\; d\;\mathrm{Im}\;\zeta}{(1+\rho\,|\zeta|)^2|\zeta|} = \int_0^\infty \frac{8\pi\rho\,\mathrm{d}r}{(1+\rho\, r)^2} = 8\pi,
\end{aligned}
$$

where $\lambda \in \mathbb{C}$, $\rho > 0$. Estimate (11.5) follows from (11.8), (11.9). □

Proof of (11.6). We have that

$$
\begin{aligned}
J_3(\lambda,\rho) &\le \left(\int\limits_{|\zeta|\le|\zeta-\lambda|} + \int\limits_{|\zeta|\ge|\zeta-\lambda|} \right) \frac{\rho\, d\,\mathrm{Re}\zeta\; d\;\mathrm{Im}\;\zeta}{(|\zeta|+\rho(|\zeta|^2+1)|\zeta-\lambda|} \\
&\le \int\limits_{\mathbb{C}} \frac{2\rho\, d\,\mathrm{Re}\zeta\; d\;\mathrm{Im}\;\zeta}{(|\zeta|+\rho(|\zeta|^2+1))|\zeta|} = \int_0^\infty \frac{4\pi\rho\,\mathrm{d}r}{r+\rho(r^2+1)} \qquad (11.10) \\
&\le \int_0^1 \frac{4\pi\rho\,\mathrm{d}r}{r+\rho} + \int_1^\infty \frac{4\pi\rho\,\mathrm{d}r}{r(1+\rho\, r)} \\
&= \int_0^1 \frac{4\pi\rho\,\mathrm{d}r}{r+\rho} = 8\pi\rho\ln\left(\frac{1+\rho}{\rho}\right),
\end{aligned}
$$

where $\lambda \in \mathbb{C}$, $\rho > 0$. Estimate (11.6) follows from (11.10).

Lemma 11.1 is proved. □

Using formulas (6.13c), (6.8), Lemmas 4.1, 5.2, 11.1 and smoothing properties of the convolution with $1/\zeta$ on the complex plane $\mathbb{C}$ we obtain properties (6.14) for $I(U_1, U_2)$ and estimate (6.15a). Properties and estimates (6.14), (6.15b), (6.15c) for $N(U)$ and $M(U)$ follow from property (6.14) and estimate (6.15a) for $I(U_1, U_2)$. Estimate (6.16a) follows from the formula

$$
N(U_1) - N(U_2) = I(U_1 - U_2, U_1) + I(U_2, U_1 - U_2) \qquad (11.11)
$$

and from estimate (6.15a). Estimate (6.16b) follows from (6.13a), (6.14a) and (6.16a).

Lemma 6.4 is proved. □

12. Proof of Lemmas 6.5 and 6.6

Proof of Lemma 6.5. Suppose that

$$U, V \in L^\infty_\mu((\mathbb{C}\backslash 0) \times (\mathbb{R}^3\backslash \mathcal{L}_\nu)), \ \ |||U|||_\mu \le r, \ \ |||V|||_\mu \le r. \tag{12.1}$$

Then using Lemma 6.4 and the assumptions of Lemma 6.5 we obtain that

$$M_{U_0}(U) \in L^\infty_\mu((\mathbb{C}\backslash 0) \times (\mathbb{R}^3\backslash \mathcal{L}_\nu)),$$
$$|||M_{U_0}(U)|||_\mu \le |||U_0|||_\mu + |||M(U)|||_\mu \le r/2 + 2c_6(\mu)r^2 < r, \tag{12.2}$$
$$|||M_{U_0}(U) - M_{U_0}(V)|||_\mu \le \alpha\, |||U - V|||_\mu, \ \ \alpha = 4c_6(\mu)r < 1, \tag{12.3}$$

where

$$M_{U_0}(U) = U_0 + M(U). \tag{12.4}$$

Due to (12.1)-(12.4), M_{U_0} is a contraction map of the ball $U \in L^\infty_\mu((\mathbb{C}\backslash 0) \times (\mathbb{R}^3\backslash \mathcal{L}_\nu))$, $|||U|||_\mu \le r$. Using now the lemma about contraction maps we obtain that (6.17) is uniquely solvable for U of the aforementioned ball by the method of successive approximations. In addition, using the formulas

$$|||U - (M_{U_0})^n(0)|||_\mu \le \sum_{j=n}^{\infty} |||(M_{U_0})^{j+1}(0) - (M_{U_0})^j(0)|||_\mu, \tag{12.5}$$

$$\begin{aligned} &|||(M_{U_0})^{j+1}(0) - (M_{U_0})^j(0)|||_\mu \\ &\overset{(12.3)}{\le} 4c_6(\mu)r|||(M_{U_0})^j(0) - (M_{U_0})^{j-1}(0)|||_\mu, \ \ j = 1, 2, 3, \dots, \end{aligned} \tag{12.6a}$$

$$\begin{aligned} &|||(M_{U_0})^{j+1}(0) - (M_{U_0})^j(0)|||_\mu \\ &\overset{(12.6a)}{\le} (4c_6(\mu)r)^j|||M_{U_0}(0) - (M_{U_0})^0(0)|||_\mu \\ &\overset{(12.4)}{=} (4c_6(\mu)r)^j|||U_0|||_\mu \le (4c_6(\mu)r)^j r/2, \ \ j = 1, 2, 3, \dots, \end{aligned} \tag{12.6b}$$

where U is the solution of (6.17) in the aforementioned ball and $(M_{U_0})^0(0) = 0$, we obtain (6.18).

Lemma 6.5 is proved. □

Proof of Lemma 6.6. We have that

$$U - \tilde{U} = U_0 - \tilde{U}_0 + M(U) - M(\tilde{U}), \tag{12.7}$$

$$M(U)(\lambda, p) - M(\tilde{U})(\lambda, p) \overset{(6.13a),(11.11)}{=} L_{U,\tilde{U}}(U - \tilde{U}), \tag{12.8}$$

where

$$L_{U,\tilde U}W = I(W,U)(\lambda,p) + I(\tilde U,W)(\lambda,p) + I(W,U)(\lambda_0(p),p) + I(\tilde U,W)(\lambda_0(p),p), \tag{12.9}$$

where $I(U_1,U_2)$ is defined by (6.13c), W is a test function on $(\mathbb{C}\backslash 0)\times(\mathbb{R}^3\backslash\mathcal{L}_\nu)$. In view of (12.8), (12.9) we can consider (12.7) as a linear integral equation for "unknown" $U-\tilde U$ with given $U_0-\tilde U_0$, U, $\tilde U$. Using (12.9), (6.14), (6.15a), and the properties $|||U|||_\mu \le r$, $|||\tilde U|||_\mu \le r$, we obtain that

$$L_{U,\tilde U}W \in L^\infty_\mu((\mathbb{C}\backslash 0)\times(\mathbb{R}^3\backslash\mathcal{L}_\nu)), \quad |||L_{U,\tilde U}W|||_\mu \le 4c_6(\mu)r|||W|||_\mu \text{ for } W\in L^\infty_\mu((\mathbb{C}\backslash 0)\times(\mathbb{R}^3\backslash\mathcal{L}_\nu)). \tag{12.10}$$

Using (12.8)-(12.10) and solving (12.7) with respect to $U-\tilde U$ by the method of successive approximations, we obtain (6.19).

Lemma 6.6 is proved. □

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