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Degree counting theorems for singular Liouville systems

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Abstract. Let (M, g) be a compact Riemann surface with no boundary and $u = (u_1, ..., u_n)$ be a solution of the following singular Liouville system:

$$\Delta_g u_i + \sum_{j=1}^{n} a_{ij}\rho_j \left(\frac{h_j e^{u_j}}{\int_M h_j e^{u_j} dV_g} - \frac{1}{vol_g(M)} \right) = \sum_{t=1}^{N} 4\pi\gamma_t \left(\delta_{p_t} - \frac{1}{vol_g(M)} \right),$$

where $i = 1, ..., n$, $h_1, ..., h_n$ are positive smooth functions, $p_1, ..., p_N$ are distinct points on M, δ_{p_t} are Dirac masses, $\rho = (\rho_1, ..., \rho_n)$ $(\rho_i \geq 0)$ and $(\gamma_1, ..., \gamma_N)$ $(\gamma_t > -1\ )$ are constant vectors. If the coefficient matrix $A = (a_{ij})_{n\times n}$ satisfies standard assumptions, we identify a family of critical hyper-surfaces Γ_k for $\rho = (\rho_1, .., \rho_n)$ so that a priori estimate of u holds if ρ is not on any of the Γ_ks. Thanks to the a priori estimate, a topological degree for u is well defined for ρ staying between every two consecutive Γ_ks. In this article we establish this degree counting formula which depends only on the Euler Characteristic of M and the location of ρ. Finally if the Liouville system is defined on a bounded domain in $\mathbb{R}^2$ with Dirichlet boundary condition, a similar degree counting formula that depends only on the topology of the domain and the location of ρ is also determined.

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