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A fresh look at the notion of normality

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Abstract. Let G be a countably infinite cancellative amenable semigroup and let (F_n) be a (left) Følner sequence in G. We introduce the notion of an (F_n)-normal set in G and an (F_n)-normal element of $\{0,1\}^G$. When $G = (\mathbb{N}, +)$ and $F_n = \{1, 2, \ldots, n\}$, the (F_n)-normality coincides with the classical notion. We prove several results about (F_n)-normality, for example:

- If (F_n) is a Følner sequence in G, such that for every $\alpha \in (0, 1)$ we have $\sum_n \alpha^{|F_n|} < \infty$, then almost every (in the sense of the uniform product measure $(\frac{1}{2}, \frac{1}{2})^G$) $x \in \{0,1\}^G$ is (F_n)-normal.
- For any Følner sequence (F_n) in G, there exists an effectively defined Champernowne-like (F_n)-normal set.
- There is a rather natural and sufficiently wide class of Følner sequences (F_n) in $(\mathbb{N}, \times)$, which we call "nice", for which the Champernowne-like construction can be done in an algorithmic way. Moreover, there exists a Champernowne-like set which is (F_n)-normal for every nice Følner sequence (F_n).

We also investigate and juxtapose combinatorial and Diophantine properties of normal sets in semigroups $(\mathbb{N}, +)$ and $(\mathbb{N}, \times)$. Below is a sample of results that we obtain:

- Let $A \subset \mathbb{N}$ be a classical normal set. Then, for any Følner sequence (K_n) in $(\mathbb{N}, \times)$ there exists a set E of (K_n)-density 1, such that for any finite subset $\{n_1, n_2, \ldots, n_k\} \subset E$, the intersection $A/n_1 \cap A/n_2 \cap \ldots \cap A/n_k$ has positive upper density in $(\mathbb{N}, +)$. As a consequence, A contains arbitrarily long geometric progressions, and, more generally, arbitrarily long "geo-arithmetic" configurations of the form $\{a(b + ic)^j, 0 \le i, j \le k\}$.
- For any Følner sequence (F_n) in $(\mathbb{N}, +)$ there exist uncountably many (F_n)-normal Liouville numbers.
- For any nice Følner sequence (F_n) in $(\mathbb{N}, \times)$ there exist uncountably many (F_n)-normal Liouville numbers.

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