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The metric at infinity on Damek-Ricci spaces

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Abstract. Let $S = NA$ be a Damek-Ricci space, identified with the unit ball B in $\mathfrak{s}$ via the Cayley transform. Let $S^{p+q} = \partial B$ be the unit sphere in $\mathfrak{s}$, $p = \dim \mathfrak{v}$, $q = \dim \mathfrak{z}$. The metric in the ball model was computed in [1] both in Euclidean (or geodesic) polar coordinates and in Cartesian coordinates on B. The induced metric on the Euclidean sphere $S(R)$ of radius R is the sum of a constant curvature term, plus a correction term proportional to h_1, where h_1 is a suitable differential expression which is smooth on $S(R)$ for $R < 1$, but becomes (possibly) singular on the unit sphere at the pole $(0, 0, 1)$. It has a simple geometric interpretation, namely $h_1 = |\Theta|^2$, where Θ is, up to a conformal factor, the pull-back of the canonical 1-form on the group N (defining the horizontal distribution on N) by the generalized stereographic projection. In the symmetric case h_1, as well as the transported distribution on $S^{p+q} \setminus \{(0, 0, 1)\}$, have a smooth extension to the whole sphere. This can be interpreted by the Hopf fibration of S^{p+q}. In the general case no such structure is allowed on the unit sphere, and the question was left open in [1] whether or not h_1 extends smoothly at the pole. In this paper we prove that h_1 does not extend, except in the symmetric case. More precisely, writing h_1 in the coordinates (V, Z) on S^{p+q} as $h_1 = \sum h_{ij}^{(\mathfrak{z})}\, dz_i\, dz_j + \sum h_{ij}^{(\mathfrak{v})}\, dv_i\, dv_j + \sum h_{ij}^{(\mathfrak{z}\mathfrak{v})}\, dz_i\, dv_j$, we prove that, in the non-symmetric case, the coefficients $h_{ij}^{(\mathfrak{z})}$ do not have a limit at the pole, but remain bounded there, whereas the coefficients $h_{ij}^{(\mathfrak{v})}$ and $h_{ij}^{(\mathfrak{z}\mathfrak{v})}$ extend smoothly at the pole. In order to do this, we obtain an explicit formula for the 1-form Θ valid for any Damek-Ricci space. From this formula we deduce that Θ does not extend to the pole, except for $q = 1$ (Hermitian symmetric case). The square of Θ and the distribution $\ker \Theta$ do not extend, unless S is symmetric. Indeed, we prove that the singular part of h_1 vanishes identically if and only if the J^2-condition holds.

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