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Mappings of smallest mean distortion and free-Lagrangians

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Abstract. Let $\mathbb{X}, \mathbb{Y} \subset \mathbb{R}^n$ be bounded domains of the same topological type. We are concerned with mappings $f : \mathbb{X} \to \mathbb{Y}$, predominately orientation preserving homeomorphisms, in the Sobolev space $\mathscr{W}^{1,p}(\mathbb{X}, \mathbb{R}^n)$. Thus at almost every $x \in \mathbb{X}$ the linear differential map $Df(x) : \mathbf{T}_x\mathbb{X} \simeq \mathbb{R}^n \longrightarrow \mathbf{T}_y\mathbb{Y} \simeq \mathbb{R}^n$, $y = f(x)$, is represented by the Jacobian matrix $Df(x) \in \mathbb{R}^{n\times n}_+$. Hereafter $\mathbb{R}^{n\times n}_+$ denotes the space of $n \times n$-matrices with positive determinant.

A little reflection on Teichmüller's theory of extremal quasiconformal mappings provokes to study homeomorphisms with smallest $\mathscr{L}^p$-norm of the distortion functions $\mathcal{K}_\ell f \stackrel{\text{def}}{=\!=} \mathcal{K}_\ell[Df(x)]$, $1 \leqslant \ell \leqslant n-1, \mathcal{K}_\ell : \mathbb{R}^{n\times n}_+ \to [1,\infty)$. This being so, we seek to compute

$$\mathbf{K}^p_\ell(\mathbb{X}, \mathbb{Y}) \stackrel{\text{def}}{=\!=} \inf_f \int_{\mathbb{X}} \big[\mathcal{K}_\ell \mathbf{M}\big]^p \,\mathrm{d}x \qquad \mathbf{M} = Df(x). \tag{0.1}$$

The infimum is subjected to Sobolev homeomorphisms $f : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ with positive Jacobian determinant, $J_f(x) = \det Df(x) > 0$ a.e. Formal change of variables leads to an energy-integral for the inverse mappings $h = f^{-1} : \mathbb{Y} \xrightarrow{\text{onto}} \mathbb{X}$. This integral takes the form

$$\mathbf{E}_{\ell,\,p}(\mathbb{Y}, \mathbb{X}) \stackrel{\text{def}}{=\!=} \inf_h \int_{\mathbb{Y}} \big[\mathcal{K}_{n-\ell}\,\mathbf{N}\big]^p \,\det(\mathbf{N})\,\mathrm{d}y\ , \quad \mathbf{N} = Dh(y). \tag{0.2}$$

Equivalence of the minimization problems for f in (0.1) and that for h in (0.2) is a matter of a change of variables for Sobolev homeomorphisms. The concept of *free-Lagrangians* becomes ever more strategic. Broadly speaking, a free Lagrangian is a nonlinear differential n-form $\mathbf{L}(x, f, Df)\mathrm{d}x$, defined for Sobolev mappings $f : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$, whose integral depends only on the homotopy class of the mapping.

Free Lagrangians proved particularly useful in solving the $\mathscr{L}^p$*-Grötzsch problem* for ring domains in $\mathbb{R}^n$. Historically, the Grötzsch problem for $p = \infty$ has been of great interest in Geometric Function Theory (GFT); for example, in the 2-dimensional theory of Teichmüller spaces. In higher dimensions GFT flourished from the pioneering work of ***Fred*** (Frederick William Gehring). Thus our $\mathscr{L}^p-$ approach to GFT commemorates Fred's paper

"Rings and Quasiconformal Mappings in Space"
Transactions of AMS, **103** (1962), 353-393, 1962.

Precisely, we ask for homeomorphisms between ring domains having smallest $\mathscr{L}^p$-mean distortion. Call them $\mathscr{L}^p$*-Teichmüller mappings*.We investigate which pairs of ring domains admit $\mathscr{L}^p$-Teichmüller mappings.

It is somewhat surprising that the minimization of the $\mathscr{L}^1$-mean distortion leads to non-surjective mappings. Equivalently, in the variational problem (0.2),

we observe the lose of injectivity when passing to the limit of the energy-minimizing sequence of homeomorphisms. In the mathematical models of Nonlinear Elasticity this phenomenon amounts to saying that *interpenetration of matter* may occur when minimizing the energy at (0.2).

More surprisingly, the expected radial symmetry of a minimal mapping turns out to be false already in dimensions $n \geqslant 3$.

In several ways our study here grew out of the conceptual principles of Nonlinear Elasticity and Calculus of Variations. The novelty lies in the proofs, based on rather tricky inequalities; seemingly elementary but in fact challenging.

The art of free Lagrangians is
not to integrate nonlinear differential expressions,
but the correct choice of such expressions.

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