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Unbounded convex polyhedra as polynomial images of Euclidean spaces

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Abstract. In a previous work we proved that each n-dimensional convex polyhedron $\mathcal{K} \subset \mathbb{R}^n$ and its relative interior are regular images of $\mathbb{R}^n$. As the image of a non-constant polynomial map is an unbounded semialgebraic set, it is not possible to substitute regular maps by polynomial maps in the previous statement. In this work we determine constructively all unbounded n-dimensional convex polyhedra $\mathcal{K} \subset \mathbb{R}^n$ that are polynomial images of $\mathbb{R}^n$. We also analyze for which of them the interior $\operatorname{Int}(\mathcal{K})$ is a polynomial image of $\mathbb{R}^n$. A discriminating object is the recession cone $\vec{\mathcal{C}}(\mathcal{K})$ of $\mathcal{K}$. Namely, $\mathcal{K}$ *is a polynomial image of* $\mathbb{R}^n$ *if and only if* $\vec{\mathcal{C}}(\mathcal{K})$ *has dimension* n. In addition, $\operatorname{Int}(\mathcal{K})$ *is a polynomial image of* $\mathbb{R}^n$ *if and only if* $\vec{\mathcal{C}}(\mathcal{K})$ *has dimension* n *and* $\mathcal{K}$ *has no bounded faces of dimension* $n-1$. A key result is an improvement of Pecker's elimination of inequalities to represent semialgebraic sets as projections of algebraic sets. Empirical approaches suggest us that there are "few" polynomial maps that have a concrete convex polyhedron as a polynomial image and that there are even fewer for which it is affordable to show that their images actually correspond to our given convex polyhedron. This search of a "needle in the haystack" justifies somehow the technicalities involved in our constructive proofs.

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