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Metrical properties of finite products of partial quotients in arithmetic progressions

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Abstract. We investigate the dynamics of continued fractions and explore the ergodic behaviour of the products of mixed partial quotients in continued fractions of real numbers. For any function $\Phi : \mathbb{N} \to [2, +\infty)$ and any integer $d \geq 1$, we determine the Lebesgue measure and the Hausdorff dimension of the set of real numbers x for which the product of partial quotients in arithmetic progressions satisfy $a_n(x)a_{2n}(x)\cdots a_{dn}(x) \geq \Phi(n)$ for infinitely many positive integers n.

Our findings shed light on the size of the set of exceptions to Bourgain's (1988) and Host and Kra's (2005) theorems concerning the convergence of multiple ergodic averages for Gauss dynamical systems. By exploring the Hausdorff dimension of these sets, we gain valuable insights into the behaviour of such exceptions. Overall, our research contributes to a deeper understanding of the dynamics of continued fractions and their connection to the convergence properties of ergodic averages in Gauss dynamical systems.

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