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Geometry of Grassmannians and optimal transport of quantum states

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Abstract. Let H be a separable Hilbert space. Building on the metric geometry of the Grassmannian $\mathsf{P}_c(\mathsf{H})$ of finite-dimensional subspaces of H, we develop a theory of optimal transport for the normal states of the von Neumann algebra of linear and bounded operators $\mathsf{B}(\mathsf{H})$. Seeing density matrices as discrete probability measures on $\mathsf{P}_c(\mathsf{H})$ (via the spectral theorem) we define an optimal transport cost and the Wasserstein distance for normal states. We prove that the transport cost induces the w^*-topology and satisfies the triangular inequality on a dense subset of normal states.

Our construction is compatible with the quantum mechanics approach of composite systems as tensor products. We provide a natural interpretation of the pure normal states of $\mathsf{B}(\mathsf{H} \otimes \mathsf{H})$ as families of transport maps. In this way we assign a Wasserstein cost to each pure normal state of $\mathsf{B}(\mathsf{H} \otimes \mathsf{H})$, consistently with the transport cost defined via $\mathsf{P}_c(\mathsf{H})$.

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