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Infinite-time bubble towers in the fractional heat equation with critical exponent

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Abstract. In this paper, we consider the fractional heat equation with critical exponent in $\mathbb{R}^n$ for $n > 6s$, with $s \in (0, 1)$,

$$u_t = -(-\Delta)^s u + |u|^{\frac{4s}{n-2s}} u, \quad (x, t) \in \mathbb{R}^n \times \mathbb{R}.$$

We construct a bubble-tower-type solution for both the forward and the backward problem by establishing the existence of a sign-changing solution with multiple blow-up at a single point, having the form

$$u(x, t) = (1 + o(1)) \sum_{j=1}^{k} (-1)^{j-1} \mu_j(t)^{-\frac{n-2s}{2}} U\left(\frac{x}{\mu_j(t)}\right) \quad \text{as} \quad t \to +\infty,$$

and of a positive solution with multiple blow-up at a single point, having the form

$$u(x, t) = (1 + o(1)) \sum_{j=1}^{k} \mu_j(t)^{-\frac{n-2s}{2}} U\left(\frac{x}{\mu_j(t)}\right) \quad \text{as} \quad t \to -\infty,$$

respectively. Here $k \geq 2$ is a positive integer,

$$U(y) = \alpha_{n,s} \left(\frac{1}{1 + |y|^2}\right)^{\frac{n-2s}{2}},$$

where $\alpha_{n,s}$ is a constant depending only on n and s, and

$$\mu_j(t) = \beta_j |t|^{-\alpha_j} (1 + o(1)) \text{ as } t \to \pm\infty, \quad \alpha_j = \frac{1}{2s} \left(\frac{n-2s}{n-6s}\right)^{j-1} - \frac{1}{2s},$$

for certain positive numbers β_j, $j = 1, \cdots, k$.

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