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Estimates for maximal functions associated to hypersurfaces in $\mathbb{R}^3$ with height $h < 2$: Part II. A geometric conjecture and its proof for generic 2-surfaces

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Abstract. In this article, we continue the study of L^p-boundedness of the maximal operator $\mathcal{M}_S$ associated to averages along isotropic dilates of a given, smooth hypersurface S in 3-dimensional Euclidean space. We focus here on small surface-patches near a given point x^0 exhibiting singularities of type $\mathcal{A}$ in the sense of Arnol'd at this point; this is the situation which had remained open and which appears to create the biggest challenges in these studies. Denoting by p_c the minimal Lebesgue exponent such that $\mathcal{M}_S$ is L^p-bounded for $p > p_c$, we are able to identify p_c for all analytic surfaces of type $\mathcal{A}$ (with the exception of a small subclass $\mathcal{A}^e$), by means of quantities which can be determined from the associated Newton polyhedra. Besides the well-known notion of height at x^0, a new quantity, called the "effective multiplicity", turns out to play a crucial role here. We also state a conjecture on how the critical exponent p_c might be determined by means of a geometric measure-theoretic condition, which quantifies in some way the order of contact of arbitrary ellipsoids with S, even for hypersurfaces in arbitrary dimension, and we show that this conjecture holds indeed true for all classes of 2-hypersurfaces S for which we have gained an essentially complete understanding of $\mathcal{M}_S$ so far. Our results lead in particular to a proof of a conjecture by Iosevich-Sawyer-Seeger for arbitrary analytic 2-surfaces. What remains open is the case of surfaces in the exceptional class $\mathcal{A}^e$.

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