Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)
Vol. XXVI (2025), 1451-1474

A Kakeya maximal inequality in the Heisenberg group

KATRIN FÄSSLER, ANDREA PINAMONTI AND PIETRO WALD

Abstract. We define the Heisenberg Kakeya maximal functions $M_\delta f$, $0 < \delta < 1$, by averaging over δ-neighborhoods of horizontal unit line segments in the Heisenberg group $\mathbb{H}$ equipped with the Korányi distance $d_{\mathbb{H}}$. We show that for $p \in [1, \infty]$ and $\varepsilon > 0$

$$\|M_\delta f\|_{L^p(S^1)} \leq C(p,\varepsilon)\delta^{-\alpha(p)-\varepsilon}\|f\|_{L^p(\mathbb{H})} \quad f \in L^p(\mathbb{H})$$

where $\alpha(p) = \max\{4/p - 1, 1/p\}$ and, moreover, that $\alpha(p)$ is the smallest exponent for which the bound holds. The proof is based on a recent variant, due to Pramanik, Yang, and Zahl, of Wolff's circular maximal-function theorem for a class of planar curves related to Sogge's cinematic curvature condition. As an application of our Kakeya maximal inequality for $p = 3$, we recover the sharp lower bound for the Hausdorff dimension of Heisenberg Kakeya sets of horizontal unit line segments in $(\mathbb{H}, d_{\mathbb{H}})$, first proven by Liu.

Mathematics Subject Classification (2020): 28A80 (primary); 28A78, 42B25, 43A80 (secondary).