Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)
Vol. XXVI (2025), 1313-1347

Stability vs. instability of singular steady states in the parabolic-elliptic Keller-Segel system on $\mathbb{R}^n$

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Abstract. The Cauchy problem in $\mathbb{R}^n$ is considered for

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v) \\ 0 = \Delta v + u. \end{cases}$$

For each $n \geq 10$, a statement on stability and attractiveness of the singular steady state given by

$$u_\star(x) := \frac{2(n-2)}{|x|^2} \qquad x \in \mathbb{R}^n \setminus \{0\},$$

is derived within classes of non-negative radial solutions emanating from initial data less concentrated than $u_\star$. In particular, for any such n it is shown that infinite-time blow-up occurs for all radial initial data which are less concentrated than $u_\star$ and satisfy

$$u_0(x) \geq \frac{2(n-2)}{|x|^2} - \frac{C}{|x|^{2+\theta}} \qquad \text{for all } x \in \mathbb{R}^n \setminus B_1(0)$$

with some $C > 0$ and some $\theta > \frac{n-2+\sqrt{(n-2)(n-10)}}{2}$.

This is complemented by a result which, in the case when $3 \leq n \leq 9$, asserts instability of $u_\star$ as well as the existence of a bounded absorbing set for all radial trajectories initially less concentrated than $u_\star$.

In particular, previous knowledge on stability properties of $u_\star$, as obtained for $n \geq 11$ in [24], is thereby extended to any dimension $n \geq 3$.

Mathematics Subject Classification (2020): 35B35 (primary); 35B44, 35B40, 35K65, 92C17 (secondary).