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Generalised Rado and Roth criteria

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Dedicated to Sean Prendiville

Abstract. We study the Ramsey properties of equations $a_1 P(x_1) + \cdots + a_s P(x_s) = b$, where $a_1, \dots, a_s, b$ are integers, and P is an integer polynomial of degree d. Provided there are at least $(1+o(1))d^2$ variables, we show that Rado's criterion and an intersectivity condition completely characterise which equations of this form admit monochromatic solutions with respect to an arbitrary finite colouring of the positive integers. Furthermore, we obtain a Roth-type theorem for these equations, showing that they admit non-constant solutions over any set of integers with positive upper density if and only if $b = a_1 + \cdots + a_s = 0$. In addition, we establish sharp asymptotic lower bounds for the number of monochromatic/dense solutions (supersaturation).

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