Erratum to “Heights of points with bounded ramification”

LUKAS POTTMEYER

Abstract. This note is to inform that Lemma 5.8 in [1] is incorrect. We give a counterexample, locate the error in the proof and discuss the consequences to Theorem 5.9 in [1]. All other results in [1] are not affected by this error.

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1. [1, Lemma 5.8] is false

All citations refer to the original paper [1] and we also use the notation therein. The following counterexample to Lemma 5.8 was found by Francesco Amoroso and Lea Terracini.

Example 1.1. The prime 3 is totally ramified in \( \mathbb{Q}(\sqrt{3})/\mathbb{Q} \). Denote by \( w_0 \) the unique place of \( \mathbb{Q}(\sqrt{3}) \) with \( w_0 | 3 \). Then \( \sqrt{1 + 2\sqrt{3}} \in \mathbb{Q}(\sqrt{3})^{nr,w_0} \).

Assume for the sake of contradiction that \( \sqrt{1 + 2\sqrt{3}} \in \mathbb{Q}^{nr,3}(\sqrt{3}) \). Then there must exist \( a, b \in \mathbb{Q}^{nr,3} \) such that \( \sqrt{1 + 2\sqrt{3}} = a + b\sqrt{3} \). Squaring this equation yields \( 1 + 2\sqrt{3} = a^2 + 3b^2 + 2ab\sqrt{3} \). Since \( \sqrt{3} \notin \mathbb{Q}^{nr,3} \), it follows that \( b = a^{-1} \) and \( a^2 + 3a^{-2} = 1 \). The latter implies that \( a \) is a root of \( x^4 - x^2 - 3 \). After a short calculation we know that the discriminant of \( \mathbb{Q}(a) \) is \(-8112\) and therefore 3 ramifies in \( \mathbb{Q}(a) \). In particular, \( a \notin \mathbb{Q}^{nr,3} \) contradicting our assumption. Hence, \( \sqrt{1 + 2\sqrt{3}} \notin \mathbb{Q}^{nr,3}(\sqrt{3}) \) and

\[
\mathbb{Q}\left(\sqrt{3}\right)^{nr,w_0} \neq \mathbb{Q}^{nr,3}\left(\sqrt{3}\right).
\]

Setting \( K = \mathbb{Q}, v = 3 \) and \( \alpha = \sqrt{3} \), this is a counterexample to the statement of Lemma 5.8.

Remark 1.2. The mistake in the proof of Lemma 5.8 is the following. It is correct that \( K^{nr,v}(\alpha) \subset \bigcap_{i=1}^n K(\alpha)^{nr,w_i} \). Moreover, it is true that \( \beta \in L^w(\alpha) \) for all

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$w \in M_{K(\alpha, \beta)}$, $w \mid v$. However, it not true that this implies the existence of a number field $L \subseteq \cap_w L^w$ such that $\beta \in L(\alpha)$, where $w$ runs through all extensions of $v$ to $K(\alpha, \beta)$.

Due to this failure, the statement of Theorem 5.9 has to be replaced by the following result.

**Theorem 1.3.** Let $E$ be an elliptic curve defined over a number field $K$. The field $K^{nr, v}(\alpha)$ has the Bogomolov property relative to $\widehat{h}_E$, if there is a $w \in M_{K(\alpha)}$, $w \mid v$, such that $E/K(\alpha)$ has bad reduction at $w$.

This follows immediately from [1, Theorem 4.1, Corollary 5.1, Proposition 5.4] and the correct inclusion $K^{nr, v}(\alpha) \subseteq \cap_{i=1}^n K(\alpha)^{nr, w_i}$.

**Remark 1.4.** The and only if-part of Theorem 5.9 is not justified anymore. Hence, the only known cases where the Bogomolov property relative to $\widehat{h}_E$ is not preserved under finite field extensions are those described in Example 5.7. It remains open whether there are further examples.

**References**