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Estimates for the Weyl coefficient of a two-dimensional canonical system

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Abstract. For a two-dimensional canonical system $y'(t) = zJH(t)y(t)$ on some interval (a, b) whose Hamiltonian H is a.e. positive semi-definite and which is regular at a and in the limit point case at b, denote by q_H its Weyl coefficient. De Branges' inverse spectral theorem states that the assignment $H \mapsto q_H$ is a bijection between Hamiltonians (suitably normalised) and Nevanlinna functions.

We give upper and lower bounds for $|q_H(z)|$ and $\operatorname{Im} q_H(z)$ when z tends to $i\infty$ non-tangentially. These bounds depend on the Hamiltonian H near the left endpoint a and determine $|q_H(z)|$ up to universal multiplicative constants. We obtain that the growth of $|q_H(z)|$ is independent of the off-diagonal entries of H and depends monotonically on the diagonal entries in a natural way. The imaginary part is, in general, not fully determined by our bounds (in a forthcoming work we shall prove that for "most" Hamiltonians also $\operatorname{Im} q_H(z)$ is fully determined).

We translate the asymptotic behaviour of q_H to the behaviour of the spectral measure μ_H of H by means of Abelian-Tauberian results and obtain conditions for membership of growth classes defined by weighted integrability condition (Kac classes) or by boundedness of the tails at $\pm\infty$ with respect to a weight function. Moreover, we apply our results to Krein strings and Sturm-Liouville equations.

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