Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)
Vol. XXV (2024), 565-604

# Equidistribution in the space of 3-lattices and Dirichlet-improvable vectors on planar lines

DMITRY KLEINBOCK, NICOLAS DE SAXCÉ, NIMISH A. SHAH AND PENGYU YANG

**Abstract.** Let $X = \mathrm{SL}_3(\mathbb{R})/\mathrm{SL}_3(\mathbb{Z})$ and $g_t = \mathrm{diag}(e^{2t}, e^{-t}, e^{-t})$. Let $\nu$ denote the push-forward of the normalized Lebesgue measure on a segment of a straight line in the expanding horosphere of $\{g_t\}_{t>0}$, under the map $h \mapsto h\,\mathrm{SL}_3(\mathbb{Z})$ from $\mathrm{SL}_3(\mathbb{R})$ to $X$. We give explicit necessary and sufficient Diophantine conditions on the line for equidistribution of each of the following families of measures on $X$:

(1) $g_t$-translates of $\nu$ as $t \to \infty$;
(2) Averages of $g_t$-translates of $\nu$ over $t \in [0, T]$ as $T \to \infty$;
(3) $g_{t_i}$-translates of $\nu$ for some $t_i \to \infty$.

We apply this dynamical result to show that Lebesgue-almost every point on the planar line $y = ax + b$ is not Dirichlet-improvable if and only if $(a, b) \notin \mathbb{Q}^2$.

**Mathematics Subject Classification (2020):** 37A17 (primary); 11J13, 11J83, 22E46, 14L24 (secondary).