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Satake-Furstenberg compactifications and gradient map

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Abstract. Let G be a real semisimple Lie group with finite center and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition of its Lie algebra. Let K be a maximal compact subgroup of G with Lie algebra $\mathfrak{k}$ and let τ be an irreducible representation of G on a complex vector space V. Let h be a Hermitian scalar product on V such that $\tau(G)$ is compatible with respect to $\mathrm{SU}(V,h)^{\mathbb{C}}$. We denote by $\mu_{\mathfrak{p}} : \mathbb{P}(V) \longrightarrow \mathfrak{p}$ the G-gradient map and by $\mathcal{O}$ the unique closed orbit of G in $\mathbb{P}(V)$, which is a K-orbit [27, 32], contained in the unique closed orbit of the Zariski closure of $\tau(G)$ in $\mathrm{SU}(V,h)^{\mathbb{C}}$. We prove that up to equivalence the set of irreducible representations of parabolic subgroups of G induced by τ is completely determined by the facial structure of the polar orbitope $\mathcal{E} = \mathrm{conv}(\mu_{\mathfrak{p}}(\mathcal{O}))$. Moreover, any parabolic subgroup of G admits a unique closed orbit in $\mathcal{O}$ which is well-adapted to $\mu_{\mathfrak{p}}$. These results are new also in the complex reductive case. The connection between $\mathcal{E}$ and τ provides a geometrical description of the Satake compactifications without root data. In this context the properties of the Bourguignon-Li-Yau map are also investigated. Given a measure γ on $\mathcal{O}$, we construct a map Ψ_γ from the Satake compactification of G/K associated to τ and $\mathcal{E}$. If γ is a K-invariant measure then Ψ_γ is a homeomorphism of the Satake compactification and $\mathcal{E}$. Finally, we prove that for a large class of measures the map Ψ_γ is surjective.

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