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Some rigidity results for Sobolev inequalities and related PDEs on Cartan-Hadamard manifolds

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Abstract. The Cartan-Hadamard conjecture states that, on every n-dimensional Cartan-Hadamard manifold $\mathbb{M}^n$, the isoperimetric inequality holds with Euclidean optimal constant, and any set attaining equality is necessarily isometric to a Euclidean ball. This conjecture was settled, with positive answer, for $n \leq 4$. It was also shown that its validity in dimension n ensures that every p-Sobolev inequality ($1 < p < n$) holds on $\mathbb{M}^n$ with Euclidean optimal constant. In this paper we address the problem of classifying all Cartan-Hadamard manifolds supporting an optimal function for the Sobolev inequality. We prove that, under the validity of the n-dimensional Cartan-Hadamard conjecture, the only such manifold is $\mathbb{R}^n$, and therefore any optimizer is an Aubin-Talenti profile (up to isometries). In particular, this is the case in dimension $n \leq 4$.

Optimal functions for the Sobolev inequality are weak solutions to the critical p-Laplace equation. Thus, in the second part of the paper, we address the classification of radial solutions (not necessarily optimizers) to such a PDE. Actually, we consider the more general critical or supercritical equation

$$-\Delta_p u = u^q \,, \quad u > 0 \,, \qquad \text{on } \mathbb{M}^n \,,$$

where $q \geq p^* - 1$. We show that if there exists a radial finite-energy solution, then $\mathbb{M}^n$ is necessarily isometric to $\mathbb{R}^n$, $q = p^* - 1$ and u is an Aubin-Talenti profile. Furthermore, on model manifolds, we describe the asymptotic behavior of radial solutions not lying in the energy space $\dot{W}^{1,p}(\mathbb{M}^n)$, studying separately the p-stochastically complete and incomplete cases.

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