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Five-dimensional para-CR manifolds and contact projective geometry in dimension three

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Abstract. We study invariant properties of 5-dimensional para-CR structures whose Levi form is degenerate in precisely one direction and which are 2-non-degenerate. We realize that *two*, out of three, primary (basic) para-CR invariants of such structures are the classical differential invariants known to *Monge* (1810) and to *Wünschmann* (1905):

$$M(G) := 40G_{ppp}^3 - 45G_{pp}G_{ppp}G_{pppp} + 9G_{pp}^2 G_{ppppp},$$
$$W(H) := 9D^2 H_r - 27DH_p - 18H_r DH_r + 18H_p H_r + 4H_r^3 + 54H_z.$$

The vanishing $M(G) \equiv 0$ provides a local necessary and sufficient condition for the graph of a function in the (p, G)-plane to be contained in a conic, while the vanishing $W(H) \equiv 0$ gives an *if-and-only-if* condition for a 3rd order ODE to define a natural Lorentzian geometry on the space of its solutions.

Mainly, we give a geometric interpretation of the *third* basic invariant of our class of para-CR structures, the simplest one, of lowest order, and of mixed nature $N(G, H) := 2G_{ppp} + G_{pp}H_{rr}$. We establish that the vanishing $N(G, H) \equiv 0$ gives an *if-and-only-if* condition for the *two* 3-dimensional quotients of the para-CR manifold by its two canonical integrable rank-2 distributions, to be equipped with contact projective geometries.

A curious transformation between the Wünschmann invariant and the Monge invariant, first noted by us in a recent publication [15], is also discussed, and its mysteries are further revealed.

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