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# Linear forms in polylogarithms

SINNOU DAVID, NORIKO HIRATA-KOHNO AND MAKOTO KAWASHIMA

**Abstract.** Let $r$, $m$ be positive integers. Let $x$ be a rational number with $0 \leq x < 1$. Consider $\Phi_s(x,z) = \sum_{k=0}^{\infty} \frac{z^{k+1}}{(k+x+1)^s}$ the $s$-th Lerch function with $s = 1, 2, \cdots, r$. When $x = 0$, this is a polylogarithmic function. Let $\alpha_1, \cdots, \alpha_m$ be pairwise distinct algebraic numbers of *arbitrary degree* over the rational number field, with $0 < |\alpha_j| < 1$ $(1 \leq j \leq m)$. In this article, we show a criterion for the linear independence, over an algebraic number field containing $\mathbb{Q}(\alpha_1, \cdots, \alpha_m)$, of all the $rm + 1$ numbers: $\Phi_1(x, \alpha_1)$, $\Phi_2(x, \alpha_1)$, $\cdots$, $\Phi_r(x, \alpha_1)$, $\Phi_1(x, \alpha_2)$, $\Phi_2(x, \alpha_2)$, $\cdots$, $\Phi_r(x, \alpha_2), \cdots\cdots$, $\Phi_1(x, \alpha_m)$, $\Phi_2(x, \alpha_m)$, $\cdots$, $\Phi_r(x, \alpha_m)$ and 1. This is the first result that gives a sufficient condition for the linear independence of values of the Lerch functions at *several distinct algebraic points*, *not necessarily lying in the rational number field nor in quadratic imaginary fields.* We give a complete proof with refinements and quantitative statements of the main theorem announced in [9].

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