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Moderate solutions of semilinear elliptic equations with Hardy potential under minimal restrictions on the potential

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Abstract. We study semilinear elliptic equations with Hardy potential

$$-\mathscr{L}_\mu u + u^q = 0 \qquad (E)$$

in a bounded smooth domain $\Omega \subset \mathbb{R}^N$. Here $q > 1$, $\mathscr{L}_\mu = \Delta + \frac{\mu}{\delta_\Omega^2}$ and $\delta_\Omega(x) = \mathrm{dist}\,(x, \partial\Omega)$. Assuming that $0 \le \mu < C_H(\Omega)$, boundary value problems with measure data and discrete boundary singularities for positive solutions of (E) have been studied in [10]. In the case of *convex* domains $C_H(\Omega) = 1/4$. In this case similar problems have been studied in [8]. In the present paper we study these problems, in *arbitrary domains*, assuming only $-\infty < \mu < 1/4$, even if $C_H(\Omega) < 1/4$. We recall that $C_H(\Omega) \le 1/4$ and, in general, strict inequality holds. The key to our study is the fact that, if $\mu < 1/4$ then in smooth domains there exist local $\mathscr{L}_\mu$-superharmonic functions in a neighborhood of $\partial\Omega$ (even if $C_H(\Omega) < 1/4$). Using this fact we extend the notion of *normalized boundary trace*, introduced in [10], to arbitrary domains, provided that $\mu < 1/4$. Further we study the b.v.p. with normalized boundary trace ν in the space of positive finite measures on $\partial\Omega$. We show that existence depends on two critical values of the exponent q and discuss the question of uniqueness. Part of the paper is devoted to the study of the linear operator: properties of local $\mathscr{L}_\mu$-subharmonic and superharmonic functions and the related notion of moderate solutions. Here we extend and/or improve results of [5] and [10] which are later used in the study of the nonlinear problem.

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