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The angular Laplacian on symmetric Damek-Ricci spaces

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Abstract. Let $S = NA$ be a symmetric Damek-Ricci space, identified with the unit ball B in $\mathfrak{s}$ via the Cayley transform. The induced metric on the geodesic sphere $S(r)$ of radius $r > 0$ can be written down and interpreted in terms of the (generalized) Hopf fibration of $S^{p+q} = \partial B$, the unit sphere in $\mathfrak{s} = \mathfrak{v} \oplus \mathfrak{z} \oplus \mathfrak{a}$ ($q = \dim \mathfrak{z}$, $p = \dim \mathfrak{v}$). We describe the bundle connection in the coordinates $(V, Z, t) \in \mathfrak{s}$. We obtain explicit formulae for a basis of horizontal-vertical vector fields and their Lie brackets. Whenever possible, we give classification independent proofs, using only the J^2-condition and related identities. We demonstrate a formula for the angular Laplacian $L_{S(r)}$ in terms of the round Laplacian on S^{p+q} and the so-called vertical Laplacian, which differentiates along the vertical directions of the fibration. The spectrum of $L_{S(r)}$ is computed by an approach which is mainly analytic. We describe the eigenspaces in terms of bihomogeneus harmonic polynomials in $\mathfrak{s} \simeq \mathbb{C}^d$, $d = (p + q + 1)/2$. The result for the octonionic Hopf fibration of S^{15} is similar to the quaternionic result, obtained recently in [1]. In the final section we use the model of S^{p+q} as the homogeneous space K/M of symmetric space theory to describe the Hopf fibration as a homogeneous fibration, and rederive the formula for $L_{S(r)}$ and its spectrum by this approach. We also discuss some open problems related to the spectrum of the angular Laplacian on the geodesic spheres of non-symmetric Damek-Ricci spaces.

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