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Comparing $\mathbb{A}^1$-h-cobordism and $\mathbb{A}^1$-weak equivalence

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Abstract. We study the problem of classifying projectivizations of rank-two vector bundles over $\mathbb{P}^2$ up to two notions of equivalence that arise naturally in $\mathbb{A}^1$-homotopy theory, namely $\mathbb{A}^1$-weak equivalence and $\mathbb{A}^1$-h-cobordism.

First, we classify such varieties up to $\mathbb{A}^1$-weak equivalence: over algebraically closed fields having characteristic unequal to two the classification can be given in terms of characteristic classes of the underlying vector bundle. When the base field is $\mathbb{C}$, this classification result can be compared to a corresponding topological result and we find that the algebraic and topological homotopy classifications agree.

Second, we study the problem of classifying such varieties up to $\mathbb{A}^1$-h-cobordism using techniques of deformation theory. To this end, we establish a deformation rigidity result for $\mathbb{P}^1$-bundles over $\mathbb{P}^2$ which links $\mathbb{A}^1$-h-cobordisms to deformations of the underlying vector bundles. Using results from the deformation theory of vector bundles we show that if X is a $\mathbb{P}^1$-bundle over $\mathbb{P}^2$ and Y is the projectivization of a direct sum of line bundles on $\mathbb{P}^2$, then if X is $\mathbb{A}^1$-weakly equivalent to Y, X is also $\mathbb{A}^1$-h-cobordant to Y.

Finally, we discuss some subtleties inherent in the definition of $\mathbb{A}^1$-h- cobordism. We show, for instance, that direct $\mathbb{A}^1$-h-cobordism fails to be an equivalence relation.

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