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Lorentzian area measures and the Christoffel problem

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Abstract. We introduce a particular class of unbounded closed convex sets of $\mathbb{R}^{d+1}$, called F-convex sets (F stands for future). To define them, we use the Minkowski bilinear form of signature $(+, \dots, +, -)$ instead of the usual scalar product, and we ask the Gauss map to be a surjection onto the hyperbolic space $\mathbb{H}^d$. Important examples are embeddings of the universal cover of some globally hyperbolic maximal flat Lorentzian manifolds.

Basic tools are first derived, similarly to the classical study of convex bodies. For example, F-convex sets are determined by their support function, which is defined on $\mathbb{H}^d$. Then the area measures of order i, with $0 \leq i \leq d$ are defined. As in the convex bodies case, they are the coefficients of the polynomial in ε which is the volume of an ε-approximation of the convex set. Here the area measures are defined with respect to the Lorentzian structure.

Then we focus on the area measure of order one. Finding necessary and sufficient conditions for a measure (here on $\mathbb{H}^d$) to be the first area measure of an F-convex set is the Christoffel Problem. We derive many results about this problem. If we restrict to F-convex set setwise invariant under linear isometries acting cocompactly on $\mathbb{H}^d$, then the problem is totally solved, analogously to the case of convex bodies. In this case the measure can be given on a compact hyperbolic manifold.

Particular attention is given on the smooth and polyhedral cases. In these cases, the Christoffel problem is equivalent to prescribing the mean radius of curvature and the edge lengths, respectively.

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