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Einstein-like geometric structures on surfaces

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Abstract. An AH (affine hypersurface) structure is a pair comprising a projective equivalence class of torsion-free connections and a conformal structure satisfying a compatibility condition which is automatic in two dimensions. They generalize Weyl structures, and a pair of AH structures is induced on a co-oriented non-degenerate immersed hypersurface in flat affine space. The author has defined for AH structures Einstein equations, which specialize on the one hand to the usual Einstein-Weyl equations and, on the other hand, to the equations for affine hyperspheres. Here these equations are solved for Riemannian signature AH structures on compact orientable surfaces, the deformation spaces of solutions are described, and some aspects of the geometry of these structures are related. Every such structure is either Einstein-Weyl (in the sense defined for surfaces by Calderbank) or is determined by a pair comprising a conformal structure and a cubic holomorphic differential, and so by a convex flat real projective structure. In the latter case it can be identified with a solution of the Abelian vortex equations on an appropriate power of the canonical bundle. On the cone over a surface of genus at least two carrying an Einstein AH structure there are Monge-Ampère metrics of Lorentzian and Riemannian signature and a Riemannian Einstein Kähler affine metric. A mean curvature zero space-like immersed Lagrangian submanifold of a para-Kähler four-manifold with constant para-holomorphic sectional curvature inherits an Einstein AH structure, and this is used to deduce some restrictions on such immersions.

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