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Approximation of complex algebraic numbers by algebraic numbers of bounded degree

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Abstract. To measure how well a given complex number ξ can be approximated by algebraic numbers of degree at most n one may use the quantities $w_n(\xi)$ and $w_n^*(\xi)$ introduced by Mahler and Koksma, respectively. The values of $w_n(\xi)$ and $w_n^*(\xi)$ have been computed for real algebraic numbers ξ, but up to now not for complex, non-real algebraic numbers ξ. In this paper we compute $w_n(\xi)$, $w_n^*(\xi)$ for all positive integers n and algebraic numbers $\xi \in \mathbf{C} \setminus \mathbf{R}$, except for those pairs (n, ξ) such that n is even, $n \geq 6$ and $n + 3 \leq \deg \xi \leq 2n - 2$. It is known that every real algebraic number of degree $> n$ has the same values for w_n and w_n^* as almost every real number. Our results imply that for every positive even integer n there are complex algebraic numbers ξ of degree $> n$ which are unusually well approximable by algebraic numbers of degree at most n, *i.e.*, have larger values for w_n and w_n^* than almost all complex numbers. We consider also the approximation of complex non-real algebraic numbers ξ by algebraic integers, and show that if ξ is unusually well approximable by algebraic numbers of degree at most n then it is unusually badly approximable by algebraic integers of degree at most $n + 1$. By means of Schmidt's Subspace Theorem we reduce the approximation problem to compute $w_n(\xi)$, $w_n^*(\xi)$ to an algebraic problem which is trivial if ξ is real but much harder if ξ is not real. We give a partial solution to this problem.

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