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# Rational fixed points for linear group actions

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**Abstract.** We prove a version of the Hilbert Irreducibility Theorem for linear algebraic groups. Given a connected linear algebraic group $G$, an affine variety $V$ and a finite map $\pi : V \to G$, all defined over a finitely generated field $\kappa$ of characteristic zero, Theorem 1.6 provides the natural necessary and sufficient condition under which the set $\pi(V(\kappa))$ contains a Zariski dense sub-semigroup $\Gamma \subset G(\kappa)$; namely, there must exist an unramified covering $p : \tilde{G} \to G$ and a map $\theta : \tilde{G} \to V$ such that $\pi \circ \theta = p$. In the case $\kappa = \mathbb{Q}$, $G = \mathbb{G}_a$ is the additive group, we reobtain the original Hilbert Irreducibility Theorem. Our proof uses a new diophantine result, due to Ferretti and Zannier [9]. As a first application, we obtain (Theorem 1.1) a necessary condition for the existence of rational fixed points for all the elements of a Zariski-dense sub-semigroup of a linear group acting morphically on an algebraic variety. A second application concerns the characterisation of algebraic subgroups of $\mathrm{GL}_N$ admitting a Zariski-dense sub-semigroup formed by matrices with at least one rational eigenvalue.

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